Modified Adomian Method for the Generalized Inhomogeneous Lane-Emden-Type Equations

M. AL-Mazmumy, A. A. Alsulami, H. O. Bakodah and N. Alzaid

Department of Mathematics, Faculty of Science
University of Jeddah
P.O. Box 80327, Jeddah, Saudi Arabia

This article is distributed under the Creative Commons by-nc-nd Attribution License.

Copyright © 2022 Hikari Ltd.

Abstract

The present study investigates certain singular Initial-Value Problems (IVPs) featuring the classical and generalized inhomogeneous Lane-Emden-type equations. These equations are very important models as they appear in many physical applications, including thermodynamics to mention a few. Further, the study proposes different forms of inverse integral operators that are based on the Adomian method to accelerate the convergence rate of the standard Adomian Decomposition Method (ADM). The method is then applied to various types of linear and nonlinear test problems and was found to be an effective modification by monitoring the rapidity of the convergence rate.

Keywords: Singular ODEs; Lane-Emden-type equations; Adomian decomposition method; inverse integral operators

1 Introduction

Various realistic applications of physics and engineering are mathematically represented through Initial-Value Problems (IVPs) of Ordinary Differential Equations (ODEs). Numerous physical models that arise from sciences, economics, management, and social sciences to state a few are modeled using ODEs. This vital reason is what causes many researchers to be more competitive with regard to the provision of reliable mathematical methods to treat
such equations. For instance, the standard Adomian Decomposition Method (ADM) [1,2] has been proven to be a skillful method that solves different forms of nonlinear ODEs iteratively, and effortlessly. This method has been modified by several researchers in order to boost the convergence process of the method, see the modifications given in [3,4,9,11] and the references therewith, to mention a few.

However, a famous class of ODEs that has not been sufficiently explored in the literature is the class of singular equations [5,8]. This class is somewhat difficult to tackle even with the ADM despite its efficiency in solving diverse forms of ODEs. Thus, Wazwaz [10] proposed a convenient modification to ADM by launching a new differential operator to tackle a Lane-Emden equation. Lane-Emden equation is a singular ODE that is derived from a more general singular equation called the Emden-Fowler equation. We also mention the work of Hosseini [7] where yet a new differential operator was proposed to solve various forms of singular and nonsingular ODEs. Hence, the present study aims at investigating certain singular IVPs featuring the classical and generalized Lane-Emden-type equations through the construction of different forms of inverse integral operators based on the standard ADM. The purpose of this construction is to enhance the convergence rate of the standard ADM, especially in the presence of singularity. The study would also assess the efficiency of these new operators on different linear and nonlinear test problems.

2 Classical Lane-Emden equation

Let us start off by recalling the definition of the classical Emden-Fowler equation. This equation is a very important model that appears in many problems of mathematical physics and engineering. Various applications have been reported in the literature to rely on this model. More precisely, the model is a singular second-order nonlinear ODE that is defined as follows

\[ v'' + 2t v' + af(t)g(v) = 0, \]

where \( a \) is a real constant, and \( f(t) \) and \( g(v) \) are prescribed functions in \( t \) and \( v \), sequentially.

However, when the functions \( f(t) \) and \( g(v) \) in the above Emden-Fowler model take the form \( f(t) = 1 \) and \( g(v) = v^n \), then the model reduces to the well-known classical Lane-Emden equation as follows

\[ v'' + \frac{2}{t} v' + av^n = 0. \]  \( (2) \)

This new model given in Eq. (2) is again a vital equation that is used in
modeling various phenomena in general thermodynamics. Additionally, several
types of this very important model are realized by considering different forms
of the function \( g(v) \). For instance, one could see diverse applications from
thermal and isothermal behaviors of spherical gas clouds, theory of harmonic
current, and the theory of stellar structure to mention [11].

What’s more, these classical models given in Eqs. (1) and (2) were gen-
eralized in the literature by considering the coefficient of \( v' \) in both models
to be \( \frac{n}{t} \), for some real number \( n, n \geq 0 \) [10]. Thus, we get the generalized
Emden-Fowler and generalized Lane-Emden equations, respectively, as follows
\[
\frac{v''}{n} + \frac{1}{t}v' + af(t)g(v) = 0, \quad n \geq 0,
\]
and
\[
\frac{v''}{n} + \frac{1}{t}v' + av^n = 0, \quad n \geq 0.
\]
Hence, in what follows, we will be presenting a modification method based
on the standard ADM to recurrently solve various IVPs featuring different
forms of the inhomogeneous Lane-Emden-type and generalized Lane-Emden-
type equations.

3 Methods for the inhomogeneous Lane-Emden
type equations

This section gives some elegant algorithms for the recurrent solution of the
inhomogeneous classical Lane-Emden-type equations. These algorithms are
based on the modified ADM for the solution of various functional equations.
Different differential operators and their corresponding inverse integral oper-
ators will be recalled and thereafter applied to treat several variants of the
classical model under consideration.

3.1 Algorithm one

To demonstrate the present algorithm, let us consider the following IVP fea-
turing an inhomogeneous Lane-Emden-type equation via [10] as follows
\[
\begin{align*}
v'' + \frac{2}{t}v' + f(t, v) &= g(t), \quad 0 < t \leq 1, \\
v(0) &= C_1, \\
v'(0) &= C_2,
\end{align*}
\]
where \( f(t, v) \) is a prescribed continuous real-valued function, \( g(t) \) is a pre-
scribed inhomogeneous term that is also continuous real-valued function, while
\( C_1 \) and \( C_2 \) are real constants.
More so, on using the ADM process, we first rewrite the ODE given in Eq. (5) through an operator notation as follows

\[ Lv = -f(t, v) + g(t), \]  
\[ \text{(6)} \]

where the differential operator \( L \) and its corresponding two-fold inverse integral operator \( L^{-1} \) are considered as follows

\[ L = t^{-2} \frac{d}{dt} \left( t^2 \frac{d}{dt} \right), \quad L^{-1}(.) = \int_0^t t^{-2} \int_0^t t^2(.) dt dt. \]
\[ \text{(7)} \]

Then, applying the inverse operator \( L^{-1} \) to Eq. (6) yields the following

\[ v(t) = C_1 + C_2 t + L^{-1} g(t) - L^{-1} f(t, v). \]
\[ \text{(8)} \]

Therefore, the ADM decomposes the solution \( v(t) \) and the nonlinear term \( f(t, v) \) as infinite series as follows

\[ v(t) = \sum_{n=0}^{\infty} v_n(t), \quad f(t, v) = \sum_{n=0}^{\infty} A_n, \]
\[ \text{(9)} \]

where \( v_n(t) \) are the components of the solution \( v(t) \); while \( A_n \)'s are the Adomian polynomials that are recurrently computed using the following relation

\[ A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ N \left( \sum_{i=0}^{n} \lambda^i v_i \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, ... \]
\[ \text{(10)} \]

Thus, substituting Eq. (9) into Eq. (8) gives

\[ \sum_{n=0}^{\infty} v_n(t) = C_1 + C_2 t + L^{-1} g(t) - L^{-1} \sum_{n=0}^{\infty} A_n, \]
\[ \text{(11)} \]

of which the recurrent solution is thus obtained via the ADM process as follows

\[ \begin{cases} 
    v_0 = C_1 + C_2 t + L^{-1} g(t), \\
    v_{k+1} = -L^{-1} A_k, \quad k \geq 0.
\end{cases} \]
\[ \text{(12)} \]

More so, we express some of the components of the above recurrent relation as follows

\[ \begin{align*}
    v_0 &= C_1 + C_2 t + L^{-1} g(t), \\
    v_1 &= -L^{-1} A_0, \\
    v_2 &= -L^{-1} A_1, \\
    v_3 &= -L^{-1} A_2, \\
    \vdots
\end{align*} \]
\[ \text{(13)} \]
such that the \( n \)-term approximation takes the form

\[
\Phi_n = \sum_{k=0}^{n-1} v_k,
\]

where the closed-form solution is finally revealed as

\[
v(t) = \lim_{n \to \infty} \Phi_n(t) = \sum_{k=0}^{\infty} v_k(t).
\]

### 3.2 Algorithm two

Let us also reconsider the IVP given in Eq. (5) via an operator denotation as follows [4]

\[
Lv = g(t) - f(t, v),
\]

where the differential operator \( L \) and its corresponding two-fold inverse integral operator \( L^{-1} \) are considered as follows

\[
L = t^{-1} \frac{d^2}{dt^2} t v, \quad L^{-1}(\cdot) = t^{-1} \int_0^t \int_0^t t(\cdot) dtdt.
\]

Now, we apply the inverse operator \( L^{-1} \) expressed above to the first two terms \( v'' + \frac{2}{t} v' \) of Eq. (5) as follows

\[
L^{-1}(v'' + \frac{2}{t} v') = t^{-1} \int_0^t \int_0^t t(v'' + \frac{2}{t} v') dtdt,
\]

\[
= t^{-1} \int_0^t (tv' + v - v(0)) dt,
\]

\[
= v - v(0).
\]

Furthermore, applying the inverse operator \( L^{-1} \) to Eq. (16) yields the following

\[
v(t) = C_1 + L^{-1} g(t) - L^{-1} f(t, v),
\]

such that on making use of the modification of ADM process by decomposing \( v(t) \) and \( f(t, v) \) as suggested above yields the following recurrent relation

\[
\begin{align*}
\{ & v_0 = C_1 + L^{-1} g(t), \\
& v_{k+1} = -L^{-1} A_k, \quad k \geq 0.
\end{align*}
\]
3.3 Algorithm three

For the present algorithm, let us consider the following IVP featuring an inhomogeneous Lane-Emden-type equation [6]

$$\begin{cases}
  v'' + \frac{2}{t}v' + av = g(t) + N(v), \\
  v(0) = C_1, \quad v'(0) = C_2,
\end{cases} \quad (21)$$

where $g(t)$ is prescribed known function, $N(v)$ is a given nonlinear term; while $C_1, C_2$ are real constants.

So, on using the ADM process, the ODE in the above system is expressed through operator notation as follows

$$Lv = g(t) + N(v), \quad (22)$$

where the differential operator $L$ and its corresponding two-fold inverse integral operator $L^{-1}$ are considered as follows

$$L(.) = \frac{1}{t \sin \sqrt{at}} \frac{d}{dt} \frac{d}{dt} \sin^2 \sqrt{at} \frac{d}{dt} \frac{t}{\sin \sqrt{at}} (.),$$

$$L^{-1}(.) = \frac{\sin \sqrt{at}}{t} \int_{0}^{t} \sin^{-2} \sqrt{at} \int_{0}^{t} \sin \sqrt{at} dt dt. \quad (23)$$

Thus, operating $L^{-1}$ on Eq. (22) yields

$$v(t) = \psi(t) + L^{-1}(g(t)) + L^{-1}(N(v)), \quad (24)$$

where the function $\psi(t)$ emanates from the prescribed initial data. such that

$$L(\psi(t)) = 0. \quad (25)$$

Accordingly, the ADM gives

$$\sum_{n=0}^{\infty} v_n = \psi(t) + L^{-1}(g(t)) + L^{-1} \sum_{n=0}^{\infty} A_n, \quad (26)$$

such that the final recurrent relation takes the following form

$$\begin{cases}
  v_0 = \psi(t) + L^{-1}(g(t)), \\
  v_{n+1} = L^{-1} A_n, \quad n \geq 0,
\end{cases} \quad (27)$$

where $A_k$'s are the Adomian polynomials resulting from the nonlinear term $N(v)$. 

4 Methods for the generalized inhomogeneous Lane-Emden type equations

This section equally outlines certain algorithms for the recurrent solution of the inhomogeneous generalized Lane-Emden-type equations. The algorithms are also based on the modified ADM. Different differential operators and their corresponding inverse integral operators are similarly devised for the treatment of several variants of the generalized model under consideration.

4.1 Algorithm four

Let us consider the following IVP featuring a generalized inhomogeneous Lane-Emden-type equation as follows [10]

\[
\begin{cases}
v'' + \frac{n}{t}v' + f(t,v) = g(t), \quad n \geq 0, \\
v(0) = C_1, \quad v'(0) = C_2,
\end{cases}
\]

where \( f(t,v) \) is a prescribed continuous real-valued function, \( g(t) \) is a prescribed inhomogeneous term that is also continuous real-valued function, \( C_1 \) and \( C_2 \) are real constants; while \( n \) is some real number \( n, n \geq 0 \).

Accordingly, we define the following differential operator \( L \) and its corresponding two-fold inverse integral operator \( L^{-1} \) are as follows

\[
L_n = t^{-n} \frac{d}{dt} \left( t^n \frac{d}{dt} \right), \quad L_n^{-1}(\cdot) = \int_0^t \int_0^s t^n(\cdot)dtdt,
\]

where the application \( L_n^{-1} \) on Eq. (28) reveals

\[
v(t) = C_1 + C_2t + L_n^{-1}g(t) - L_n^{-1}f(t,v).
\]

Without further delay as in the preceding methodology, we get the following recurrent relation

\[
\begin{cases}
v_0 = C_1 + C_2t + L_n^{-1}g(t), \\
v_{k+1} = -L_n^{-1}A_k, \quad k \geq 0,
\end{cases}
\]

where \( A_k \)'s are the Adomian polynomials resulting from the nonlinear term \( f(t,v) \).
5 Algorithm five

Consider the following IVP featuring a generalized inhomogeneous Lane-Emden-type equation as follows [4]

\[
\begin{align*}
\begin{cases}
v'' + \frac{2n}{t} v' + \frac{n(n-1)}{t^2} v + f(t, v) = g(t), & n \geq 0, \\
v(0) = C_1, & v'(0) = C_2,
\end{cases}
\end{align*}
\]

(32)

where \( g(t) \) is a prescribed inhomogeneous function that is continuous real-valued function over \([0, 1]\), \( C_1 \) and \( C_2 \) are real constants; while \( n \) is some real number \( n, n \geq 0 \).

Accordingly, we define the following new differential operator \( L_n \) and its corresponding two-fold inverse integral operator \( L_n^{-1} \) are as follows

\[
L = t^{-n} \frac{d^2}{dt^2} t^n v, \quad L^{-1}(.) = t^{-n} \int_0^t \int_0^t t^n(.) dt dt. \tag{33}
\]

So, writing the given problem in an operator form reduces to

\[
Lv = g(t) - f(t, v), \tag{34}
\]

such that after employing the inverse integral operator \( L_n^{-1} \) to the first three terms \( v'' + \frac{2n}{t} v' + \frac{n(n-1)}{t^2} v \) of the present model gives

\[
L^{-1} (v'' + \frac{2n}{t} v' + \frac{n(n-1)}{t^2} v) = t^{-n} \int_0^t \int_0^t t^n(v'' + \frac{2n}{t} v' + \frac{n(n-1)}{t^2} v) dt dt = t^{-n} \int_0^t (t^n v' + n t^{n-1} v) dt = v. \tag{35}
\]

Therefore, upon operating \( L_n^{-1} \) on Eq. (34), we obtain

\[
v(t) = C_1 + L_n^{-1} g(t) - L_n^{-1} f(t, v). \tag{36}
\]

Also, without much delay, we get the following recurrent relation

\[
\begin{align*}
\begin{cases}
v_0 = C_1 + L_n^{-1} g(t), \\
v_{k+1} = -L_n^{-1} A_k, & k \geq 0,
\end{cases}
\end{align*}
\]

(37)

where \( A_k \)'s are the Adomian polynomials resulting from the nonlinear term \( f(t, v) \).
6 Algorithm six

Consider the following IVP featuring a generalized inhomogeneous Lane-Emden-type equation as follows [5]

\[
\begin{align*}
\begin{cases}
v'' + \frac{n}{t}v' + \frac{m}{t^2}v &= g(t) + f(t, v), \quad n > 1, \quad m \geq 0, \\
v(0) &= C_1, \quad v'(0) = C_2, 
\end{cases}
\end{align*}
\]

where the functions \(g(t)\) and \(f(t, v)\) are as explained above; while \(C_1, C_2, n\) and \(m\) are real constants.

Firstly, let us make a transformation using \((h-1)(h+k) = m, \text{ and } 2h+k = n\). Thus, Eq. (38) reduces to the following

\[
v'' + \frac{2h+k}{t}v' + \frac{(h-1)(h+k)}{t^2}v = g(t) + F(t, v), \quad h \geq 1, \quad k \geq -1, \tag{39}\]

where \(h\) and \(k\) are real constants.

Thus, we express Eq. (39) in an operator form as follows

\[
Lv = g(t) + F(t, v), \tag{40}
\]

where the operator \(L\) and its corresponding two-fold inverse are constructed as follows

\[
L(.) = t^{-h} \frac{d}{dt} \left( t^{-k} \frac{d}{dt} t^h \right) (.), \quad L^{-1}(.) = t^{-(h+k)} \int_0^t t^k \int_0^t t^h (.) dt dt. \tag{41}
\]

More so, applying \(L^{-1}\) on Eq. (40) to the third terms \(v'' + \frac{2h+k}{t}v' + \frac{(h-1)(h+k)}{t^2}v\) of Eq. (39) yields

\[
v = \psi(t) + L^{-1}(g(t)) + L^{-1}(F(t, v)), \tag{42}\]

with the satisfaction of

\[
L(\psi(t)) = 0. \tag{43}\]

Thus, without loss of generality, we obtain the following recurrent relation

\[
\begin{cases}
v_0 = \psi(t) + L^{-1}(g(t)), \\
v_{n+1} = L^{-1}A_n, \quad n \geq 0, \tag{44}
\end{cases}
\]

where \(A_k\)'s are the Adomian polynomials resulting from the nonlinear term \(f(t, v)\).

Remark
Algorithm six is a generalization of algorithm 5, because in algorithm five the coefficients of \(\frac{1}{t}v\) and \(\frac{1}{t^2}v\) are only even, but in algorithm six the coefficients
of $\frac{1}{t}v$ and $\frac{1}{t^2}v$ can be both even and odd. If we apply algorithm six to an equation in which the coefficients of $\frac{1}{t}v$ and $\frac{1}{t^2}v$ are even, we get the same inverse operator as in algorithm five.

7 Numerical illustrations

The current section demonstrates the application of the proposed modification method on various IVPs featuring both the classical and generalized inhomogeneous linear and nonlinear Lane-Emden-type equations.

Example 7.1. Consider the following IVP featuring a linear inhomogeneous Lane-Emden-type equation [10]

\[
\begin{cases}
v'' + \frac{2}{t}v' + v = 6 + 12t + t^2 + t^3, \\
v(0) = 0, \quad v'(0) = 0.
\end{cases}
\]

Algorithm one

Let us define a differential operator $L$ together with its corresponding two-fold inverse integral operator $L^{-1}$ as follows

\[
L = t^{-2} \frac{d}{dt} \left( t^2 \frac{d}{dt} \right), \quad L^{-1} = \int_0^t \int_0^t t^2 dt dt.
\]

Next, we express Eq. (45) in an operator form as follows

\[
Lv = 6 + 12t + t^2 + t^3 - v,
\]

such that after applying $L^{-1}$ to both sides of Eq. (47) yields

\[
v = t^2 + t^3 + \frac{1}{20} t^4 + \frac{1}{30} t^5 - L^{-1} v.
\]

Therefore, without loss of generality, we obtain the following recurrent relation

\[
\begin{cases}
v_0 = t^2 + t^3 + \frac{1}{20} t^4 + \frac{1}{30} t^5, \\
v_{k+1} = -L^{-1}(v_k), \quad k \geq 0,
\end{cases}
\]

Consequently, the first few components are expressed as follows

\[
\begin{align*}
v_0(t) &= t^2 + t^3 + \frac{1}{20} t^4 + \frac{1}{30} t^5, \\
v_1(t) &= -L^{-1}(v_0) = -\frac{1}{20} t^4 - \frac{1}{30} t^5 - \frac{1}{840} t^6 - \frac{1}{1680} t^7, \\
v_2(t) &= -L^{-1}(v_1) = \frac{1}{840} t^6 + \frac{1}{1680} t^7 + \frac{1}{60480} t^8 + \frac{1}{151200} t^9, \\
v_3(t) &= -L^{-1}(v_2) = -\frac{1}{60480} t^8 - \frac{1}{151200} t^9 - \cdots,
\end{align*}
\]
More so, other components can equally be determined iteratively in the same way. We mention here the presence of the noise terms $\frac{1}{20}t^4$ and $\frac{1}{30}t^5$ in $v_0$ and $v_1$; having opposite signs. Thus, upon canceling the effects of these terms from the first component $v_0$, the following exact solution is obtained

$$v(x) = t^2 + t^3.$$  \hspace{1cm} (51)

Lastly, one can justify the exactness of this solution by substituting the obtained exact solution in the governing model. Also, it can be noted that these noise terms appearing in other components equally vanish in the limit.

**Algorithm two**

Let us consider the differential operator $L$ together with its corresponding two-fold inverse integral operator $L^{-1}$ as follows

$$L = t^{-1} \frac{d^2}{dt^2} tv, \quad L^{-1} = t^{-1} \int_0^t \int_0^t t dt dt,$$  \hspace{1cm} (52)

then, we rewrite Eq. (45) in an operator form as follows

$$Lv = 6 + 12t + t^2 + t^3 - v,$$  \hspace{1cm} (53)

such that after operating $L^{-1}$ on the latter equation reveals

$$v = t^2 + t^3 + \frac{1}{20}t^4 + \frac{1}{30}t^5 - L^{-1}v.$$  \hspace{1cm} (54)

As preceded, the following recurrent solution is obtained

$$v_0(t) = t^2 + t^3 + \frac{1}{20}t^4 + \frac{1}{30}t^5,$$

$$v_1(t) = -L^{-1}(v_0) = -\frac{1}{20}t^4 - \frac{1}{30}t^5 - \frac{1}{840}t^6 - \frac{1}{1680}t^7,$$

$$v_2(t) = -L^{-1}(v_1) = \frac{1}{840}t^6 + \frac{1}{1680}t^7 + \frac{1}{60480}t^8 + \frac{1}{151200}t^9,$$

$$v_3(t) = -L^{-1}(v_2) = \frac{1}{60480}t^8 - \frac{1}{151200}t^9 - \cdots,$$  \hspace{1cm} (55)

which similarly yields the exact solution

$$v(t) = t^2 + t^3.$$  \hspace{1cm} (56)

**Algorithm three**

Here, we equally define yet a new differential operator $L$ together with its corresponding two-fold inverse integral operator $L^{-1}$ as follows

$$L(.) = \frac{1}{t \sin t} \frac{d}{dt} \sin^2 t \frac{d}{dt} \frac{t}{\sin t} (.), \quad L^{-1}(.) = \frac{\sin t}{t} \int_0^t \sin^{-2}t \int_0^t \sin t dt dt,$$  \hspace{1cm} (57)
Then, Eq. (45) in this new operator becomes

\[ Lv = 6 + 12t + t^2 + t^3, \]  

(58)
such that after operating \( L^{-1} \) on the latter equation reveals

\[ v = L^{-1}(6 + 12t + t^2 + t^3), \]
\[ = \frac{\sin t}{t} \int_0^t \sin^{-2} t \int_0^t t \sin t(6 + 12t + t^2 + t^3) dt, \]
\[ = t^2 + t^3, \]  

(59)
which also reveals the exact analytical solution.

**Example 7.2.** Consider the following IVP featuring a nonlinear inhomogeneous Lane-Emden-type equation [10]

\[
\begin{aligned}
& v'' + \frac{2}{t} v' - 6v = 4v \ln v, \\
& v(0) = 1, \quad v'(0) = 0.
\end{aligned}
\]  

(60)

**Algorithm one**

Let us define a differential operator \( L \) together with its corresponding two-fold inverse integral operator \( L^{-1} \) as follows

\[ L = t^{-2} \frac{d}{dt} \left( t^2 \frac{d}{dt} \right), \quad L^{-1} = \int_0^t t^{-2} \int_0^t t^2 dt dt. \]  

(61)
Expressing Eq. (60) in an operator form, we get

\[ Lv = 6v + 4v \ln v, \]  

(62)
such that after operating \( L^{-1} \) on the latter equation reveals

\[ v = 1 + 6L^{-1}(v) + 4L^{-1}(v \ln v). \]  

(63)
Accordingly, the following recurrent relation is obtained

\[
\begin{aligned}
& v_0 = 1, \\
& v_{k+1} = 6L^{-1}(v_k) + 4L^{-1}(A_k), \quad k \geq 0,
\end{aligned}
\]  

(64)
where \( A_k \)'s comes from the nonlinear term \( F(v) = 4v \ln v \) such that its Adomian polynomials are recurrently obtained with some few components as fol-
Modified Adomian method for the generalized inhomogeneous ... 

... 

Thus, upon substituting the above polynomials in the recurrent relation determined in Eq. (64) yields the following

\[ v_0 = 1, \]
\[ v_1 = 6L^{-1}(v_0) + 4L^{-1}(A_0) = t^2, \]
\[ v_2 = 6L^{-1}(v_1) + 4L^{-1}(A_1) = \frac{1}{2}t^4, \]
\[ v_3 = 6L^{-1}(v_2) + 4L^{-1}(A_2) = \frac{1}{3!}t^6, \]

Thus, summing the above iterates gives

\[ v(t) = 1 + t^2 + \frac{1}{2!}t^4 + \frac{1}{3!}t^6 + \cdots \] (67)

which converges to the following exact solution

\[ v(t) = e^{t^2}. \] (68)

**Algorithm two**

Consider the differential operator \( L \) together with its corresponding two-fold inverse integral operator \( L^{-1} \) as follows

\[ L = t^{-1} \frac{d^2}{dt^2} tv, \quad L^{-1} = t^{-1} \int_0^t \int_0^s t \, dt \, ds, \] (69)

such that after operating \( L^{-1} \) on Eq. (62) reveals

\[ v = 1 + 6L^{-1}(v) + 4L^{-1}(v \ln v). \] (70)

As preceded, the following recurrent solution is obtained

\[ v_0 = 1, \]
\[ v_1 = 6L^{-1}(v_0) + 4L^{-1}(A_0) = t^2, \]
\[ v_2 = 6L^{-1}(v_1) + 4L^{-1}(A_1) = \frac{1}{2}t^4, \]
\[ v_3 = 6L^{-1}(v_2) + 4L^{-1}(A_2) = \frac{1}{3!}t^6, \]

...
where upon summing the iterates gives

\[ v(t) = 1 + t^2 + \frac{1}{2!} t^4 + \frac{1}{3!} t^6 + \cdots, \] (72)

that also leads to the exact solution earlier obtained as follows

\[ v(t) = e^{t^2}. \] (73)

**Algorithm three**

Consider the differential operator \( L \) together with its corresponding two-fold inverse integral operator \( L^{-1} \) as follows

\[
L(.) = \frac{1}{t \sin \sqrt{-6t}} \frac{d}{dt} \sin^2 \sqrt{-6t} \frac{d}{dt} \sin \sqrt{-6t}(.),
\]

\[
L^{-1}(.) = \frac{\sin \sqrt{-6t}}{t} \int_0^t \sin^{-2} \sqrt{-6t} \int_0^t t \sin \sqrt{-6t}(.) dt dt.
\] (74)

The given ODE in an operator form thus becomes

\[ Lv = 4v \ln v, \] (75)

such that after operating \( L^{-1} \) on the above equation gives

\[ v = \frac{\sin \sqrt{-6t}}{t} + 4L^{-1}(v \ln v). \] (76)

Now, upon using of the Taylor’s series expansion on \( \frac{\sin \sqrt{-6t}}{t} \) of order 9 results in the following recurrent solution

\[
v_0 = 1 + t^2 + \frac{3}{10} t^4 + \frac{3}{70} t^6 + \frac{1}{280} t^8 + \cdots ,
v_1 = \frac{1}{8} t^4 + \frac{11}{105} t^6 + \frac{1}{54} t^8 + \cdots ,
v_2 = \frac{2}{105} t^6 + \frac{1}{54} t^8 + \cdots ,
\] (77)

where upon summing the iterative components gives

\[ v(t) = 1 + t^2 + \frac{1}{2} t^4 + \frac{1}{6} t^6 + \frac{1}{24} t^8 + \cdots , \] (78)

which converges to the exact analytical solution as follows

\[ v(t) = e^{t^2}. \] (79)

**Example 7.3.** Consider the following IVP featuring a linear inhomogeneous
generalized Lane-Emden-type equation [5]

\[
\begin{aligned}
  v'' + \frac{5}{t} v' + \frac{3}{t^2} v &= 15, \\
  v(0) = 0, & v'(0) = 0.
\end{aligned}
\]  

(80)

Algorithm six

Firstly, let us make a transformation using \((h-1)(h+k) = 3\), and \(2h+k = 5\). This gives \(k = 1\), \(h = 2\). Therefore, we devise the following differential operator \(L\) and its inverse \(L^{-1}\) by substituting the values of \(k\) and \(h\) into Eq. (41) as follows

\[
L(\cdot) = t^{-2} \frac{d}{dt} \left( \frac{t^{-1} d}{dt} t^3 \right) (\cdot), \quad L^{-1}(\cdot) = t^{-3} \int_0^t \int_0^t t^2 (\cdot) dt dt.
\]  

(81)

So, the given model in an operator form thus becomes

\[
Lv = 15,
\]  

(82)

such that after operating \(L^{-1}\) on the above equation gives

\[
v = L^{-1}(15),
\]  

(83)

that means

\[
v = t^2.
\]  

(84)

Indeed, the present method reveals the exact solution of the IVP so easily.

Example 7.4. Consider the following IVP featuring a nonlinear inhomogeneous generalized Lane-Emden-type equation [10]

\[
\begin{aligned}
  v'' + \frac{6}{t} v' + 14v &= -4v \ln v, \quad n \geq 0, \\
  v(0) = 1, & v'(0) = 0.
\end{aligned}
\]  

(85)

Algorithm four

Let us define a differential operator \(L_n\) together with its corresponding two-fold inverse integral operator \(L_n^{-1}\) as follows

\[
L_n = t^{-6} \frac{d}{dt} \left( t^6 \frac{d}{dt} \right), \quad L_n^{-1}(\cdot) = \int_0^t \int_0^t t^{-6} \int_0^t t^6 (\cdot) dt dt.
\]  

(86)

Therefore, Eq. (85) in an operator form becomes

\[
L_n v = -14v - 4v \ln v,
\]  

(87)
such that after operating $L^{-1}$ on the latter equation reveals

$$v(t) = 1 - 14L_n^{-1}(v) - 4L_n^{-1}(v \ln v)$$  \hspace{1cm} (88)

As preceded, the following recurrent relation is obtained

$$\begin{cases}
v_0 = 1, \\
v_{k+1} = -14L_n^{-1}(v_k) - 4L_n^{-1}(A_k), \quad k \geq 0,
\end{cases}$$  \hspace{1cm} (89)

where $A_k$’s comes from the nonlinear term $F(v) = v \ln v$ such that its Adomian polynomials are recurrently obtained with some few components as follows

$$\begin{align*}
A_0 &= v_0 \ln v_0, \\
A_1 &= v_1 F'(v_0) = v_1(1 + \ln v_0), \\
A_2 &= F'(v_0) v_2 + \frac{1}{2} F''(v_0) v_1^2 = v_2(1 + \ln v_0) + \frac{v_1^2}{2v_0}, \\
& \vdots
\end{align*}$$  \hspace{1cm} (90)

Thus, upon substituting the above polynomials in the recurrent relation determined in Eq. (89) yields the following iterates

$$\begin{align*}
v_0 &= 1, \\
v_1 &= -14L_n^{-1}(v_0) - 4L_n^{-1}(A_0) = -t^2, \\
v_2 &= -14L_n^{-1}(v_1) - 4L_n^{-1}(A_1) = \frac{1}{2!}t^4, \\
v_3 &= -14L_n^{-1}(v_2) - 4L_n^{-1}(A_2) = -\frac{1}{3!}t^6, \\
v_4 &= -14L_n^{-1}(v_3) - 4L_n^{-1}(A_3) = \frac{1}{4!}t^8, \\
& \vdots
\end{align*}$$  \hspace{1cm} (91)

Thus, summing the above iterates gives

$$v(t) = 1 - t^2 + \frac{1}{2!}t^4 - \frac{1}{3!}t^6 + \frac{1}{4!}t^8 + \cdots ,$$  \hspace{1cm} (92)

which converges to the following exact solutions

$$v(t) = e^{-t^2}.$$  \hspace{1cm} (93)

**Algorithm six**

We start by making a transformation using $(h-1)(h+k) = 0$, and $2h+k = 6$. This gives $k = 4, h = 1$. Therefore, upon substituting the values of $k$ and $h$ into Eq. (41), we get the following differential operator $L_n$ together with its corresponding two-fold inverse integral operator $L_n^{-1}$ as follows

$$L_n(\cdot) = t^{-1} \frac{d}{dt} \left( t^{-4} \frac{d}{dt} t^5 \right)(\cdot), \quad L_n^{-1}(\cdot) = t^{-5} \int_0^t t^4 \int_0^t \cdot dt dt.$$  \hspace{1cm} (94)
More so, we express Eq. (85) via the differential operator $L$ as follows
\[ L_n v = -14v - 4v \ln v, \tag{95} \]
such that when applying $L_n^{-1}$ on Eq. (95) yields
\[ v(t) = 1 - 14L_n^{-1}(v) - 4L_n^{-1}(v \ln v). \tag{96} \]

Accordingly, the following iterative solution is obtained
\[
\begin{align*}
v_0 &= 1, \\
v_1 &= -14L_n^{-1}(v_0) - 4L_n^{-1}(A_0) = -t^2, \\
v_2 &= -14L_n^{-1}(v_1) - 4L_n^{-1}(A_1) = \frac{1}{2!}t^4, \\
v_3 &= -14L_n^{-1}(v_2) - 4L_n^{-1}(A_2) = -\frac{1}{3!}t^6, \\
v_4 &= -14L_n^{-1}(v_3) - 4L_n^{-1}(A_3) = \frac{1}{4!}t^8, \\
&\vdots
\end{align*}
\tag{97}
\]

The solution in series form is thus expressed as follows
\[ v(t) = 1 - t^2 + \frac{1}{2!}t^4 - \frac{1}{3!}t^6 + \frac{1}{4!}t^8 + \cdots, \tag{98} \]

that converges to the following exact solution
\[ v(t) = e^{-t^2}. \tag{99} \]

**Example 7.5.** Consider the following IVP featuring a nonlinear inhomogeneous generalized Lane-Emden-type equation \([4]\)
\[
\begin{cases}
  v'' + \frac{6}{t} v' + \frac{6}{t^2}v + v^2 = 20 + t^4, \\
  v(0) = 0, \quad v'(0) = 0.
\end{cases}
\tag{100}
\]

**Algorithm four**

Accordingly, we consider the following differential operator $L$ together with its corresponding two-fold inverse integral operator $L^{-1}$ as follows
\[
L(.) = t^{-6} \frac{d}{dt} t^6 \frac{d}{dt}(.), \quad L^{-1}(.) = \int_0^t t^{-6} \int_0^t t^6(\cdot) dt dt.
\tag{101}
\]

As preceded, we get the following equation in an operator form
\[ Lv = 20 + t^4 - \frac{6}{t^2}v - v^2, \tag{102} \]
such that the latter equation in the presence of $L^{-1}$ becomes

$$v = L^{-1}(20 + t^4) - L^{-1}\left(\frac{6}{t^2}v\right) - L^{-1}(v^2) \quad (103)$$

As before, the following recurrent relation is obtained

$$\begin{cases}
v_0 = \frac{10}{7}t^2 + \frac{t^6}{66}, \\
v_{k+1} = L^{-1}\left(\frac{6}{t^2}v_k\right) - L^{-1}(A_k), \quad k \geq 0,
\end{cases} \quad (104)$$

where Adomian polynomials $A_k$’s of nonlinear term $F(v) = v^2$ are given for some few components as follows

$$\begin{align*}
A_0 &= v_0^2, \\
A_1 &= 2v_0v_1, \\
A_2 &= 2v_0v_2 + v_1^2, \\
A_3 &= 2v_1v_2 + 2v_0v_3, \\
&\vdots
\end{align*} \quad (105)$$

Thus, substituting Eq. (105) into Eq. (104) gives the following components

$$\begin{align*}
v_0 &= \frac{10}{7}t^2 + \frac{t^6}{66} \\
v_1 &= -L^{-1}\left(\frac{6}{t^2}v_0\right) - L^{-1}(A_0) = -\frac{30}{49}t^2 - \frac{383}{11858}t^6 - \frac{1}{3465}t^{10} - \frac{1}{1158696}t^{14} \\
v_2 &= -L^{-1}\left(\frac{6}{t^2}v_1\right) - L^{-1}(A_1) = \frac{90}{345}t^2 + \frac{26881}{913066}t^6 + \cdots \\
&\vdots
\end{align*} \quad (106)$$

Lastly, it is noted from the above equation that this algorithm fails as the solution diverges; however, the same problem happens to have a convergent solution using the proposed method via Algorithm five as shown in the next step.

**Algorithm five**

Consequently, we consider the differential operator $L$ together with its corresponding two-fold inverse integral operator $L^{-1}$ as follows

$$L(\cdot) = t^{-3} \frac{d^2}{dt^2} t^3 v(\cdot), \quad L^{-1}(\cdot) = t^{-3} \int_0^t \int_0^t t^3 (\cdot) dt dt. \quad (107)$$

Then, we rewrite Eq. (100) as follows

$$Lv = 20 + t^4 - v^2, \quad (108)$$
such that after applying $L^{-1}$ to the latter equation gives

$$v = L^{-1}(20 + t^4) - L^{-1}(v^2),$$

$$v = t^2 + \frac{t^6}{72}. \quad (109)$$

More so, dividing $t^2 + \frac{t^6}{72}$ into two parts gives the following recurrent relation

$$\begin{cases}
  v_0 = t^2, \\
  v_1 = \frac{t^6}{72} - L^{-1}(A_0) = 0.
\end{cases} \quad (110)$$

In view of the above equation, the exact analytical solution is thus obtained as follows

$$v(t) = t^2. \quad (111)$$

In fact, this is the expected exact analytical solution via the proposed scheme.

**Algorithm six**

As in the previous case, we start by making a transformation using $(h - 1)(h + k) = 6$, and $2h + k = 6$. This gives $k = 0, h = 3$. Therefore, upon substituting the values of $k$ and $h$ into Eq. (41), we get the following differential operator $L_n$ together with its corresponding two-fold inverse integral operator $L_n^{-1}$ as follows

$$L(.) = t^{-3} \frac{d^2}{dt^2} t^3(.), \quad L^{-1}(.) = t^{-3} \int_0^t \int_0^t t^3(.) dt dt. \quad (112)$$

However, since the obtained inverse operator in the above equation is the same as that of Algorithm five, we will eventually get the solution in the same way.

**8 Conclusion**

In conclusion, the present study investigated certain forms of singular IVPs featuring the classical and generalized inhomogeneous Lane-Emden-type equations. These equations are very important models as they appear in many physical applications, including thermodynamics to mention a few. Further, the study proposed different forms of inverse integral operators based on the standard ADM and effectively applied them to various linear and nonlinear test problems of the governing models. Finally, we assessed the effectiveness of the proposed method by monitoring the convergence rate in the presence of the analytically obtained exact solutions whenever they are attainable. Moreover, the method is fast in reaching the exact closed-form solution and can be
relied upon.

References


Received: June 7, 2022; Published: July 16, 2022