Stability Analysis of a Discrete Economic System

Fan Bu, Zongyu Peng and Ming Zhao 1

School of Science, China University of Geosciences (Beijing)
Beijing, 100083, China

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Abstract

In this work, we investigate the dynamical behaviors of a discrete nonlinear economic model. First, by applying Lemma 2.1, we analyze the existence and stability of the fixed point of the system. Second, we show that the system passes through a Neimark-Sacker bifurcation with the first Lyapunov coefficient. Moreover, numerical simulations are implemented not only to validate theoretical analysis, but also to exhibit chaotic behaviors. Finally, we consider the causes of drastic economic fluctuations and draw a conclusion.

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1 Introduction

As economy develops, the theory of nonlinear dynamic system plays an important role in economy. Since Tugan Balarovsky, a Russian economist, first put forward the concept of multiplier, many economical models have been investigated. For example, American economist Samuelson [1] combined the multiplier principle with the acceleration principle and introduced the balance of payments model of national economy. Goodwin [2] put forward nonlinear cycle models, which considered the impact of savings and consumption in macro-economy. Especially, Puu [3] improved the investment function and proposed an economic cycle model with cubic term and nonlinear investment function.

1Corresponding author
Recently, the following nonlinear cubic model (1) has been proposed in literature [4].

\[
\begin{align*}
    y(k+1) &= my(k) + (v-n)z(k) - vz^3(k), \\
    z(k+1) &= (m-1)y(k) + (v-n)z(k) - vz^3(k),
\end{align*}
\]

where \( y(k) \) and \( z(k) \) represent income and month-over-month growth of income respectively, \( 0 \leq m \leq 1 \) is the marginal propensity to consume, \( 0 \leq n \leq 1 \) is the supplementary savings rate, and \( v > 1 \) is the “accelerator” which stands for the accelerating effect of income change on investment.

In paper [4], the authors studied the bifurcation and Marotto’s chaos. In this paper, we will further investigate this discrete economic model by different approach. To simplify the model, we make \( t = v - n \), so \( 0 < t \leq v \) and we get

\[
\begin{align*}
    y(k+1) &= my(k) + tz(k) - vz^3(k), \\
    z(k+1) &= (m-1)y(k) + tz(k) - vz^3(k).
\end{align*}
\]

We will analyze the dynamics of system (2) in the rest of this article.

This paper is organized as follows. In Section 2, we discuss the existence and stability of the fixed point of system (2) by using the method proposed by Li et al [5]. In Section 3, we use the first Lyapunov coefficient to obtain parameter conditions for Neimark-Sacker bifurcation. This method is different from literature [4] where center manifold theorem was used. Actually, we gain a similar conclusion as literature [4]. In Section 4, we perform the numerical simulations, which further illustrate the correctness of the above theoretical analysis. Finally, we carry out a conclusion in Section 5.

2 Stability analysis

In this section, we will discuss the existence and stability of the fixed point \( P(y^*, z^*) \) of system (2). Obviously, the fixed point \( P(y^*, z^*) \) satisfies the following equations :

\[
\begin{align*}
    y^* &= my^* + tz^* - vz^3, \\
    z^* &= (m-1)y^* + tz^* - vz^3.
\end{align*}
\]

By simple calculations, we know that system (2) has only one fixed point \( O(0,0) \). Next, we analyze the local stability of fixed point \( O(0,0) \). We now introduce Lemma 2.1 [5] which will be used to prove the stability of the fixed point \( O(0,0) \).

**Lemma 2.1.** Let \( F(\lambda) = \lambda^2 + B\lambda + C \), where \( B \) and \( C \) are two real constants. Let \( \lambda_1 \) and \( \lambda_2 \) are two roots of \( F(\lambda) = 0 \). There are the following results

(i) If \( F(1) > 0 \), we have
Stability analysis of a discrete economic system

(i.1) \(|\lambda_1| < 1 \text{ and } |\lambda_2| < 1 \text{ if and only if } F(-1) > 0 \text{ and } C < 1; \)

(i.2) \(|\lambda_1| < 1 \text{ and } |\lambda_2| > 1 \text{ if and only if } F(-1) < 0; \)

(i.3) \(|\lambda_1| > 1 \text{ and } |\lambda_2| > 1 \text{ if and only if } F(-1) > 0 \text{ and } C > 1; \)

(i.4) \(\lambda_1 \text{ and } \lambda_2 \text{ are a pair of conjugate complex roots and } |\lambda_1| = |\lambda_2| = 1 \)
\(\text{if and only if } -2 < B < 2 \text{ and } C = 1; \)

(i.5) \(\lambda_1 = -1 \text{ and } \lambda_2 \neq -1 \text{ if and only if } F(-1) = 0 \text{ and } B \neq 2; \)

(i.6) \(\lambda_1 = \lambda_2 = -1 \text{ if and only if } F(-1) = 0 \text{ and } B = 2. \)

(ii) If \(F(1) = 0, \text{ namely, } 1 \text{ is a root of } F(\lambda) = 0 \text{ and the other root } \lambda \text{ satisfies} \)
\(|\lambda| = (\langle, \rangle)1 \text{ if and only if } |C| = (\langle, \rangle)1. \)

(iii) If \(F(1) < 0, \text{ then } F(\lambda) = 0 \text{ has one root lying in } (1, \infty). \text{ Moreover,} \)

(iii.1) \(\text{the other root } \lambda \text{ satisfies } \lambda < (=) -1 \text{ if and only if } F(-1) < (=)0; \)

(iii.2) \(\text{the other root } \lambda \text{ satisfies } -1 < \lambda < 1 \text{ if and only if } F(-1) > 0. \)

By using Lemma 2.1, we obtain the following theorem for the stability of the fixed point \(O(0, 0). \)

**Theorem 2.2.** The characteristic polynomial of Jacobian matrix of system (2) at the fixed point \(O(0, 0)\) is

\[ F(\lambda) = \lambda^2 - (m + t)\lambda + t, \]  \(4\)

and the stability of the fixed point \(O(0, 0)\) is as follows.

(i) When \(0 < t < 1 \text{ and } 0 \leq m < 1, \text{ the modulus of all characteristic roots of } F(\lambda) \text{ are less than 1. At this point, the fixed point } O(0, 0) \text{ is the sink, which is stable;} \)

(ii) When \(t > 1 \text{ and } 0 \leq m < 1, \text{ the modulus of all characteristic roots of } F(\lambda) \text{ are larger than 1. At this point, the fixed point } O(0, 0) \text{ is the source, which is unstable;} \)

(iii) When \(t = 1 \text{ and } 0 \leq m < 1, \text{ system (2) generates a pair of conjugate complex roots whose modulus } |\lambda_{1,2}| = 1. \text{ And the fixed point } O(0, 0) \text{ is a non-hyperbolic fixed point, where system (2) may generate a Neimark-Sacker bifurcation.} \)

**Proof.** Obviously, Jacobian matrix of system (2) at the fixed point \(O(0, 0)\) is

\[ A = \begin{pmatrix} m & t \\ m - 1 & t \end{pmatrix}, \]  \(5\)
so the characteristic equation of $A$ is
\[ F(\lambda) = \lambda^2 - (m + t)\lambda + t. \]

Let $F(\lambda) = \lambda^2 + B\lambda + C$, $\lambda_1$ and $\lambda_2$ be the two roots of $F(\lambda) = 0$. It is not difficult to get that $F(1) = 1 - m$ and $F(-1) = 1 + m + 2t$. According to Lemma 2.1, we obtain the value of $\lambda_1$ and $\lambda_2$ only exist in three cases as follows, and other circumstances are not valid.

(i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(1) > 0$ and $F(-1) > 0$ and $C < 1$, namely, $0 \leq m < 1$ and $0 < t < 1$.

(ii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(1) > 0$ and $F(-1) > 0$ and $C > 1$, namely, $0 \leq m < 1$ and $t > 1$.

(iii) $\lambda_1$ and $\lambda_2$ are a pair of conjugate complex roots, and $|\lambda_1| = |\lambda_2| = 1$ if and only if $F(1) > 0$, $-2 < B < 2$ and $C = 1$, namely, $0 \leq m < 1$ and $t = 1$.

So we complete the proof of Theorem 2.2.

3 Neimark-Sacker bifurcation analysis

According to the Theorem 2.2, we have that when $t = 1$ and $0 \leq m < 1$, the system (2) generates a pair of conjugate complex roots $\lambda_{1,2}$ whose modulus are $|\lambda_{1,2}| = 1$, and system (2) may produce Neimark-Sacker bifurcation. Next, we will prove that system (2) undergoes a supercritical Neimark-Sacker bifurcation. For further discussion, we first introduce some preliminary definitions.

Definition 3.1. $\langle x, y \rangle$ is called the standard scalar product, it is defined as
\[ \langle x, y \rangle = x^T y = \sum_{i=1}^{n} x_i y_i. \]  

Remark 3.2. It is easy to get that the standard scalar product has following property
\[ \langle x, y \rangle = \overline{\langle y, x \rangle}. \]  

Definition 3.3. We remark discrete-time nonlinear dynamic system
\[ x \mapsto Ax + N(x), \quad A = (a_{ij})_{n \times n}, \quad x \in \mathbb{R}^n, \]  
where $N(x) = O(\|x\|^2)$ is a smooth function. Let the nonlinearity term $N(x)$ by
\[ N(x) = \frac{1}{2}B(x, x) + \frac{1}{6}C(x, x, x) + O(\|x\|^4). \]
Then we give the definitions of $B_i(x,y)$ and $C_i(x,y,z)$. In the coordinate system, there are

$$B_i(x,y) = \sum_{j,k=1}^{n} \frac{\partial^2 N_i(\xi)}{\partial \xi_j \partial \xi_k} \bigg|_{\xi=0} x_j y_k, \quad C_i(x,y,z) = \sum_{j,k,l=1}^{n} \frac{\partial^3 N_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l} \bigg|_{\xi=0} x_j y_k z_l.$$  

(10)

**Definition 3.4.** Let $A$ be the Jacobian matrix of system (8), vector $p, q, \bar{p}, \bar{q}$ satisfy the following equation (vector $q$ is the eigenvector of matrix $A$ and vector $p$ is the eigenvector of matrix $A^T$)

$$Aq = e^{i\theta_0} q, \quad A\bar{q} = e^{-i\theta_0} \bar{q}, \quad A^T p = e^{-i\theta_0} p, \quad A^T \bar{p} = e^{i\theta_0} \bar{p}. \quad (11)$$

We next introduce the following theorem [6, 7] to obtain parameter conditions for system (2) to undergo a supercritical Neimark-Sacker bifurcation.

**Theorem 3.5.** Suppose a two-dimensional discrete-time system

$$x \mapsto f(x, \alpha) \quad x \in \mathbb{R}^2, \alpha \in \mathbb{R}^1,$$  

(12)

with smooth $f$, has, for all sufficiently small $|\alpha|$, the fixed point $x = 0$ with multipliers

$$\lambda_{1,2}(\alpha) = r(\alpha) e^{\pm i\varphi(\alpha)}, \quad (13)$$

where $r(0) = 1$ and $\varphi(0) = \theta_0$.

Then the system occurs supercritical Neimark-Sacker bifurcation when the following conditions are satisfied

(i) $|\lambda_{1,2}| = 1$;

(ii) $\frac{d|\lambda_{1,2}|}{d\alpha} \bigg|_{\alpha=0} \neq 0$;

(iii) $e^{ik\theta_0} \neq 1$, for $k = 1, 2, 3, 4$;

(iv) $d < 0$, where

$$d = \frac{1}{2} \text{Re} \left\{ e^{-i\theta_0} \left[ \langle p, C(q, q, \bar{q}) \rangle + 2 \langle p, B(q, (I_n - A)^{-1} B(q, \bar{q})) \rangle \right] + \left. \langle p, B(\bar{q}, (e^{2i\theta_0} I_n - A)^{-1} B(q, q)) \right] \right\}. \quad (14)$$

Applying Theorem 3.5, we get the following result for system (2).
Theorem 3.6. When $0 < m < 1$ and $t = 1$, system (2) undergoes a Neimark-Sacker bifurcation. Moreover $d < 0$, so the bifurcation is a supercritical Neimark-Sacker bifurcation and there exists an attracting invariant closed curve for $t > 1$.

Proof. (i) Based on the previous analysis and discussion, we will choose $t$ as the bifurcation parameter in this section. Considering system (2), let $t = 1$, then we have
\[
\begin{align*}
y &\mapsto my + z - vz^3, \\
z &\mapsto (m - 1)y + z - vz^3.
\end{align*}
\] (15)

Giving parameter $t$ a small perturbation $\tilde{t}$, then we get system (16) as follows,
\[
\begin{align*}
y &\mapsto my + (1 + \tilde{t})z - vz^3, \\
z &\mapsto (m - 1)y + (1 + \tilde{t})z - vz^3,
\end{align*}
\] (16)

where $|\tilde{t}| \ll 1$. The characteristic equation of the system (16) at the fixed point $O(0,0)$ is
\[
\bar{F}(\lambda) = \lambda^2 - a(\tilde{t})\lambda + b(\tilde{t}) = 0,
\] (17)

where $a(\tilde{t}) = m + 1 + \tilde{t}$ and $b(\tilde{t}) = 1 + \tilde{t}$. When $\tilde{t}$ changes in the neighborhood of $\tilde{t} = 0$, the roots of the characteristic equation are
\[
\lambda_{1,2} = \frac{1}{2} \left( a(\tilde{t}) \pm i\sqrt{4b(\tilde{t}) - a^2(\tilde{t})} \right). \tag{18}
\]

Then we get $|\lambda_{1,2}| = \sqrt{b(\tilde{t})}\bigg|_{\tilde{t}=0} = 1$.

(ii) According to the above analysis, we have $\left. \frac{d|\lambda_{1,2}|}{dt} \right|_{\tilde{t}=0} = \frac{1}{2} (1 + \tilde{t})^{-\frac{3}{2}} \bigg|_{\tilde{t}=0} = \frac{1}{2} \neq 0$.

(iii) It’s easy to check $e^{ik\theta_0} \neq 1$, for $k = 1, 2, 3, 4$.

(iv) For system (2), we have
\[
A = \begin{pmatrix} m & t \\ m - 1 & t \end{pmatrix}, \quad N(z) = \begin{pmatrix} -vz^3 \\ -vz^3 \end{pmatrix}. \tag{19}
\]

Let $t = 1$, then
\[
A = \begin{pmatrix} m & 1 \\ m - 1 & 1 \end{pmatrix}. \tag{20}
\]
By calculating, we get following eigenvectors \( p, q \) and \( \bar{p}, \bar{q} \).

\[
q = \begin{pmatrix}
\frac{1-m-i\sqrt{-(m-1)(m+3)}}{2(1-m)} \\
1 \\
\frac{(m-1)(m+3)-i(m+1)\sqrt{-(m-1)(m+3)}}{2(m+3)} \\
\frac{m+3-i\sqrt{-(m-1)(m+3)}}{2(m+3)}
\end{pmatrix}, \quad \bar{q} = \begin{pmatrix}
\frac{1-m+i\sqrt{-(m-1)(m+3)}}{2(1-m)} \\
1 \\
\frac{i\sqrt{-(m-1)(m+3)}}{2(m+3)} \\
\frac{m+3-i\sqrt{-(m-1)(m+3)}}{2(m+3)}
\end{pmatrix},
\]

\[
p = \begin{pmatrix}
\frac{1-m-i\sqrt{-(m-1)(m+3)}}{2(1-m)} \\
1 \\
\frac{(m-1)(m+3)-i(m+1)\sqrt{-(m-1)(m+3)}}{2(m+3)} \\
\frac{m+3-i\sqrt{-(m-1)(m+3)}}{2(m+3)}
\end{pmatrix}, \quad \bar{p} = \begin{pmatrix}
\frac{1-m+i\sqrt{-(m-1)(m+3)}}{2(1-m)} \\
1 \\
\frac{i\sqrt{-(m-1)(m+3)}}{2(m+3)} \\
\frac{m+3-i\sqrt{-(m-1)(m+3)}}{2(m+3)}
\end{pmatrix}.
\]

Then we can derive \( B(x, y) = (0, 0)^T \), \( C(x, y, z) = (-6vxyz, -6vxyz)^T \) and \( C(q, q, \bar{q}) = (-6v, -6v)^T \).

Substituting them into equation (14), we have

\[
d = \frac{1}{2} \text{Re} \left( -3 - \frac{3i\sqrt{1-m}}{\sqrt{m+3}} \right) v = \frac{-3}{2} v < 0. \quad (22)
\]

Hence, if \( 0 < m < 1 \) and \( t = 1 \), the system occurs Neimark-Sacker bifurcation. Furthermore, because of \( d = \frac{-3}{2} v < 0 \), system (2) undergoes a supercritical Neimark-Sacker bifurcation and generates an attracting invariant closed curve for \( t > 1 \).

4 Numerical simulation

In this section, we give the bifurcation diagram Figure 1 and phase portraits Figures 2(a)-(d) of system (2) under different parameters. These diagrams show the correctness of above theoretical analysis, and also show more complex dynamic behaviors of system (2).

Figure 1: Bifurcation diagram of system (2) for \( m = 0.5, v = 3 \), and \( t \in (0.8, 2.05) \).
The parameters of bifurcation diagram are considered for \( m = 0.5, v = 3 \) and \( t \in (0.8, 2.05) \) in Figure 1. Based on Theorem 3.6, we know that a supercritical Neimark-Sacker bifurcation occurs at the fixed point \( O(0, 0) \) for \( t = 1 \), and Figure 1 shows the above results. From Figure 1, we can observe that the fixed point \( O(0, 0) \) is stable at \( t < 1 \) and it gradually loses its stability with the increase of \( t \).

Figures 2(a)-(d) show the phase portraits which are corresponding to Figure 1. The phase portraits clearly describe how the circle varies from a fixed point \( O(0, 0) \) and a limit cycle to chaotic attractors. Figure 2(a) shows the fixed point \( O(0, 0) \) of system (2) for \( t = 0.99 \), and the fixed point \( O(0, 0) \) is a stable fixed point. Figure 2(b) depicts system (2) generates a stable limit cycle when \( t = 1.01 \). Figure 2(c) shows system (2) generates a quasi-periodic orbit when \( t = 1.5 \). Figure 2(d) describes system (2) generates chaos when \( t = 1.7 \).

![Phase portraits of system (2) for various values of t associating with Figure 1.](image)

From the results of numerical simulation, it can be seen that the supercritical Neimark-Sacker bifurcation occurs when the parameters of the system are \( 0 < m < 1 \) and \( t = 1 \). With the change of parameter \( t \), the system will be in periodic state, and when the value of \( t \) is larger, the system will enter chaotic state.
5 Conclusion

In this paper, we investigate the dynamical behaviors of the discrete non-linear economic model and we prove that system (2) can undergo a Neimark-Sacker bifurcation. What’s more, we find when parameter $t$ exceeds 1, system (2) will go from periodic state to chaotic state with serious imbalance, which should be avoided in actual production and life. Thus, based on our above analysis, decision makers may avoid drastic economic fluctuations by controlling marginal propensity to consume in practical application. The government and other relevant functional departments can also take measures to carry out micro-control in advance according to the characteristics of economic fluctuations, such as reducing the impact of cyclical fluctuations on economic development or making the economy more steadily. Even if appropriate measures are not taken in advance, the economic trend can also be predicted after the crisis, which can provide reference for the steady development of government, enterprises and economies.

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