Abstract

This note complements a previous paper by the author in two ways. Firstly, errors in this previous paper for the sufficient conditions for sums of two solutions of specific equations to satisfy the equation are underlined. Secondly, in this work, the class of nonlinear partial differential equations that can be decomposed into sums of terms that are products of not necessarily the same number of linear partial differential operators is considered and necessary and sufficient conditions on the operators and on the solutions are established for the linear combinations of two solutions to satisfy the equation. In the mentioned previous paper a necessary and sufficient condition was obtained for the same superposition property for equations that are composed of sums of products of the same number of linear partial differential operators. Also an example is included for the problem taken up.

Keywords: Superposition, nonlinear partial differential equation

1 Introduction

Construction of new solutions is possible by the superposition property in nonlinear differential equations. This concept has been treated in the literature and other than specific approaches for specific equations such as in [1], Lie’s symmetry algorithm has been the basis for finding nonlinear superposition principles.
A brief review of the past work related to the nonlinear superposition principles can be found in [3] whereas reference [2] also provides some physically motivated examples for them.

Unlike linear differential equations, the linear combination of solutions of the nonlinear equation needs not satisfy the differential equation. However with some classes of equations such as equations of Navier-Stokes, Burgers, Hamilton-Jacobi type, pseudo-linear combination of solutions (in which the operations of addition and multiplication are replaced by pseudo-operations) is again a solution [4, 5].

In [6] the author considered classes of nonlinear partial differential equations that can be decomposed into sums of terms that are products of linear partial differential operators. Therein in Section 2, a necessary and a sufficient condition for the linear combinations of the two solutions of the equation that is a sum of terms that are products of the same number of linear partial differential operators to satisfy the equation was obtained.

Also in Section 3 of [6], specific equations that are sums of terms that are products of not necessarily the same number of linear partial differential operators were considered and sufficient conditions were proposed for the sums of two solutions to satisfy the equation. However other than an error in the expression of the differential equation written for the sum $u = u_1 + u_2$ of the solutions $u_1$ and $u_2$ of the third structure taken up, for all four structures considered, the proposed sufficient conditions were not compatible with the original differential equations written for $u = u_1$ and $u = u_2$. This invalidates Section 3 of [6].

In this work again we investigate the conditions on the solutions of classes of nonlinear partial differential equations and on the differential operators that constitute these equations, so that the linear combinations of two solutions also satisfy the equation. This is the specific superposition principle that this work is focused on.

In this paper we consider equations that have a structure made up of a sum of terms such that each term is composed of the products of the results of the application of not necessarily the same number of linear partial differential operators on the dependent variable $u$. In Section 2 a necessary and sufficient condition on the solutions and the differential operators is proved for the linear combinations of the two solutions to satisfy equations with the mentioned structure. An example is also included at the end of the section. In this way this paper complements the work reported in [6].
2 Necessary and sufficient conditions for the linear combinations of two solutions of the differential equation to satisfy the equation

In the following as a convention we shall denote a linear partial differential operator by the symbol \( L \) with proper subscripts. We shall use the notation \( f_{pk}(u_i) \) to represent the product \( f_{pk}(u_i) = \prod_{j=1}^{m_p} L_{kj}(u_i) \) where \( i \) is 1 or 2 and \( u_i \) is a solution of the differential equation. Then the differential equation considered can be represented as

\[
\sum_{k=1}^{n_1} \prod_{j=1}^{m_1} L_{kj}(u) + \sum_{k=n_1+1}^{n_2} \prod_{j=1}^{m_2} L_{kj}(u) + \cdots + \sum_{k=n_{q-1}+1}^{n_q} \prod_{j=1}^{m_q} L_{kj}(u) = 0. \tag{1}
\]

Using the notation employing \( f_{pk}(u) \), the equation considered has the structure;

\[
\sum_{k=1}^{n_1} f_{1k}(u) + \sum_{k=n_1+1}^{n_2} f_{2k}(u) + \cdots + \sum_{k=n_{q-1}+1}^{n_q} f_{qk}(u) = 0. \tag{2}
\]

In other words for \( n_{p-1} + 1 \leq k \leq n_p \) the term \( f_{pk} \) is made up of a product of \( m_p \) linear operators \( L_{kj} \), where \( 1 \leq j \leq m_p \) and \( 1 \leq p \leq q \) with \( n_0 = 0 \).

In Section 2 we shall discuss necessary and sufficient conditions under which a linear combination \( u = au_1 + bu_2 \) of the solutions \( u_1 \) and \( u_2 \) of (1), also satisfies (1).

2.1 Obtaining the necessary conditions

Suppose that along with solutions \( u_1 \) and \( u_2 \) of (1), also \( u = au_1 + bu_2 \) satisfies (1) where \( a \) and \( b \) are arbitrary constants. Substitution of \( u = au_1 + bu_2 \) in (1) or (2) yields,

\[
\sum_{k=1}^{n_1} f_{1k}(au_1 + bu_2) + \sum_{k=n_1+1}^{n_2} f_{2k}(au_1 + bu_2) + \cdots + \sum_{k=n_{q-1}+1}^{n_q} f_{qk}(au_1 + bu_2) = 0. \tag{3}
\]

Here

\[
f_{pk}(au_1 + bu_2) = L_{k1}(au_1 + bu_2)L_{k2}(au_1 + bu_2) \cdots L_{km_p}(au_1 + bu_2) = a^{mp} f_{pk}(u_1) \sum_{i=0}^{m_p} \frac{b^i}{a^i} h_{pki}(u_1, u_2) \tag{4}
\]
The function $h_{pki}(u_1, u_2)$ is the sum of all terms each of which is a product of $i$ choices without repetition of the fractions $\frac{L_{kj}(u_2)}{L_{kj}(u_1)}$, $1 \leq j \leq m_p$. For example when $m_p = 3$,

$$h_{pk2}(u_1, u_2) = \frac{L_{k1}(u_2) L_{k2}(u_2)}{L_{k1}(u_1) L_{k2}(u_1)} + \frac{L_{k1}(u_2) L_{k3}(u_2)}{L_{k1}(u_1) L_{k3}(u_1)} + \frac{L_{k2}(u_2) L_{k3}(u_2)}{L_{k2}(u_1) L_{k3}(u_1)},$$

while $h_{pk0}(u_1, u_2) = 1$. Then (3) can be written in the form

$$\sum_{p=1}^{q} a^{m_p} \left\{ \sum_{k=n_{p-1}+1}^{n_p} [f_{pk}(u_1) \sum_{i=0}^{m_p} \left( \frac{b}{a} \right)^i h_{pki}(u_1, u_2)] \right\} = 0. \quad (5)$$

In (5) the quantity $\frac{b}{a}$ is arbitrary and its powers are linearly independent functions. Therefore (5) can hold only if the coefficients of powers of $\frac{b}{a}$ vanish. This allows us to write down the following condition for the coefficient of power zero of the fraction $\frac{b}{a}$:

$$\sum_{p=1}^{q} a^{m_p} \left\{ \sum_{k=n_{p-1}+1}^{n_p} f_{pk}(u_1) h_{pk0}(u_1, u_2) \right\} = 0. \quad (6)$$

We rewrite (4) with the far right side in the following alternative form;

$$f_{pk}(au_1 + bu_2) = L_{k1}(au_1 + bu_2)L_{k2}(au_1 + bu_2) \cdots L_{km_p}(au_1 + bu_2) = b^{m_p} f_{pk}(u_2) \sum_{i=0}^{m_p} \left( \frac{a}{b} \right)^i h_{pki}(u_2, u_1).$$

Then the equation corresponding to (5) written using this alternative form will have coefficient of $\frac{a}{b}$ of power zero only if

$$\sum_{p=1}^{q} b^{m_p} \left\{ \sum_{k=n_{p-1}+1}^{n_p} f_{pk}(u_2) h_{pk0}(u_2, u_1) \right\} = 0, \quad (7)$$

holds. Noting that $f_{pk}(u_1) h_{pk0}(u_1, u_2) = f_{pk}(u_1)$ and $f_{pk}(u_2) h_{pk0}(u_2, u_1) = f_{pk}(u_2)$, from (6) and (7) we write;

$$\sum_{p=1}^{q} a^{m_p} \left\{ \sum_{k=n_{p-1}+1}^{n_p} f_{pk}(u_1) \right\} = 0, \quad (8)$$

$$\sum_{p=1}^{q} b^{m_p} \left\{ \sum_{k=n_{p-1}+1}^{n_p} f_{pk}(u_2) \right\} = 0. \quad (9)$$

These two relations state that if $u_1$ and $u_2$ are solutions of (1), as a necessary condition we must have the property that $au_1$ and $bu_2$ must also satisfy
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(1), where \(a, b\) are arbitrary constants. In Equations (8) and (9) because \(a, b\) are arbitrary, their powers are linearly independent functions. Then these two equations can hold if and only if coefficients of powers of \(a, b\) in them, vanish. Then this condition of vanishing of coefficients of powers of \(a, b\) in (8) and (9) can be expressed as

\[
\sum_{k=n_p-1+1}^{n_p} f_{pk}(u_1) = 0, \tag{10}
\]

\[
\sum_{k=n_p-1+1}^{n_p} f_{pk}(u_2) = 0, \tag{11}
\]

where \(1 \leq p \leq q\).

In the light of (8) through (11), we first simplify (5):

\[
\sum_{p=1}^{q} a^{m_p} \left\{ \sum_{k=n_p-1+1}^{n_p} \left[ f_{pk}(u_1) \sum_{i=1}^{m_p-1} (-\frac{b}{a})^i h_{pk_i}(u_1, u_2) \right] \right\} = 0. \tag{12}
\]

by also utilizing the fact that \(f_{pk}(u_1)h_{pk_m}(u_1, u_2) = f_{pk}(u_2)\). Then we proceed by writing the conditions for the vanishing coefficients of the other powers of \(\frac{b}{a}\) in (12):

\[
\sum_{p=1}^{q} a^{m_p} \left\{ \sum_{k=n_p-1+1}^{n_p} \left[ f_{pk}(u_1)h_{pk1}(u_1, u_2) \right] \right\} = 0, \tag{13}
\]

\[
\sum_{p=1}^{q} a^{m_p} \left\{ \sum_{k=n_p-1+1}^{n_p} \left[ f_{pk}(u_1)h_{pk2}(u_1, u_2) \right] \right\} = 0, \tag{14}
\]

\[
\sum_{p=1}^{q} a^{m_p} \left\{ \sum_{k=n_p-1+1}^{n_p} \left[ f_{pk}(u_1)h_{pk(m_p-1)}(u_1, u_2) \right] \right\} = 0. \tag{15}
\]

In Equations (13) through (15), if for a \(p\) value, \(m_p - 1\) is less than index \(i\) of an equation, then the associated \(f_{pk}(u_1)h_{pk_i}(u_1, u_2)\) quantity will cease to appear in the equation. Also the parameter \(a\) is again arbitrary, so that its powers are linearly independent functions, causing coefficients of these powers to vanish in the said equations. If we enforce this condition in (13) through (15), we will obtain the following relations for \(1 \leq p \leq q\).

\[
\sum_{k=n_p-1+1}^{n_p} \left[ f_{pk}(u_1)h_{pk1}(u_1, u_2) \right] = 0, \tag{16}
\]

\[
\sum_{k=n_p-1+1}^{n_p} \left[ f_{pk}(u_1)h_{pk2}(u_1, u_2) \right] = 0, \tag{17}
\]

\[
\sum_{k=n_p-1+1}^{n_p} \left[ f_{pk}(u_1)h_{pk(m_p-1)}(u_1, u_2) \right] = 0. \tag{18}
\]
Equations (10), (11) and (16) through (18) constitute the necessary conditions sought after.

### 2.2 Sufficiency of the obtained necessary conditions

In (12) the triple sum consists of only what we shall name 'mixed' product terms. For given \( p \) and \( k \) a mixed product term is the term \( \prod_{j=1}^{m_p} L_{kj}(u_δ) \) where \( u_δ \) may attain the values \( au_1 \) or \( bu_2 \), but \( u_δ \) is never all \( au_1 \) or all \( bu_2 \) in the term. This character of (12) is due to the fact that the 'nonmixed' terms which correspond to the expressions in the left side of (6) and (7) have already been subtracted from (5) while writing (12).

On the other hand Equations (16) through (18) imply that for a given \( p \), the full sum of the possible mixed product terms for all \( k \) such that \( n_p-1+1 \leq k \leq n_p \), vanishes. This fact can be represented as:

\[
\sum_{k=n_p-1+1}^{n_p} [f_{pk}(au_1 + bu_2) - f_{pk}(au_1) - f_{pk}(bu_2)] = 0, \tag{19}
\]

where in the general term of the series, the nonmixed product terms \( f_{pk}(au_1) \) and \( f_{pk}(bu_2) \) are subtracted from the overall product \( f_{pk}(au_1 + bu_2) \).

Equations (10) and (11) which are part of our necessary conditions imply that \( au_1 \) and \( bu_2 \) satisfy (1), or that:

\[
\sum_{p=1}^{q} \left\{ \sum_{k=n_p-1+1}^{n_p} f_{pk}(au_1) \right\} = 0, \tag{20}
\]

\[
\sum_{p=1}^{q} \left\{ \sum_{k=n_p-1+1}^{n_p} f_{pk}(bu_2) \right\} = 0. \tag{21}
\]

Adding (20) and (21) we get;

\[
\sum_{p=1}^{q} \left\{ \sum_{k=n_p-1+1}^{n_p} [f_{pk}(au_1) + f_{pk}(bu_2)] \right\} = 0. \tag{22}
\]

Observing (19) we obtain;

\[
\sum_{p=1}^{q} \left\{ \sum_{k=n_p-1+1}^{n_p} f_{pk}(au_1 + bu_2) \right\} = 0, \tag{23}
\]

QED.
2.3 An example

As an example to illustrate the ideas discussed, consider the differential equation;

\[ u^2 + \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial y^2} - \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial y^2} = 0. \]  

(24)

For this equation

\[ p = 0, n_0 = 0, \]  

(25)

\[ p = 1, n_1 = 2, m_1 = 2, \sum_{k=1}^{2} f_{1k}(u) = L_{11}(u)L_{12}(u) + L_{21}(u)L_{22}(u) \]  

(26)

\[ p = 2, n_2 = 4, m_2 = 1, \sum_{k=3}^{4} f_{2k}(u) = L_{31}(u) + L_{41}(u). \]  

(27)

Here

\[ L_{11} = 1; L_{12} = 1; L_{21} = \frac{\partial}{\partial x}; L_{22} = \frac{\partial^2}{\partial y^2} \]  

(28)

\[ L_{31} = -\frac{\partial}{\partial x}; L_{41} = -\frac{\partial^2}{\partial y^2}. \]  

(29)

Two solutions of this equation are \( u_1 = \exp(x) \sin y \) and \( u_2 = \exp(x) \cos y \). These solutions satisfy (10) and (11), because, for \( p = 1 \)

\[ \sum_{k=1}^{n_1} f_{1k}(u_1) = L_{11}(u_1)L_{12}(u_1) + L_{21}(u_1)L_{22}(u_1) \]

\[ = \exp(2x) \sin^2 y - \exp(2x) \sin^2 y = 0, \]  

(30)

\[ \sum_{k=1}^{n_1} f_{1k}(u_2) = L_{11}(u_2)L_{12}(u_2) + L_{21}(u_2)L_{22}(u_2) \]

\[ = \exp(2x) \cos^2 y - \exp(2x) \cos^2 y = 0, \]  

(31)

and for \( p = 2 \)

\[ \sum_{k=3}^{n_2} f_{2k}(u_1) = L_{31}(u_1) + L_{41}(u_1) \]

\[ = -\exp(x) \sin y + \exp(x) \sin y = 0, \]  

(32)

\[ \sum_{k=3}^{n_2} f_{2k}(u_2) = L_{31}(u_2) + L_{41}(u_2) \]

\[ = -\exp(x) \cos y + \exp(x) \cos y = 0. \]  

(33)
These solutions also satisfy (16) through (18), because, for \( p = 1 \)

\[
\sum_{k=1}^{2} [f_{1k}(u_1)h_{1k1}(u_1, u_2)] = f_{11}(u_1)h_{111}(u_1, u_2) + f_{12}(u_1)h_{121}(u_1, u_2)
\]

\[
= L_{11}(u_1)L_{12}(u_1)[\frac{L_{11}(u_2)}{L_{11}(u_1)} + \frac{L_{12}(u_2)}{L_{12}(u_1)}] + L_{21}(u_1)L_{22}(u_1)[\frac{L_{21}(u_2)}{L_{21}(u_1)} + \frac{L_{22}(u_2)}{L_{22}(u_1)}]
\]

\[
= 2 \exp(2x) \cos y \sin y - \exp(2x) \sin y \cos y - \exp(2x) \sin y \cos y = 0.
\]

(34)

For \( p = 2 \), \( m_p - 1 = 0 \) so that (16) through (18) which are for \( 1 \leq i \leq m_p - 1 \) do not apply. Hence the necessary and sufficient conditions are satisfied. Indeed it can be checked that the linear combination \( u = au_1 + bu_2 \) also satisfies (24).

On the other hand \( u_3 = \exp(-x) \sinh y \) is another solution of (24). To see if a linear combination \( u = au_1 + bu_3 \) also satisfies (24), we check the conditions (10) and (11) and (16) through (18) for \( u_1 \) and \( u_3 \). These two solutions satisfy (10) and (11). But the test for (16) with \( p = 1 \) gives

\[
\sum_{k=1}^{2} [f_{1k}(u_1)h_{1k1}(u_1, u_3)] = L_{11}(u_1)L_{12}(u_1)[\frac{L_{11}(u_3)}{L_{11}(u_1)} + \frac{L_{12}(u_3)}{L_{12}(u_1)}]
\]

\[
+ L_{21}(u_1)L_{22}(u_1)[\frac{L_{21}(u_3)}{L_{21}(u_1)} + \frac{L_{22}(u_3)}{L_{22}(u_1)}] = 2 \sin y \sinh y + 2 \sin y \sinh y \neq 0.
\]

(35)

Indeed it can be checked that \( u = au_1 + bu_3 \) does not satisfy (24).

3 Conclusion and Results

Nonlinear partial differential equations that can be decomposed into sums of terms that are products of the results of application of not necessarily the same number of linear partial differential operators have been considered. Necessary and sufficient conditions have been established for the linear combinations of two solutions to satisfy the equation. In [6] a necessary and sufficient condition had been established for the linear combinations of two solutions to satisfy the equation when the equation was composed of sums of terms that are products of the results of application of the same number of linear partial differential operators. In this way we complete the task of finding necessary and sufficient conditions for the linear combinations of two solutions to satisfy an equation when the equation is composed of sums of products of arbitrary numbers of linear partial differential operators.

One example has also been included to illustrate the use of the technique presented in the paper.

As one possible application of the result of this work, we can mention the case when the two solutions of a differential equation are given not analytically, but as two data sets. Then, to be able to determine whether an arbitrary and
not a specific linear combination of these two solutions will also satisfy the differential equation, the presented result can be used.

References


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