Existence of Solutions to Impulsive Fractional Differential Equations with Mixed Boundary Value Conditions

Jie Yang
College of Mathematics and Statistics, Jishou University
Jishou, Hunan 416000, P.R. China

Guoping Chen
School of Civil Engineering and Architecture
Jishou University
Jishou, Hunan 416000, P.R. China

Jingli Xie
College of Mathematics and Statistics, Jishou University
Jishou, Hunan 416000, P.R. China

This article is distributed under the Creative Commons by-nc-nd Attribution License. Copyright © 2020 Hikari Ltd.

Abstract
In this paper, the existence of solutions for a class of Caputo fractional differential equations with impulsive mixed boundary value problem is discussed. And then the conclusions are based on Krasnoselskii’s fixed point theorem. Finally, an example is given to illustrate the main result.

Mathematics Subject Classifications: 34B15, 34B18, 34B37, 58E30

Keywords: Impulsive fractional differential equations; Mixed boundary value; Krasnoselskii’s fixed point theorem

1This work was supported by Scientific Research Fund of Jishou University (No: Jdy19005).
2Corresponding author
1 Introduction

Recently, many scholars have carried out research on the related work of boundary value problems of fractional impulsive differential equations. The scholars have adopted different research methods under certain conditions and have obtained good research results, this also shows the importance of studying such issues. In [1–4], these authors have adopted the fixed point theorem to study the existence of solutions for Caputo or Riemann fractional differential equations BVP’s. There are many methods to solve the boundary value problems of fractional order impulsive differential equations. For example, the numerical method [5], the Mawhin continuation method [6], the integral operator method [7], the Green function method [8], the upper and lower solution method [9] and their references. This article is under the influence of [3], and the paper increases the weight coefficients \( a \) and \( c \) of \( u(1) \) and \( u'(1) \), which promotes the research results of [3] to a certain extent. On the contrary, if \( a = c = 1 \), it degenerates to the results of [3] research.

Precisely, we consider the existence and uniqueness of solutions for a mixed boundary value problem of impulsive differential equations given by

\[
\begin{cases}
C D_0^\alpha u(t) = f(t, u(t)), & 1 < \alpha < 2, \ t \in J', \\
\Delta u |_{t=t_k} = I_k(u(t_k)), \Delta u' |_{t=t_k} = Q_k(u(t_k)), & k = 1, 2, \cdots, p, \\
au(0) + bu'(1) = \delta_1, cu'(0) + du(1) = \delta_2,
\end{cases}
\]

(1.1)

where \( C D_0^\alpha \) is a Caputo fractional derivative of order \( \alpha \in (1, 2) \), \( a \geq b > 0, c \geq d > 0 \), \( \delta_1 \) and \( \delta_2 \) are constants, \( f \in C(J \times R, R), I_k, Q_k \in C(R, R) \), \( J = [0, 1] \) and \( t_k \) satisfy \( 0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1, J' = J \setminus \{t_1, t_2, \cdots, t_p\} \), \( \Delta u(t_k) = u(t_k^+) - u(t_k^-) \), \( \Delta u'(t_k) = u'(t_k^+) - u'(t_k^-) \). Here, respectively, the left and the right limits of \( u(t) \) at \( t = t_k(k = 1, 2, \cdots, p) \) are represented by \( u(t_k^+) \) and \( u(t_k^-) \).

The structure of this article is as follows. In Sect 2, The definitions and theorems related to Caputo’s fractional derivative and integral are given. In Sect 3, the existence of solutions to mixed impulsive boundary value problems are proved by using the fixed point theorem. In Sect 4, an example is provided to illustrate the main research result.

2 Preliminaries

In this section, we mainly introduce related definitions, theorems, lemmas and necessary symbol descriptions.

Let \( PC(J) = \{ u : [0, 1] \rightarrow R | u \in C(J'), u(t_k^+), u(t_k^-) \text{ exist, and } u(t_k^-) = u(t_k), 1 \leq k \leq p \} \). Thus, \( PC(J) \) is a Banach space with the norm \( \| u \|_{PC} = \sup_{0 \leq t \leq 1} |u(t)| \).
For convenience, let $J_0 = [0, t_1], J_1 = [t_1, t_2], \ldots, J_{p-1} = [t_{p-1}, t_p], J_p = [t_p, 1]$.

**Definition 2.1.** ([3]) For a function $f : [0, +\infty) \to \mathbb{R}$ given on the interval $[a, b]$, the Caputo fractional-order derivative of $f : [0, +\infty) \to \mathbb{R}$ is defined by

$$
C^D_{0+}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) \, ds, \quad t > 0, n = [\alpha] + 1.
$$

(2.1)

Where $[\alpha]$ denotes the integer part of real number $\alpha$, and $\Gamma(\cdot)$ is the gamma function.

**Definition 2.2.** ([3]) The fractional integral of order $\alpha$ of a function $f : [0, +\infty) \to \mathbb{R}$ is defined as

$$
I_{0+}^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) \, ds, \quad t > 0, \alpha > 0,
$$

(2.2)

provided that the right hand side of the integral is point-wise defined on $(0, \infty)$.

**Lemma 2.1.** ([10]) For $\alpha > 0$, the general solution of the fractional differential equation $C^D_{0+}^\alpha u(t) = 0$ is given by

$$
u(t) = k_0 + k_1 t + k_2 t^2 + \cdots + k_{n-1} t^{n-1}, \quad k_i \in \mathbb{R},
$$

(2.3)

and

$$
I_{0+}^\alpha (D_{0+}^\alpha u)(t) = u(t) + k_0 + k_1 t + k_2 t^2 + \cdots + k_{n-1} t^{n-1},
$$

(2.4)

for some $k_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n-1, n = [\alpha] + 1$.

where $[\alpha]$ denotes the integer part of the real number $\alpha$.

**Lemma 2.2.** ([10]) The set $G \subset PC([0, 1], R^n)$ is relatively compact set if and only if $G$ is bounded, therefore, $\|x\| \leq M$ for each $x \in G$ and some $M > 0$; the $G$ is quasi-equicontinuous in $[0, 1]$, in other words, for any $\varepsilon > 0$ there exists $\gamma > 0$ such that if $x \in G, k \in N, \chi_1, \chi_2 \in (t_{k-1}, t_k]$ and $|\chi_1 - \chi_2| < \gamma$, we have $|x(\chi_1) - x(\chi_2)| < \varepsilon$.

**Lemma 2.3.** (Krasnoselskii’s) Assume $C$ is a closed, convex and non-empty subset of a Banach space $H$, and the operators $A$ and $B$ be such that: $Ax + By \in C$, whenever $x, y \in C$; $A$ is compact and continuous; and $B$ is a contraction mapping. Therefore, there exists $z \in C$ such that $z = Az + Bz$.

**Lemma 2.4.** Let $\alpha \in (1, 2)$, and $y : J \to \mathbb{R}$ be continuous. A functional $u$ is a solution of the following impulsive mixed boundary value

$$
\begin{cases}
C^D_{0+}^\alpha u(t) = y(t), & 1 < \alpha \leq 2, \ t \in J', \\
\Delta u |_{t=t_k} = I_k(u(t_k)), \Delta u' |_{t=t_k} = Q_k(u(t_k)), & k = 1, 2, \ldots, p, \\
u(0) + b u'(1) = \delta_1, cu'(0) + du(1) = \delta_2,
\end{cases}
$$

(2.5)
if and only if $u$ is a solution of the impulsive fractional integral equation

$$u(t) = \begin{cases} 
\frac{\Gamma(\alpha)}{(t-s)^{\alpha-1}} \int_0^t y(s) \, ds - d\lambda_1(t) \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha-1)} y(s) \, ds \\
- b\lambda_1(t) \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha-1)} y(s) \, ds + \lambda_1(t) \delta_1 + \lambda_2(t) \delta_2, & t \in [0, t_1]; \\
\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha-1)} y(s) \, ds - d\lambda_1(t) \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha-1)} y(s) \, ds + \lambda_1(t) \delta_1 + \lambda_2(t) \delta_2 \\
- b\lambda_1(t) \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s) \, ds + c\lambda_1(t) \sum_{j=1}^p Q_j(u(t_j)) - \lambda_3(t) \sum_{j=1}^p Q_j(u(t_j)) t_j \\
+ \lambda_4(t) \sum_{j=1}^p I_j(u(t_j)), & t \in [t_p, t_{p+1}],
\end{cases}$$

where

$$\lambda_1(t) = \frac{c + d(1 - t)}{ad - bd + ac}, \quad \lambda_2(t) = \frac{at - b}{ad - bd + ac},$$

$$\lambda_3(t) = \frac{ac + d(c - at)}{ad - bd + ac}, \quad \lambda_4(t) = \frac{c + d(1 - at)}{ad - bd + ac}.$$  

**Proof.** With the Lemma 2.1, a general solution $u$ of the equation

$$C^{\alpha}D^\alpha u(t) = y(t)$$

on each interval $(t_k, t_{k+1}](k = 0, 1, 2, \cdots, p)$ is given by

$$u(t) = \Gamma^\alpha y(t) + w_k + y_k t = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds + w_k + y_k t, \quad t \in J_0, \quad (2.6)$$

for some $w_k, y_k \in R$, where $t_0 = 0$ and $t_{p+1} = 1$. Then

$$u'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s) \, ds + y_k, \quad t \in (t_k, t_{k+1}].$$

We have

$$u(0) = w_0, u'(0) = y_0,$$

$$u(1) = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds + w_p + y_p, u'(1) = \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s) \, ds + y_p.$$

We use the boundary conditions $au(0) + bu'(0) = \delta_1$ and $cu' + du(1) = \delta_2$ to get

$$aw_0 + b\left(\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s) \, ds + y_p\right) = \delta_1, \quad cy_0 + d\left(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds + w_p + y_p\right) = \delta_2. \quad (2.7)$$
Next, using the condition of \( \Delta u' \mid_{t=t_k} = Q_k(u(t_k)) = u'(t^+_k) - u'(t^-_k) \), we derive

\[
y_k = y_{k-1} + Q_k(u(t_k)), \quad y_k = y_p - \sum_{j=k+1}^{p} Q_j(u(t_j)). \tag{2.8}
\]

In the same way, using the condition of \( \Delta u \mid_{t=t_k} = I_k(u(t_k)) = u(t^+_k) - u(t^-_k) \), we obtain

\[
w_k + y_k t_k = w_{k-1} + y_{k-1} t_k + I_k(u(t_k)), \tag{2.9}
\]

which by (2.8) implies that

\[
w_k = w_p + \sum_{j=k+1}^{p} Q_j(u(t_j)) t_j - \sum_{j=k+1}^{p} I_j(u(t_j)). \tag{2.10}
\]

By combining (2.7), (2.8), (2.9) and (2.10), we have

\[
by_p + aw_p + \frac{b}{\Gamma(\alpha - 1)} \int_0^1 (1 - s)^{\alpha-2} y(s) ds + a \sum_{j=1}^{p} Q_j(u(t_j)) t_j - a \sum_{j=1}^{p} I_j(u(t_j)) = \delta_1,
\]

and

\[
(c + d)y_p + dw_p + d \int_0^1 (1 - s)^{\alpha-1} y(s) ds - c \sum_{j=1}^{p} Q_j(u(t_j)) = \delta_2.
\]

Then

\[
w_p = \frac{1}{d} \left[ \delta_2 - (c + d)y_p - d \int_0^1 (1 - s)^{\alpha-1} y(s) ds + c \sum_{j=1}^{p} Q_j(u(t_j)) \right]
\]

\[
= \frac{\delta_2}{d} - \frac{d}{d} y_p - d \int_0^1 (1 - s)^{\alpha-1} y(s) ds + \frac{c}{d} \sum_{j=1}^{p} Q_j(u(t_j)).
\]

Therefore \( w_p \) and \( y_p \) are found as follows:

\[
y_p = - \left( \frac{d}{ad + ac - bd} \right) \delta_1 + \left( \frac{a}{ad + ac - bd} \right) \delta_2 - \left( \frac{ad}{ad + ac - bd} \right) \int_0^1 (1 - s)^{\alpha-1} y(s) ds
\]

\[
+ \left( \frac{bd}{ad + ac - bd} \right) \int_0^1 (1 - s)^{\alpha-2} y(s) ds + \left( \frac{ac}{ad + ac - bd} \right) \sum_{j=1}^{p} Q_j(u(t_j))
\]

\[
+ \left( \frac{ad}{ad + ac - bd} \right) \sum_{j=1}^{p} Q_j(u(t_j)) t_j - \left( \frac{ad}{ad + ac - bd} \right) \sum_{j=1}^{p} I_j(u(t_j)). \tag{2.11}
\]

and

\[
w_p = \left( \frac{d + c}{ad + ac - bd} \right) \delta_1 - \left( \frac{b}{ad + ac - bd} \right) \delta_2 + \left( \frac{bd}{ad + ac - bd} \right) \int_0^1 (1 - s)^{\alpha-1} y(s) ds
\]
\[- \left( \frac{bc + bd}{ad + ac - bd} \right) \int_0^1 (1 - s)^{\alpha - 2} y(s) ds - \left( \frac{bc}{ad + ac - bd} \right) \sum_{j=1}^p Q_j(u(t_j)) \]
\[- \left( \frac{ac + cd}{ad + ac - bd} \right) \sum_{j=1}^p Q_j(u(t_j)) t_j + \left( \frac{c + d}{ad + ac - bd} \right) \sum_{j=1}^p I_j(u(t_j)). \]

Combining (2.8), (2.10), (2.11) and (2.12), we obtain

\[
y_k = y_p - \sum_{j=k+1}^p Q_j(u(t_j)) \]
\[
= - \left( \frac{d}{ad + ac - bd} \right) \delta_1 + \left( \frac{a}{ad + ac - bd} \right) \delta_2 - \left( \frac{bd}{ad + ac - bd} \right) \int_0^1 (1 - s)^{\alpha - 1} y(s) ds \]
\[
+ \left( \frac{bd}{ad + ac - bd} \right) \int_0^1 (1 - s)^{\alpha - 2} y(s) ds + \left( \frac{ac}{ad + ac - bd} \right) \sum_{j=1}^p Q_j(u(t_j)) \]
\[
+ \left( \frac{ad}{ad + ac - bd} \right) \sum_{j=1}^p Q_j(u(t_j)) t_j - \left( \frac{ad}{ad + ac - bd} \right) \sum_{j=1}^p I_j(u(t_j)) - \sum_{j=k+1}^p Q_j(u(t_j)), \]

and

\[
w_k = w_p + \sum_{j=k+1}^p Q_j(u(t_j)) t_j - \sum_{j=k+1}^p I_j(u(t_j)) \]
\[
= \left( \frac{d + c}{ad + ac - bd} \right) \delta_1 - \left( \frac{b}{ad + ac - bd} \right) \delta_2 + \left( \frac{bd}{ad + ac - bd} \right) \int_0^1 (1 - s)^{\alpha - 1} y(s) ds \]
\[
- \left( \frac{bc + bd}{ad + ac - bd} \right) \int_0^1 (1 - s)^{\alpha - 2} y(s) ds - \left( \frac{bc}{ad + ac - bd} \right) \sum_{j=1}^p Q_j(u(t_j)) \]
\[
- \left( \frac{ac + cd}{ad + ac - bd} \right) \sum_{j=1}^p Q_j(u(t_j)) t_j + \left( \frac{c + d}{ad + ac - bd} \right) \sum_{j=1}^p I_j(u(t_j)) \]
\[
+ \sum_{j=k+1}^p Q_j(u(t_j)) t_j - \sum_{j=k+1}^p I_j(u(t_j)), \]

for \( k = 0, 1, \ldots, p - 1 \). By using (2.13) and (2.14), we get

\[
w_k + y_k t = \left( \frac{c + d(1 - t)}{ad + ac - bd} \right) \delta_1 + \left( \frac{at - b}{ad + ac - bd} \right) \delta_2 + \left( \frac{-d(at - b)}{ad + ac - bd} \right) \int_0^1 (1 - s)^{\alpha - 1} y(s) ds \]
\[
+ \left( \frac{-b(c + d(1 - t))}{ad + ac - bd} \right) \int_0^1 (1 - s)^{\alpha - 2} y(s) ds + \left( \frac{c(at - b)}{ad + ac - bd} \right) \sum_{j=1}^p Q_j(u(t_j)) \]
\[
- \left( \frac{ac + d(c - at)}{ad + ac - bd} \right) \sum_{j=1}^p Q_j(u(t_j)) t_j + \left( \frac{c + d(1 - at)}{ad + ac - bd} \right) \sum_{j=1}^p I_j(u(t_j)) \]
\[
+ \sum_{j=k+1}^p Q_j(u(t_j))(t_j - t) - \sum_{j=k+1}^p I_j(u(t_j)).\]
Therefor, by the (2.6), we get
\[ u(t) = \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \lambda_1(t) \delta_1 + \lambda_2(t) \delta_2 - d\lambda_2(t) \int_0^1 \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \]
\[ - b\lambda_1(t) \int_0^1 \frac{(1 - s)^{\alpha-2}}{\Gamma(\alpha - 1)} y(s) ds + c\lambda_1(t) \sum_{j=1}^p Q_j(u(t_j)) - \lambda_3(t) \sum_{j=1}^p Q_j(u(t_j)) t_j \]
\[ + \lambda_4(t) \sum_{j=1}^p I_j(u(t_j)) + \sum_{j=k+1}^p Q_j(u(t_j))(t_j - t) - \sum_{j=k+1}^p I_j(u(t_j)), \]

where \( \lambda_1(t) = \frac{c + d(1-t)}{ad + ac - bd} \), \( \lambda_2(t) = \frac{at - b}{ad + ac - bd} \), \( \lambda_3(t) = \frac{ac + d(c-at)}{ad + ac - bd} \), \( \lambda_4(t) = \frac{c + d(1-at)}{ad + ac - bd} \).

If \( a = c = 1 \), we have \( \lambda_1(t) = \lambda_3(t) = \lambda_4(t) \), the questions of (1.1) degenerates to problems (1) in literature [3]. □

### 3 Main Results

In this section, we will state and prove the existence and uniqueness of problem (1.1).

By Lemma 2.4 with the \( y(t) = f(t, u(t)) \), and define an operator \( T : PC(J) \to PC(J) \), we have
\[ (Tu)(t) = \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) ds + \lambda_1(t) \delta_1 + \lambda_2(t) \delta_2 - d\lambda_2(t) \int_0^1 \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) ds \]
\[ - b\lambda_1(t) \int_0^1 \frac{(1 - s)^{\alpha-2}}{\Gamma(\alpha - 1)} f(s, u(s)) ds + c\lambda_1(t) \sum_{j=1}^p Q_j(u(t_j)) - \lambda_3(t) \sum_{j=1}^p Q_j(u(t_j)) t_j \]
\[ + \lambda_4(t) \sum_{j=1}^p I_j(u(t_j)) + \sum_{j=k+1}^p Q_j(u(t_j))(t_j - t) - \sum_{j=k+1}^p I_j(u(t_j)), \]
\[ t \in (t_k, t_{k+1}], \quad k = 0, 1, 2, \cdots, p, \quad (3.1) \]

for convenience, we quote the following symbols throughout this paper:
\[
\begin{align*}
\lambda_1(t) &= \frac{c + d(1-t)}{ad + ac - bd} \leq \lambda_1 := \frac{|c + 2d|}{|ad + ac - bd|}, \\
\lambda_2(t) &= \frac{at - b}{ad + ac - bd} \leq \lambda_2 := \frac{|a + b|}{|ad + ac - bd|}, \\
\lambda_3(t) &= \frac{ac + d(c-at)}{ad + ac - bd} \leq \lambda_3 := \frac{|ac + d(c+a)|}{|ad + ac - bd|}, \\
\lambda_4(t) &= \frac{c + d(1-at)}{ad + ac - bd} \leq \lambda_4 := \frac{|c + d(1+a)|}{|ad + ac - bd|}.
\end{align*}
\]

The problem (1.1) reduces to a fixed point problem \( u(t) = Tu(t) \), where \( T \) is given by (3.1). Therefor, problem (1.1) has a solution if and only if the operator \( T \) has a fixed point.

**Theorem 3.1.** Assume \( |f(t, u)| \leq \eta(t) \) for all \( (t, u) \in J \times R \) where \( \eta \in L^\infty(J, R)(0 < \mu < \alpha - 1) \), and

\( (A_1) \ |f(t, x) - f(t, y)| \leq L_1|x - y|, \) for all \( t \in [0, 1], \) \( x, y \in R; \)
\((A_2)\) \(|Q_k(x) - Q_k(y)| \leq L_2|x - y|, |I_k(x) - I_k(y)| \leq L_3|x - y|, |Q_k(x)| \leq M_1, |I_k(x)| \leq M_2, x, y \in R, k = 1, 2, \ldots, p, \) hold with
\[ p(|c| \lambda_1 + \lambda_3 + 2)L_2 + p(\lambda_4 + 1)L_3 < 1. \]  
(3.2)

Then problem (1.1) has at least solution on \([0,1]\).

**Proof.** Let us choose
\[ r \geq \|\eta\|_{L^\frac{1}{\mu}(J)} \left[ \frac{(1 + |d|\lambda_2)}{\Gamma(\alpha)(\frac{\alpha - \mu}{1 - \mu})^{1-\mu}} + \frac{|b|\lambda_1}{\Gamma(\alpha - 1)(\frac{\alpha - \mu - 1}{1 - \mu})^{1-\mu}} \right] \]
\[ + p\left[ (|c| \lambda_1 + \lambda_3 + 2)M_1 + (\lambda_4 + 1)M_2 \right], \]
and denote
\[ Z_r = \{ u \in PC(J,R) \|\| u \|_{PC} \leq r \}. \]

Define the operators \(N\) and \(K\) on \(Z_r\) as
\[
(Nu)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) ds - \lambda_2(t) \int_0^t \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} f(s, u(s)) ds
\]
\[- \lambda_1(t) \int_0^t \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) ds,
\]
\[
(Ku)(t) = c\lambda_1(t) \sum_{j=1}^p Q_j(u(t_j)) - \lambda_3(t) \sum_{j=1}^p Q_j(u(t_j)) t_j + \lambda_4(t) \sum_{j=1}^p I_j(u(t_j))
\]
\[- \sum_{j=k+1}^p Q_j(u(t_j))(t_j - t) - \sum_{j=k+1}^p I_j(u(t_j)).
\]

For any \(u, v \in Z_r(t \in J)\), using the condition that \(|f(t, u) \leq \eta(t)\)| and the Hölder inequality,
\[
\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds \leq \frac{1}{\Gamma(\alpha)} \left( \int_0^t (t-s)^{\alpha-1} \eta(s) ds \right)^{1-\mu} \left( \int_0^t \eta(s) \frac{1}{\eta(s)} ds \right)^{\mu}
\]
\[ \leq \frac{\|\eta\|_{L^{\frac{1}{\mu}}(J)}}{\Gamma(\alpha)(\frac{\alpha - \mu}{1 - \mu})^{1-\mu}}.
\]
\[
\frac{1}{\Gamma(\alpha)} \int_0^t (1-s)^{\alpha-1} f(s, u(s)) ds \leq \frac{1}{\Gamma(\alpha)} \left( \int_0^t (1-s)^{\alpha-1} \eta(s) ds \right)^{1-\mu} \left( \int_0^t \eta(s) \frac{1}{\eta(s)} ds \right)^{\mu}
\]
\[ \leq \frac{\|\eta\|_{L^{\frac{1}{\mu}}(J)}}{\Gamma(\alpha)(\frac{\alpha - \mu}{1 - \mu})^{1-\mu}}.
\]
and
\[ \frac{1}{\Gamma(\alpha-1)} \int_0^t (1-s)^{\alpha-2} f(t,u(s))ds \leq \frac{1}{\Gamma(\alpha-1)} \left( \int_0^t (1-s)^{\alpha-2} ds \right)^{1-\mu} \left( \int_0^t (\eta(s))^{\frac{1}{\mu}} ds \right)^\mu. \]

Thus,
\[ \| (Nu) + (Kv) \| \leq \| \eta \|_{L^\frac{1}{\mu}(J)} \left( \frac{1 + |d|\lambda_2}{\Gamma(\alpha)(\frac{\alpha-\mu}{1-\mu})^{1-\mu}} + \frac{|b|\lambda_1}{\Gamma(\alpha-1)(\frac{\alpha-\mu-1}{1-\mu})^{1-\mu}} \right) + p \left[ |c|\lambda_1 + \lambda_3 + 2 \right] M_1 + (\lambda_4 + 1) M_2. \]

Therefore, \( Nu + Kv \in Z_r \). By the (3.2), it is obvious that \( N \) is a contraction mapping. And the continuity of \( f \) implies that the operator \( K \) is continuous. Thus, \( K \) is uniformly bounded on \( Z_r \) where
\[ \| (Ku) \| \leq \frac{(1 + |d|\lambda_2)\| \eta \|_{L^\frac{1}{\mu}(J)}}{\Gamma(\alpha)(\frac{\alpha-\mu}{1-\mu})^{1-\mu}} + \frac{|b|\lambda_1\| \eta \|_{L^\frac{1}{\mu}(J)}}{\Gamma(\alpha-1)(\frac{\alpha-\mu-1}{1-\mu})^{1-\mu}} \leq r. \]

Next the quasi-equicontinuity of the operator \( K \) is proved. Let \( \Psi = J \times Z_r, f_{\text{max}} = \sup_{(t,u) \in \Psi} |f(t,u)|. \) For any \( t_k < \chi_2 < \chi_1 \leq t_{k+1} \), we have
\[ \| (Ku)(\chi_2) - (Ku)(\chi_1) \| \leq f_{\text{max}} \left[ \frac{2(\chi_1 - \chi_2)\alpha + \chi_1^\alpha - \chi_2^\alpha}{\Gamma(\alpha + 1)} + |d| \left( \frac{\chi_1^\alpha - \chi_2^\alpha}{\Gamma(\alpha + 1)} \right) + \frac{|b|\|d\|}{\Gamma(\alpha)} \right]. \]

Which tends to zero as \( \chi_2 \to \chi_1 \). Thus \( K \) is quasi-equicontinuous on the interval \( (t_k, t_{k+1}] \). By Lemma 2.2, it is obvious that \( K \) is compact and is relatively on \( Z_r \). And all the assumptions of Lemma 2.3 are satisfied therefor BVP of the (1.1) has at least one solution on \( J = [0, 1] \). \( \square \)

### 4 Example

**Example 4.1.** Consider the following mixed fractional boundary value problem with \( 1 < \alpha \leq 2, T = [0, 1] \)

\[
\begin{align*}
\frac{CD^\frac{3}{5}}{\Gamma(\frac{3}{5})} u(t) &= \frac{\cos^3 u(t)}{4(t + 2)^3(1 + u^\gamma(t))}, \quad 0 < t < 1, t \neq \frac{1}{5} \\
\Delta u(\frac{1}{5}) &= \frac{|u(\frac{1}{5})|}{50 + |u(\frac{1}{5})|}, \quad \Delta u'(\frac{1}{5}) = \frac{|u'(\frac{1}{5})|}{60 + |u'(\frac{1}{5})|}, \\
u(0) + \frac{1}{2} u'(1) &= 0, u'(0) + \frac{1}{3} u(1) = 0.
\end{align*}
\]
Here $0 < t < 1$, let $\alpha = \frac{3}{2}$, $t = \frac{1}{5}$, $a = c = 1$, $b = \frac{1}{2}$, $d = \frac{1}{3}$, $p = 1$, $\delta_1 = \delta_2 = 0$, $L_1 = \frac{1}{20}$, $L_2 = \frac{1}{50}$, $L_3 = \frac{1}{60}$, $M_2 = \frac{1}{50}$, $M_3 = \frac{1}{60}$. Then $\lambda_1 := \frac{|c+2d|}{ad+ac-bd} = \frac{10}{7}$, $\lambda_2 := \frac{|a+b|}{ad+ac-bd} = \frac{9}{7}$, $\lambda_3 := \frac{|ac+d(c+a)|}{ad+ac-bd} = \frac{10}{7}$, $\lambda_4 := \frac{|c+d(1+a)|}{ad+ac-bd} = \frac{10}{7}$. By the condition (3.2), we have $p(|c|\lambda_1 + \lambda_3 + 2)L_2 + p(\lambda_4 + 1)L_3 \approx 0.2834 < 1$.

Therefore, all assumptions in the Theorem 3.1 is satisfied. Hence, the fractional impulsive mixed boundary value problem of (4.1) at least one solution on $[0, 1]$.

References


**Received:** September 11, 2020; **Published:** October 3, 2020