Limit Cycle Bifurcations of Some Liénard System with Symmetry

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Abstract

In this paper, we study limit cycle bifurcations of some Liénard system with symmetry. Using the Melnikov functions and bifurcation theories, we found that $H(8, 7) \geq 11$.

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1 Introduction

Hilbert [1] proposed 23 mathematical problems at the Second Congress of Mathematicians, of which the second part of the 16th one is to find the maximal number of limit cycles and their relative locations for polynomial vector fields. It is one of the few unsolved problems so far, and mathematicians have done a lot of work to promote the thorough solution to this problem, see [3]-[7]. A large part of the work is concerned with the Liénard system [2]

$$\begin{align*}
\dot{x} &= y, \\
\dot{y} &= g(x) + \varepsilon f(x)y,
\end{align*}$$

(1)

where $\varepsilon$ is a small parameter, $f(x)$ and $g(x)$ are polynomials of $x$ with $\deg f = m$ and $\deg g = n$. Let $H(m, n)$ denote the maximum number of limit cycles for system (1), there are a lot of results about it. For $n = 1$, Blows and Lloyd [3] proved $H(m, 1) \geq \lceil \frac{m}{2} \rceil$; For $n = 2$, Han[4] proved $H(m, 2) \geq \lceil \frac{2m+1}{3} \rceil$
for \( m \geq 3 \); for \( n = 3 \), Yang, Han and Romanovski [5] proved \( H(3,3) \geq 5 \), \( H(4,3) \geq 6 \), \( H(5,3) \geq 6 \), \( H(6,3) \geq 8 \), \( H(7,3) \geq 8 \), \( H(8,3) \geq 9 \); for \( n = 4 \), Han, Yan et al. [6] proved \( H(m,4) \geq m + 3 \), for \( m = 2, 3, 5, 6, 7, 8 \) and \( H(4,4) \geq 6 \); for \( n = 5 \), Xu and Li [7] proved \( H(2,5) \geq 3 \), \( H(4,5) \geq 5 \), \( H(6,5) \geq 10 \), \( H(8,5) \geq 10 \); for \( n = 6 \), Asheghi and Bakhshalizadeh [8] proved \( H(5,6) \geq 9 \), \( H(6,6) \geq 10 \), \( H(7,6) \geq 11 \).

In this paper, we consider a class of system (1) about \( n = 7 \),

\[
\dot{x} = y, \quad \dot{y} = -x(x^2 - 1)(x^2 - \frac{1}{4})(x^2 - 2) + \varepsilon f(x, \delta)y, \tag{2}
\]

where

\[
f(x, \delta) = a_0 + a_1x^2 + a_2x^4 + a_3x^6 + a_4x^8 \tag{3}
\]

is polynomial of degree 8. Then we get the following result

**Theorem 1.1** Let \( a_4 \neq 0 \), the system (2) has at least 11 limit cycles.

## 2 Preliminary

Consider the following system

\[
\dot{x} = H_y(x, y) + \varepsilon p(x, y, \varepsilon, \delta), \quad \dot{y} = -H_x(x, y) + \varepsilon q(x, y, \varepsilon, \delta), \tag{4}
\]

where \( H, p \) and \( q \) are analytic functions, \( \varepsilon \geq 0 \) is small enough and \( \delta \in D \subset R^n \) is a vector parameter with \( D \) compact. When \( \varepsilon = 0 \), (4) is a near-Hamiltonian system and has the following assumption (see fig.1) that (4) is symmetric and has a double homoclinic loop \( L = L_1 \cup L_2 \) inside and a 2-poly-cycle \( \Gamma^2 \) and two homoclinic loops \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) outside, where \( L_1 = L_{|x \geq 0} \) and \( L_2 = L_{|x \leq 0} \) are homoclinic loops with hyperbolic saddle \( O(0,0) \) and \( L \) is defined by \( H(x,y) = 0 \); Centers \( C_l(x_c, y_c) \), \( l = 1, 2 \) are surrounded by the homoclinic loop \( L_l \) and \( H(C_l) = \eta \); A 2-poly-cycle \( \Gamma^2 \) is connected by 2 hyperbolic saddles \( S_l \) with 2 heteroclinic orbits \( \mathcal{L}_l \) and \( H(x,y) = \mu \); Hyperbolic saddles \( S_l \), \( l = 1, 2 \) connect the homoclinic loop \( \mathcal{L}_l \) respectively and \( H(x,y) = \mu \); Centers \( D_l(x_{D_l}, y_{D_l}) \), \( l = 1, 2 \) are surrounded by the homoclinic loop \( \mathcal{L}_l \) and \( H(D_l) = \beta \); 2-poly-cycle \( \Gamma^2 \) and two homoclinic loops \( \mathcal{L}_l \), \( l = 1, 2 \) are together denoted

\[
\Gamma^* = \Gamma^2 \cup \mathcal{L}_1 \cup \mathcal{L}_2. \tag{5}
\]

Then there exist four families of periodic orbits given by

\[
\begin{align*}
L_l(h) & : \quad H(x,y) = h, \quad h \in (\eta,0), \\
\mathcal{L}_l(h) & : \quad H(x,y) = h, \quad h \in (0,\mu), \\
\mathcal{L}_l(h) & : \quad H(x,y) = h, \quad h \in (\beta,\mu), \\
\mathcal{L}_l(h) & : \quad H(x,y) = h, \quad h \in (\mu, +\infty). \tag{6}
\end{align*}
\]
Let
\[ M_l(h, \delta) = \oint_{L_l(h)} (qdx - pdy)|_{\varepsilon=0}, \quad h \in (\eta, 0), \quad l = 1, 2, \]
\[ M(h, \delta) = \oint_{L(h)} (qdx - pdy)|_{\varepsilon=0}, \quad h \in (0, \mu), \]
\[ \overline{M}_l(h, \delta) = \oint_{\Gamma} (qdx - pdy)|_{\varepsilon=0}, \quad h \in (\beta, \mu), \quad l = 1, 2, \]
\[ \overline{M}(h, \delta) = \oint_{\Gamma} (qdx - pdy)|_{\varepsilon=0}, \quad h \in (\mu, +\infty), \]
which are called Melnikov functions. Moreover by the symmetry, we have
\[ M_1(h, \delta) = M_2(h, \delta) \text{ and } \overline{M}_1(h, \delta) = \overline{M}_2(h, \delta). \]

By [9], for \((x, y)\) near the saddle \(O(0, 0)\), we have

**Lemma 2.1** Suppose
\[ H(x, y) = \frac{\lambda}{2} (y^2 - x^2) + \sum_{i+j \geq 3} h_{ij} x^i y^j, \]
then
\[ M_1(h, \delta) = c_{01} + c_1 h \ln |h| + c_{21} h + c_3 h^2 \ln |h| + O(h^2), \quad 0 < -h \ll 1, \]
\[ M(h, \delta) = c_0 + 2c_1 h \ln |h| + c_2 h + 2c_3 h^2 \ln |h| + O(h^2), \quad 0 < h \ll 1, \]
where
\[ c_{01} = M_1(0, \delta), \quad c_0 = 2c_{01}, \]
\[ c_1 = -\frac{1}{|\lambda|} (a_{10} + b_{01}) = -\frac{1}{|\lambda|} (p_x + q_y)(0, 0, 0, \delta), \]
\[ c_{21} = \oint_{L_1} (p_x + q_y - a_{10} - b_{01})|_{\varepsilon=0} dt|_{c_1=0}, \quad c_2 = 2c_{21}, \]
\[ c_{3|c_1=0} = \frac{-1}{2|\lambda|^2} \left\{ (-3a_{30} - b_{21} + a_{12} + 3b_{03}) - \frac{1}{\lambda} [2b_{02} + a_{11})(3h_{03} - h_{21}) + (2a_{20} + b_{11})(3h_{30} - h_{12})] \right\}. \]

Figure 1: The portrait of \((4)|_{\varepsilon=0}.\)
Lemma 2.2 \cite{10} In Lemma 2.1 denote
\[
\tilde{c}_0 = c_{01}, \quad \tilde{c}_1 = c_1, \quad \tilde{c}_2 = c_{21}, \quad \tilde{c}_3 = c_3 \mid \tilde{c}_1 = 0.
\tag{10}
\]
If there exists $\delta_0 \in D$ and $1 \leq k \leq 3$ such that
\[
\tilde{c}_j(\delta_0) = 0, \quad j = 0, \ldots, k - 1, \quad \tilde{c}_k \neq 0,
\]
then system (4) can have $\lceil \frac{3k}{2} \rceil$ limit cycles near $L$ for some $(\varepsilon, \delta)$ near $(0, \delta_0)$.

By \cite{11}, for $(x, y)$ near $(x_{c_1}, y_{c_1})$, we have

Lemma 2.3 Suppose
\[
H(x, y) = \eta + \frac{1}{2}((x - x_{c_1})^2 + (y - y_{c_1})^2) + \sum_{i+j \geq 3} h_{ij}(x - x_{c_1})^i(y - y_{c_1})^j,
\]
and
\[
(p_x + q_y) \mid \varepsilon = 0 = \sum_{i+j \geq 0} c_{ij}(x - x_{c_1})^i(y - y_{c_1})^j.
\]
Then for $0 < h - \eta \ll 1$, we have
\[
M_l(h, \delta) = b_{01}(h - \eta) + b_{11}(h - \eta)^2 + \cdots + b_{k1}(h - \eta)^{k+1} + O(|h - \eta|^{k+2}), \tag{11}
\]
where
\[
\begin{aligned}
b_{01} &= 2\pi c_{00}, \\
b_{11} &= c_{00} \pi \left[ \frac{15}{2} (h_{30}^2 + h_{03}^2) + \frac{3}{2} (h_{21}^2 + h_{12}^2 + 2h_{30}h_{12} + 2h_{21}h_{03}) \right] - c_{10} \pi (h_{12} + 3h_{30}) \\
&\quad - c_{01} \pi (h_{21} + 3h_{03}) + c_{20} \pi + c_{02} \pi.
\end{aligned}
\tag{12}
\]

By \cite{10}, for $(x, y)$ near $(x_{S_l}, y_{S_l})$, $l = 1, 2$, we have

Lemma 2.4 Suppose
\[
H(x, y) = \mu + \frac{\lambda_l}{2}((y - y_{S_l})^2 - (x - x_{S_l})^2) + \sum_{i+j \geq 3} h_{ij}(x - x_{S_l})^i(y - y_{S_l})^j,
\]
then
\[
\begin{aligned}
M_l(h, \delta) &= \tilde{c}_0 + \tilde{c}_1(h - \mu) \ln |h - \mu| + O(h - \mu), \quad 0 < \mu - h \ll 1, \\
\overline{M}_l(h, \delta) &= \tilde{c}_0 + \tilde{c}_1(h - \mu) \ln |h - \mu| + O(h - \mu), \quad 0 < \mu - h \ll 1, \\
\overline{M}(h, \delta) &= (\tilde{c}_0 + \tilde{c}_0 + \tilde{c}_{02} + (\tilde{c}_1 + 2\tilde{c}_1)(h - \mu) \ln |h - \mu| + O(h - \mu), \quad 0 < h - \mu \ll 1.
\end{aligned}
\tag{13}
\]
where
\[
\begin{align*}
\tilde{c}_0(\delta) &= \int_{E_1} qdx - pdy|_{\varepsilon=0} + \int_{E_2} qdx - pdy|_{\varepsilon=0}, \quad l = 1, 2, \\
\tau_0(\delta) &= \int_{E_1} qdx - pdy|_{\varepsilon=0}, \quad l = 1, 2
\end{align*}
\]

(14)

By [?], for \((x, y)\) near \((x_{D_l}, y_{D_l})\), we have

**Lemma 2.5** Suppose

\[
H(x, y) = \beta + \frac{1}{2}((x - x_{D_l})^2 + (y - y_{D_l})^2) + \sum_{i+j\geq 3} h_{ij}(x - x_{D_l})^i(y - y_{D_l})^j,
\]

and

\[
(p_x + q_y)|_{\varepsilon=0} = \sum_{i+j\geq 0} c_{ij}(x - x_{D_l})^i(y - y_{D_l})^j.
\]

Then for \(0 < h - \beta \ll 1\), we have

\[
\bar{M}_i(h, \delta) = \bar{b}_{01}(h - \beta) + O(|h - \beta|^2), \quad \bar{b}_{01} = 2\pi c_{00}.
\]

(15)

According to the above lemmas, we can get the following theorem

**Theorem 2.1** Let (7)-(15) hold. Suppose that there exists \(\delta_0 \in D\) and \(k_1 = 1, k_2 = k_3 = k_4 = 0, k = 3\) such that

\[
\begin{align*}
\tilde{c}_0(\delta_0) &= \tilde{c}_1(\delta_0) = \tilde{c}_2(\delta_0) = 0, \quad \tilde{c}_3(\delta_0) \neq 0, \\
b_{01}(\delta_0) &= 0, \quad b_{11}(\delta_0) \neq 0, \quad \tilde{c}_4(\delta_0) \neq 0, \quad \tau_{01}(\delta_0) \neq 0, \quad \bar{b}_{01}(\delta_0) \neq 0,
\end{align*}
\]

and

\[
\text{rank} \frac{\partial(\tilde{c}_0, \tilde{c}_1, \tilde{c}_2, b_{01})}{\partial(\delta_1, \delta_2, \delta_3, \delta_4)}(\delta_0) = 4.
\]

Let \(\rho_1 = b_{11}(\delta_0)\tilde{c}_3(\delta_0), \quad \rho_2 = \tilde{c}_3(\delta_0)\tilde{c}_0(\delta_0), \quad \rho_3 = -\bar{b}_{01}(\delta_0)\tau_{01}(\delta_0).\) Suppose \(\sigma(l) = 1\) if \(\rho_l > 0\) and \(\sigma(l) = 0\) if \(\rho_l < 0, \ l = 1, 2, 3\), then system (4) can have \(9 + 2\sigma(1) + \sigma(2) + 2\sigma(3)\) limit cycles for some \((\varepsilon, \delta)\) near \((0, \delta_0)\).

**Proof.** When (8)-(15) hold, we have

\[
\begin{align*}
M_1(h, \delta_0) &= b_{11}(\delta_0)(h - \eta)^2 + O(|h - \eta|^3), \quad 0 < h - \eta \ll 1, \\
M_1(h, \delta_0) &= \tilde{c}_3(\delta_0)h^2 \ln |h| + O(h^2), \quad 0 < -h \ll 1, \\
M(h, \delta_0) &= 2\tilde{c}_3(\delta_0)h^2 \ln |h| + O(h^2), \quad 0 < h \ll 1, \\
M(h, \delta_0) &= \tilde{c}_0(\delta_0) + O(|h - \eta| \ln |h - \eta|), \quad 0 < \mu - h \ll 1, \\
\bar{M}_1(h, \delta_0) &= \bar{b}_{01}(\delta_0)(h - \eta) + O((h - \eta)^2), \quad 0 < h - \beta \ll 1, \\
\bar{M}(h, \delta_0) &= \tau_{01}(\delta_0) + O(|h - \eta| \ln |h - \eta|), \quad 0 < \mu - h \ll 1, \\
\bar{M}(h, \delta_0) &= (\bar{c}_0(\delta_0) + 2\bar{c}_{01}(\delta_0)) + O(|h - \eta| \ln |h - \eta|), \quad 0 < \mu - h \ll 1.
\end{align*}
\]

If \(b_{11}(\delta_0)\tilde{c}_3(\delta_0) > 0, \quad \tilde{c}_3(\delta_0)\tilde{c}_0(\delta_0) > 0, \quad \bar{b}_{01}(\delta_0)\tau_{01}(\delta_0) < 0\), then there exists \(h_1 \in (\eta, 0), \ h \in (0, \mu)\) and \(\bar{h}_1 \in (\beta, \mu)\) such that

\[
\begin{align*}
M_1(h_1, \delta_0) &= 0, \quad M_1(h_1 - \varepsilon_0, \delta_0)M_1(h_1 + \varepsilon_0, \delta_0) < 0, \\
M(h, \delta_0) &= 0, \quad M(h - \varepsilon_0, \delta_0)M(h + \varepsilon_0, \delta_0) < 0, \\
\bar{M}_1(\bar{h}_1, \delta_0) &= 0, \quad \bar{M}(\bar{h}_1 - \varepsilon_0, \delta_0)\bar{M}(\bar{h}_1 + \varepsilon_0, \delta_0) < 0,
\end{align*}
\]

(16)
when $\varepsilon_0$ is sufficiently small.

Choose $b_{01}, \bar{c}_0, \bar{c}_1, \bar{c}_2$ as free parameters, Then (8)-(15) can be rewritten as

\[
\begin{align*}
M_1(h, \delta) &= b_0(1 - \eta) + b_{11}^*(1 - \eta)^2 + O((1 - \eta)^3), \quad 0 < h - \eta \ll 1, \\
M_1(h, \delta) &= \bar{c}_0 + \bar{c}_1 h \ln |h| + \bar{c}_2 h + \bar{c}_3^2 h^2 \ln |h| + O(h^2), \quad 0 < h \ll 1, \\
M(h, \delta) &= 2\bar{c}_0 + 2\bar{c}_1 h \ln |h| + 2\bar{c}_2 h + 2\bar{c}_3^2 h^2 \ln |h| + O(h^2), \quad 0 < h \ll 1, \\
M(h, \delta) &= \bar{c}_0 + O((h - \mu) \ln |h - \mu|), \quad 0 < \mu - h \ll 1, \\
\overline{M}(h, \delta) &= \bar{c}_0 + O((h - \mu) \ln |h - \mu|), \quad 0 < \mu - h \ll 1, \\
\overline{M}(h, \delta) &= \bar{b}_{01}(h - \beta) + O((h - \beta)^2), \quad 0 < h - \beta \ll 1, \\
\overline{M}(h, \delta) &= (\bar{c}_0 + 2\bar{c}_{01}) + O((h - \mu) \ln |h - \mu|), \quad 0 < h - \mu \ll 1.
\end{align*}
\]

where

\[
\begin{align*}
b_{11}^* &= b_{1, 1}(\delta_0) + O((b_{01}, \bar{c}_0, \bar{c}_1, \bar{c}_2)), \quad \bar{c}_3^* = \bar{c}_3(\delta_0) + O((b_{01}, \bar{c}_0, \bar{c}_1, \bar{c}_2)), \\
\bar{c}_0^* &= \bar{c}_0(\delta_0) + O((b_{01}, \bar{c}_0, \bar{c}_1, \bar{c}_2)), \quad \bar{c}_{01}^* = \bar{c}_{01}(\delta_0) + O((b_{01}, \bar{c}_0, \bar{c}_1, \bar{c}_2)).
\end{align*}
\]

Then it is easy to see that if

\[
b_{01}b_{11}^* < 0, \quad 0 < |b_{01}| \ll |b_{11}| \ll 1 \tag{17}
\]

$M_1(h, \delta)$ has $k_1$ zeros near $h = \eta$. If

\[
0 < \bar{c}_0 \ll -\bar{c}_1 \ll -\bar{c}_2 \ll 1 \tag{18}
\]

$M_1(h, \delta)$ has 3 negative zeros and $M(h, \delta)$ has 1 positive zero near $h = 0$.

Moreover, it follows from (16) that under (17) and (18), there exists $h_1^*$ near $h_1$, $h^*$ near $h$ and $\overline{h}_1^*$ near $\overline{h}_1$ such that

\[
\begin{align*}
M_1(h_1^*, \delta_0) &= 0, \quad M_1(h_1^* - \varepsilon_0, \delta_0)M_1(h_1^* + \varepsilon_0, \delta_0) < 0, \\
M(h_1^*, \delta_0) &= 0, \quad M(h_1^* - \varepsilon_0, \delta_0)M(h_1^* + \varepsilon_0, \delta_0) < 0, \\
\overline{M}(\overline{h}_1^*, \delta_0) &= 0, \quad \overline{M}(\overline{h}_1^* - \varepsilon_0, \delta_0)\overline{M}(\overline{h}_1^* + \varepsilon_0, \delta_0) < 0.
\end{align*}
\]

Then for $0 < |\varepsilon_0| \ll 1$, system (4) can have at least $7 + 2(1 + 1) + (0 + 1) + 2 \times 0 + 2(0 + 1) = 9 + 2\sigma(1) + \sigma(2) + 2\sigma(3)$ limit cycles.

## 3 Proof of Theorem 1.1.

We consider system (2). For $\varepsilon = 0$, (2) is a Hamiltonian system with

\[
H(x, y) = \frac{1}{2} y^2 - \frac{1}{4} x^2 + \frac{11}{16} x^4 - \frac{13}{24} x^6 + \frac{1}{8} x^8 \tag{19}
\]

and has 7 equilibria: saddles $O(0, 0)$, $S_1(1, 0)$, $S_2(-1, 0)$ and centers $C_1(\frac{1}{2}, 0)$, $C_2(-\frac{1}{2}, 0)$, $D_1(\sqrt{2}, 0)$, $D_2(-\sqrt{2}, 0)$. Note that $H(O) = 0$, $H(S_i) = \frac{1}{48}$, $H(C_l) = -\frac{169}{6144}$, $H(D_l) = -\frac{1}{12}$, $l = 1, 2$. 
The equation $H(x, y) = 0$ defines a double homoclinic loop $L = L_1 \cup L_2$ with $L_1 = L_{x \ge 0}$, $L_2 = L_{x \le 0}$. The equation $H(x, y) = \frac{1}{48}$ defines a 2-polycycle $\Gamma^2$ consisting of 2 hyperbolic saddles $S_1, S_2$ and 2 heteroclinic orbits $\hat{L}_1$ and $\hat{L}_2$ satisfying $\omega(\hat{L}_1) = S_1, \alpha(\hat{L}_1) = S_2$ and $\omega(\hat{L}_2) = S_2, \alpha(\hat{L}_2) = S_1$, respectively and the equation $H(x, y) = \frac{1}{48}$ also defines two homoclinic loops $\hat{L}_1$ and $\hat{L}_2$ consisting of 2 hyperbolic saddles $S_1, S_2$, respectively.

Then the equation $H(x, y) = h$ defines three families of periodic orbits $L_l(h), l = 1, 2$ for $-\frac{169}{6144} < h < 0$ and $L(h)$ for $0 < h < \frac{1}{48}$. The equation $\mathcal{H}(x, y) = h$ defines two families of periodic orbits $\mathcal{L}_l(h), l = 1, 2$ for $-\frac{1}{12} < h < \frac{1}{48}$. And the equation $H^*(x, y) = h$ defines a family of periodic orbits $L^*(h)$ for $h > \frac{1}{48}$.

By (7) and (19), we have the Melnikov functions as follows

$$M_l(h, \delta) = \oint_{L_l(h)} f(x, \delta) y \, dx, \quad -\frac{169}{6144} < h < 0, \quad l = 1, 2,$$

$$M(h, \delta) = \oint_{L(h)} f(x, \delta) y \, dx, \quad 0 < h < \frac{1}{48},$$

$$\overline{M}_l(h, \delta) = \oint_{\mathcal{L}_l(h)} f(x, \delta) y \, dx, \quad -\frac{1}{12} < h < \frac{1}{48}, \quad l = 1, 2,$$

$$\overline{M}(h, \delta) = \oint_{L^*(h)} f(x, \delta) y \, dx, \quad h > \frac{1}{48},$$

where $M_1(h, \delta) = M_2(h, \delta)$ and $\overline{M}_1(h, \delta) = \overline{M}_2(h, \delta)$.

By Lemma 2.1, for $0 < -h \ll 1$, we have

$$M_1(h, \delta) = c_0 + c_1 h \ln |h| + c_2 h + c_3 (O, \delta) h^2 \ln |h| + O(h^2), \quad (20)$$

where $\lambda = \frac{\sqrt{2}}{2}$ denotes an eigenvalue of $O$. By (19), the homoclinic orbit $L_1$ has the expression:

$$L_1: \quad y_\pm = \pm \frac{1}{12} x \sqrt{72 - 198x^2 + 156x^4 - 36x^6}, \quad 0 \le x \le x_0, \quad x_0 = \frac{1}{2} \sqrt{6 - 2\sqrt{3}}.$$ 

Then by Lemma 2.1 and (3) we have

$$c_{01} = \oint_{L_1} f(x, \delta) y \, dx = 2 \int_0^{x_0} (a_0 + a_1 x^2 + a_2 x^4 + a_3 x^6 + a_4 x^8) y_+ \, dx = 2 \sum_{j=0}^{4} a_j I_j \quad (21)$$

with

$$I_0 = 0.1269637110, \quad I_1 = 0.0297371178, \quad I_2 = 0.01027524854, \quad I_3 = 0.00420844189, \quad I_4 = 0.001897364641,$$

$$c_1 = -\sqrt{2} a_0, \quad (22)$$
\[ c_{21} = \oint_{L_1} f(x, \delta) dt |_{c_1=0} = 2 \int_0^{x_0} (a_1 x^2 + a_2 x^4 + a_3 x^6 + a_4 x^8) dt \]
\[ = 2 \int_0^{x_0} (a_1 x^2 + a_2 x^4 + a_3 x^6 + a_4 x^8) \frac{1}{y_+} dx = 2 \sum_{j=1}^{4} a_j J_j \]  
(23)

with

\[ J_1 = 1.225023372, J_2 = 0.5503773932, J_3 = 0.2869119956, J_4 = 0.1584015511, \]
\[ c_3(O, \delta) = \sqrt{2} a_1. \]  
(24)

Next, we discuss the property of \( M_1(h, \delta) \) for \( 0 < h + \frac{169}{6144} \ll 1 \). We move the center \( C_1(\frac{1}{2}, 0) \) into the origin by letting \( x = x_1 + \frac{1}{2}, y = \sqrt{\frac{32}{42}} y_1 \) and \( t = \frac{8}{\sqrt{42}} \tau \), system (19) has the following Hamiltonian function

\[ \tilde{H}(x_1, y_1) = -\frac{169}{6032} + \frac{1}{2} (x_1^2 + y_1^2) + \frac{23}{63} x_1^3 - \frac{17}{14} x_1^4 - \frac{8}{7} x_1^5 + \frac{32}{63} x_1^6 + \frac{16}{21} x_1^7 + \frac{4}{21} x_1^8 \]
and Melnikov function

\[ \tilde{M}_1(h, \delta) = \oint_{\tilde{H}(x_1, y_1) = h} \frac{8}{\sqrt{42}} f(x_1 + \frac{1}{2}, \delta) y_1 dx_1. \]

Then by Lemma 2.3, we have

\[ \tilde{M}_1 = \tilde{b}_{01}(h + \frac{169}{4032}) + \tilde{b}_{11}(h + \frac{169}{4032})^2 + O((h + \frac{169}{4032})^3). \]

Note that

\[ M_1(h, \delta) = \frac{21}{32} \tilde{M}_1(h + \frac{1}{2}, \delta), \]

we have

\[ b_{01}(\delta) = \tilde{b}_{01}(\delta) = \frac{2\sqrt{7\pi}}{21} (4a_0 + a_1 + \frac{1}{4} a_2 + \frac{1}{16} a_3 + \frac{1}{64} a_4), \]
\[ b_{11}(\delta) = \frac{32}{21} \tilde{b}_{11}(\delta) = \frac{32}{21} \sum_{i=0}^{4} a_i N_i, \]  
(25)

where

\[ N_0 = \frac{5290}{27783}, N_1 = \frac{1637}{55566}, N_2 = \frac{42965}{222264}, N_3 = \frac{126629}{889056}, N_4 = \frac{252629}{3556224}. \]

Then by Lemma 2.4, for \( 0 < \frac{1}{48} - h \ll 1 \) and by Lemma 2.5, for \( 0 < h + \frac{1}{12} \ll 1 \), we have

\[ M(h, \delta) = \tilde{c}_0 + O((h - \frac{1}{48}) \ln |h - \frac{1}{48}|), \]
\[ \frac{M}{M_1}(h, \delta) = \tilde{c}_{01} + O((h - \frac{1}{48}) \ln |h - \frac{1}{48}|), \]
\[ \frac{M_1}{M_1}(h, \delta) = \tilde{b}_{01}(h + \frac{1}{12}) + O((h + \frac{1}{12})^3). \]
Similarly the computation of (21) and (25), we have
\[
\hat{c}_0 = 2 \sum_{j=0}^{4} a_j K_j, \quad \tau_{01} = 2 \sum_{i=0}^{4} a_i I_i, \quad \bar{b}_{01} = \frac{2\pi}{\sqrt{t}} (a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4),
\]
where
\[
K_0 = 0.467507027, K_1 = 0.1259831529, K_2 = 0.0596594218, K_3 = 0.034771258,
K_4 = 0.022742298, I_0 = 0.1584238578, I_1 = 0.285211086, I_2 = 0.5304582796,
I_3 = 1.014945240, I_4 = 1.989723792.
\]

In (21)-(23) and (25), letting \( c_{01} = c_1 = c_{21} = b_{01} = 0 \), we can obtain
\[
a_0 = 0, a_1 = -0.1660724681a_4, a_2 = 1.080920174a_4, a_3 = -1.916521206a_4.
\]
Then substituting (27) into (24)-(26), we have
\[
c_3 = -0.2348619367a_4, \quad b_{11} = 0.04321917751a_4, \quad \hat{c}_0 = -0.00033281587a_4,
\tau_{01} = 11.06515966a_4, \quad \bar{b}_{01} = 0.5705770639a_4.
\]
By (21)-(23) and (25), it is obvious that
\[
\det \frac{\partial (c_{01}, c_1, c_{21}, b_{01})}{\partial (a_0, a_1, a_2, a_3)} = \begin{vmatrix}
2I_0 & 2I_1 & 2I_2 & 2I_3 \\
-\sqrt{2} & 0 & 0 & 0 \\
0 & 2J_1 & 2J_2 & 2J_3 \\
8\sqrt{12\pi} & 2\sqrt{12\pi} & 2\sqrt{12\pi} & 2\sqrt{12\pi}
\end{vmatrix}
\approx 0.0002625666033 \neq 0.
\]
Then by (27), we have \( a_0 \equiv a_0^*, a_1 \equiv a_1^*, a_2 \equiv a_2^* \) and \( a_3 \equiv a_3^* \). Let us take \( \delta_0 = (a_0^*, a_1^*, a_2^*, a_3^*) \). Thus, with \( a_4 \neq 0 \), we have \( c_3(\delta_0)b_{11}(\delta_0) < 0 \), which indicates that there exist no root \( h_1^* \in (-\frac{169}{6144}, 0) \). And we also have \( c_3(\delta_0)\hat{c}_0(\delta_0) > 0 \), which indicates that there exists a root \( h^* \in (0, \frac{1}{48}) \) such that \( M(h^*, \delta_0) = 0 \) under (28). Moreover, we also have \( \tau_{01}(\delta_0)\bar{b}_{01}(\delta_0) > 0 \), which indicates that there exist no roots \( h^* \in (-\frac{1}{12}, \frac{1}{36}) \) such that \( M(h^*, \delta_0) = 0 \) under (28). So by Theorem 2.1 and its proof, we get 10 limit cycles.

Then we can get one more limit cycle for \( h > \frac{1}{48} \). Let \( G(x) = \int_0^x g(x)dx \), \( F(x, \delta_0) = \int_0^x f(x, \delta_0)dx \) hold and \( x^* > \frac{\sqrt{M}}{2} \) satisfy \( G(x^*(h)) = h \) for \( h > \frac{1}{48} \).

Let \( \bar{M}(h, \delta) = \int_{\Gamma^*} f(x, \delta)dydx \). By the symmetry and using (3), we have
\[
\bar{M}(h, \delta) = \oint_{\Gamma^*} f(x, \delta)dydx = -\oint_{\Gamma^*} F(x, \delta)dy = -\oint_{\Gamma^*} \frac{F(x, \delta)}{y}dG(x),
\]
then
\[
\bar{M}(h, \delta_0) = -4 \int_0^{x^*(h)} \frac{F(x, \delta_0)}{\sqrt{2(h + G(x))}}dG(x), \quad h > \frac{1}{48}.
\]
Suppose $a_4 > 0$, let $x_0 > \sqrt{10}/2$ be such that $F(x, \delta) > a_4$ for $x > x_0$. Then we can write

$$M(h, \delta_0) = m_1(h) + m_2(h),$$

where

$$m_1(h) = -4\int^{x_0}_0 \frac{F(x, \delta_0)}{\sqrt{2(h + G(x))}} dG(x),$$

$$m_2(h) = -4\int^{x}_0 \frac{F(x, \delta_0)}{\sqrt{2(h + G(x))}} dG(x),$$

From the mean value theorem for integral, it is obvious that $m_1(h) \to 0$ as $h \to \infty$. And we have $m_2(h) < -4\int^{h}_{G(x_0)} \frac{a_4}{\sqrt{2(h + z)}} dz \to -\infty$ as $h \to \infty$. It follows that $M(h, \delta_0) \to -\infty$ as $h \to \infty$ for $a_4 > 0$. On the other hand, under (28), we have $M(h, \delta_0) = \tilde{c}_0(\delta_0) + 2\tilde{\sigma}_0(\delta_0) = 1.140821312a_4 > 0$ for $0 < h - \frac{1}{48} \ll 1$. Then we can get one more limit cycle for $h > \frac{1}{48}$. Therefore, the system (2) can have 11 limit cycles for $(\varepsilon, a_0, a_1, a_2, a_3, a_4)$ near $(0, 0, -0.1660724681a_4, 1.080920174a_4, -1.916521206a_4, a_4)$.

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**References**


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