Economic Stability of Non-Smooth Periodic Orbits in the Plane: Omega Limit Sets Part II

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Abstract

In this paper we enhance the stability theory of non-smooth economic periodic orbits provided by theorem 3 in Stiefenhofer and Giesl ([14], p.516). This requires to construct a Poincaré-like map $\mathcal{P}$ and show its contraction. We then show existence and stability of a non-smooth period orbit $\Omega$. The results of this paper complete the theory of switching economic regimes.

Mathematics Subject Classification: 34D20

Keywords: Omega Limit Sets, Filippov Differential Equations, Exponential Asymptotic Stability

1 Introduction

The analysis of dynamic economic behavior lies at the heart of modern economics. Economists frequently find their models to be systems of differential equations. Recent theoretical developments have shown that such systems might depend on differential equations with discontinuous right hand side.
Such equations demand a new solution concept because the standard theory of differential equations is not sufficiently general to guarantee the basic requirements desired by economists such as existence and uniqueness of solutions. The usual existence and uniqueness theorem of a solution path requires the continuity of the right-hand side as well as the Lipschitz condition. This problem was first recognized in the 60’s by Filippov [2] who also provides a definite answer to this problem. Mallivaud’s planning model with non-negative constraints is an example of an economic model defined by a system of differential equations with discontinuous right-hand side [9]. Satisfactory solution concepts for this model were known at the time but little progress was made on uniqueness and stability of such models. Itô [7] suggests the construction of a Filippov solution to the class of dynamic disequilibrium macroeconomic models, showing that a unique solution is determined despite discontinuities at the boundaries of different economic regimes. His theory, however, lacks the conditions for determining stability and the basin of attraction of stable solutions. In a series of papers Stiefenhofer and Giesl address the problem of economic stability of models defined by Filippov equations [11][14][13][12] and [3]. Stiefenhofer and Giesl [11] introduce the basic theory of existence, uniqueness, exponentially asymptotically stability of non-smooth periodic orbits, and provide a formula for the basin of attraction. The advantage of this theory is that it does not require the explicit calculation of solutions of the system in order to determine all these properties. This is a desirable convenience, since for most economic models it is not easy, or often not even possible, to find explicit solutions. An explicit example showing how to determine stability and the basin of attraction for a Filippov differential equation without solving the differential equation is provided in [12]. The same example is then replicated using Poincaré theory in [13] in order to compare the merits of the theory.

In this paper, we complete the stability proof. It remains to complete the proof of theorem 2 which requires to (i) characterize solutions near a point $x \in K$, where $K$ is a compact set, (ii) define a correction mapping $\pi$, (iii) define a Poincaré-like map $\mathcal{P}$, and finally (iv) show existence and stability of the period orbit $\Omega$. Part (i-ii) are shown in a companion paper of this journal. Section three shows (iii), section four shows (iv). A conclusion and an outlook for future work is provided at the end of this paper.

Let’s consider a nonsmooth dynamical system defined by an autonomous ordinary differential equation

$$\dot{x} = f(x)$$

where $f$ is a discontinuous function at $x_2 = 0$ and $x \in \mathbb{R}^2$ (Stiefenhofer and Giesl 2019a). The discontinuity of $f \in C^1(\mathbb{R}^2 \setminus (\mathbb{R} \times \{0\}), \mathbb{R}^2)$ implies that the phase space $X = \mathbb{R}^2$ is divided into subspaces $X = X^+ \cup X^0 \cup X^-$, where $X^+ = \{x \in \mathbb{R}^2 : x_2 > 0\}$, $X^- = \{x \in \mathbb{R}^2 : x_2 < 0\}$, and $X^0 = \{x \in \mathbb{R}^2 : x_2 = 0\}$. By defining $f := f^\pm$ where $f(x) = f^+(x)$ if $x \in X^+$, and $f(x) = f^-(x)$ if $x \in X^-$,
we have
\[ \dot{x} = f(x) = \begin{cases} f^+(x) & \text{if } x \in \mathbb{R}^+ \\ f^-(x) & \text{if } x \in \mathbb{R}^- \end{cases} \] (1)

An initial value condition for this system at time \( t = 0 \) is given by \( x(0) = x_0 \in \mathbb{R} \). We restrict ourselves to a set of assumptions which according to a sequence of results by Filippov (1988) guarantees global existence, uniqueness, and continuous dependence on the initial condition of solutions of the differential equation (1). We assume \( f \in C^1(\mathbb{R}^2 \setminus (\mathbb{R} \times \{0\}), \mathbb{R}) \). Each function \( f^\pm(x) \) with \( x \in \mathbb{R}^+ \) or \( x \in \mathbb{R}^- \) can be extended to a continuous function up to \( x \in \mathbb{R}^0 \). Each function \( D^\pm f(x) \) with \( x \in \mathbb{R}^+ \) or \( x \in \mathbb{R}^- \) can be extended to a continuous function up to \( x \in \mathbb{R}^0 \). \( f^+(x_1,0) - f^-(x_1,0) \) is a \( C^1 \) function with respect to \( x_1 \). For all \( (x_1,0) \in \mathbb{R}^0 \) it holds that \( f^2_2(x_1,0) \cdot f^2_2(x_1,0) > 0 \). The assumption \( f^2_2(x_1,0) \cdot f^2_2(x_1,0) > 0 \) for all \( (x_1,0) \in \mathbb{R}^0 \) states that \( f^2_2, f^2_2 < 0 \), or \( f^2_2, f^2_2 > 0 \). This assumption excludes all sliding phenomena on the \( \mathbb{R}^0 \) manifold and shall be relaxed in future work. It implies a discontinuity in \(+/-\) direction if both \( f^2_2, f^2_2 < 0 \) or a discontinuity in \(-/+\) direction if both \( f^2_2, f^2_2 > 0 \). Let the flow of the system given by (1) be defined by \( S_t(x_0) := (x_1(t), x_2(t)) \in \mathbb{R}, \) where \( (x_1(t), x_2(t)) \in \mathbb{R} \) is its solution with initial value \( ((x_1(0), x_2(0)) = x_0 \). Hence the flow \( S_t x_0 \) maps the initial point \( x_0 \) at time \( t = 0 \) to a point \( x(t) \) at time \( t \geq 0 \). An adjacent trajectory is defined by \( S_{\theta}(x_0 + \eta) := (y_1(\theta), y_2(\theta)) \in \mathbb{R}, \) where \( (y_1(\theta), y_2(\theta)) \in \mathbb{R} \), and \( ||\eta|| > 0 \) and time \( \theta \geq 0 \). Let \( K \subseteq \mathbb{R} \) be positively invariant if \( S_t x_0 \in K \) for all \( t \geq 0 \) and all \( x_0 \in K \). A periodic orbit \( \Omega \) of the system (1) is a set defined by \( \Omega := \{ S_t(x_0) : t \in [0,T] \}, \) such that \( S_T(x_0) = x_0 \) \( \subset \mathbb{R}, \) with minimal period \( T > 0 \). Let \( K \subseteq \mathbb{R} \) and \( K \neq \emptyset \) be a compact, connected and positively invariant set which contains no equilibria. Moreover, set \( K^+ := K \cap \{ x \in \mathbb{R}^2 : x_2 > 0 \} \) and \( K^- := K \cap \{ x \in \mathbb{R}^2 : x_2 < 0 \} \). We define a neighborhood \( \mathcal{A}(\Omega) \) of \( \Omega \) consisting of a set of points \( x_0 \in \mathbb{R} \) such that the distance between \( S_t(x_0) \) and \( \Omega \) vanishes as \( t \to \infty \). The basin of attraction \( \mathcal{A}(\Omega) \) of an exponentially asymptotically stable orbit \( \Omega \) is the set defined by \( \mathcal{A}(\Omega) := \{ x_0 \in \mathbb{R} : \text{dist}(S_t x_0, \Omega) \xrightarrow{t \to \infty} 0 \} \).

**Theorem 1.** Let assumptions above hold, and let \( \emptyset \neq K \subset \mathbb{R}^2 \) be a compact, connected and positively invariant set with \( f^\pm(x) \neq 0 \) for all \( x \in K^\pm \). Moreover, assume that \( W^\pm : \mathbb{R}^2 \to \mathbb{R} \) are continuous functions and let the orbital derivatives \( (W^\pm)' \) exist and be continuous functions in \( \mathbb{R}^2 \) and continuously extendable up to \( \mathbb{R}^0 \). We set \( K^0 := \{ x \in K : x_2 = 0 \} \). Let the following conditions hold:

1. \( L_{W^\pm}(x) := \max_{||v^\pm|| = e^{-W^\pm(x)}, v^\pm \perp f^\pm(x)} L_{W^\pm}(x, v^\pm) \leq -\nu < 0 \)
2. \( L_{W^\pm}(x, v^\pm) := e^{2W^\pm(x)} \left( (v^\pm)^T [Df^\pm(x)] v^\pm + (\nabla W^\pm(x), f^\pm(x)) ||v^\pm||^2 \right) \)
   for all \( x \in K^\pm \).
2. \[ \frac{f^+_2(x)}{f^-_2(x)} \sqrt{\left( f^+_2(x) \right)^2 + \left( f^-_2(x) \right)^2} e^{W^+(x) - W^-(x)} < 1 \]

for all \( x \in K^0 \) with \( f^+_2(x) < 0 \), \( f^-_2(x) < 0 \).

Then there is one and only one periodic orbit \( \Omega \subset K \). Moreover, \( \Omega \) is exponentially asymptotically stable with exponent \(-\nu < 0\) and for its basin of attraction the inclusion \( K \subset A(\Omega) \) holds.

The next result generalizes theorem 3 in Stiefenhofer and Giesl ([14], p.516) to a full neighborhood of \( x \). It provides the conditions for a point \( x \) to belong to an exponentially asymptotically stable periodic orbit. The theorem below is also stated in a companion paper in this journal called ”Economic Stability of Non-Smooth Periodic Orbits in the Plane: Omega Limit Sets Part I”, where we characterize solutions near a point \( x \in K \), define a correction mapping \( \pi \) and show its contraction property. It remains to define a Poincaré-like map \( \mathcal{P} \) and study its stability property and to complete the proof of existence and uniqueness of the period orbit \( \Omega \). We define \( F(t) \) as a smooth function in a direction which is not perpendicular to \( f(S_t x) \).

\[ F(t) := \frac{f(S_t x)}{\| f(S_t x) \|}. \]

**Theorem 2.** Let assumptions of theorem 1 hold. Let \( x \in K^\pm \) satisfy \( x \in \omega(x) \). Assume there is a continuous map \( F : \mathbb{R}^+_0 \rightarrow \mathbb{R}^2 \) with \( \| F(t) \| = 1 \) and \( \langle F(t), f(S_t x) \rangle > 0 \) for all \( t \geq 0 \). Furthermore, assume that there are constants \( \delta, \nu > 0 \) and \( C \geq 1 \) such that for all \( \eta \in \mathbb{R}^2 \) with \( \eta \perp F(0) \) and \( \| \eta \| \leq \delta \) there is a piecewise multi valued mapping \( T^{x+\eta} : \mathbb{R}^+_0 \rightarrow \mathbb{R}^+_0 \) such that \( T^{x+\eta}(t_i) \) depends continuously on \( \eta \) and satisfies

\[ \frac{2}{3} \cdot t \leq T(t_i) \leq \frac{4}{3} \cdot t \forall t_i \in \mathcal{G}^\pm. \] (2)

\[ \frac{1}{3} \cdot t \leq T(t_i) \leq \frac{2}{3} \cdot t \forall t_i \in \mathcal{I}^\pm \] (3)

\[ \frac{2}{3} \left( \frac{1}{2 + \frac{c_1}{c_2}} \right) \cdot t \leq T(t_i) \leq \frac{4}{3} \left( \frac{1}{2 + \frac{c_1}{c_2}} \right) \cdot t \forall t_i \in \mathcal{J}^\pm \] (4)

\[ \langle S_{T^{x+\eta}(t_i)}(x + \eta) - S_t x, F(t) \rangle = 0 \] (5)

and

\[ \| S_{T^{x+\eta}(t_i)}(x + \eta) - S_t x \| \leq C e^{-\mu t} \| \eta \| \] (6)

for all \( t \geq 0 \). Then \( x \) is a point of an exponentially asymptotically stable period orbit \( \Omega \).
2 A Poincaré-like map $\mathcal{P}$

In the remaining two steps of the proof of theorem 2 we need to show that a period orbit $\Omega$ exists and that it is exponentially asymptotic stable. Existence of a periodic orbit $\Omega$ requires to define a Poincaré-like map $\mathcal{P}$ which maps open sets into themselves, $\mathcal{P} : U_0 \rightarrow U_0$, where the compact sets $U_0$ are subsets of the hyperplane $H$. Once $\mathcal{P}$ is defined, we show that it is contracting, which shows existence of $\Omega$. Now we define a Poincaré-like map $\mathcal{P}$ and show that the diameter of the sets defined by the return map decrease. Let’s consider a point $x \in \omega(x)$. Set $\delta := \frac{\delta_2}{2(k_0+1)}$. Thus there is a minimal time period $T^* \geq 3t_0 + 2\ln(2C(k_0+1))$ so that $S_{T^*}x \in B_{3\delta_2}(x)$. By the correction mapping lemma there is a $T^* \in [T^* - t_0, T^* + t_0]$ such that the point $x_1 := S_{T^*}x = \pi(S_{T^*}x)$ that by condition (25) of the correction mapping lemma it holds that $x_1 \in H$ and $\| x_1 - x \| \leq \frac{\delta_3}{2}$. Observe that $t_1 \geq 2(t_0 + 2\ln(2C(k_0+1)))$. Now, set $\delta_3 := 2\| x_1 - x \| \leq \delta_2 \leq \delta_1 \leq \delta$. By condition (6) of theorem 2 a point $q$ in the compact set $U_0$ with $U_0 := H \cap B_{\delta_3}(x)$ will move to a point $q_1$ by $S_{T^*(T_1)}q$ which satisfies $\| q_1 - x_1 \| \leq Ce^{-\nu T_1} \| q - x \| \leq \frac{\delta_3}{2(k_0+1)} < \delta_2$. The point $\pi(q_1)$ satisfies $\pi(q_1) \in H$ by the correction mapping lemma. Moreover, by condition (25) we have that $\| \pi(q_1) - x_1 \| \leq (k_0 + 1) \| q_1 - x_1 \| \leq \frac{\delta_3}{2}$. Hence, we are now in a position to define the Poincaré-like map by

$$\mathcal{P} : \begin{cases} U_0 &\rightarrow & U_0 \\ q &\mapsto & \pi(S_{T^*(T_1)}q). \end{cases}$$

(7)

**Remark 1.** The Poincaré-like map $\mathcal{P}$ is a return map. However, it is not necessarily the first return map to the hyperplane $H$.

It remains to prove that $\mathcal{P}(U_0) \subset U_0$. By calculation

$$\| \mathcal{P}(x) - q \| \leq \| \mathcal{P}(x) - x_1 \| + \| x_1 - x \| \leq \frac{\delta_3}{2} + \frac{\delta_3}{2}.$$ 

Note that $\mathcal{P}$ is continuous. Continuity of $\mathcal{P}$ directly follows from continuity of $\pi, S_T$ and $T^*_x$. By definition of the projection map $\pi$ we have $\mathcal{P}(q) = S_{\tau(q)}q$ for a continuous map $\tau$ with $\tau(q) \geq \frac{T_1}{2} - t_0 \geq \frac{\ln(2C(k_0+1))}{\nu} > 0$ for all $q \in U_0$. We now show that the diameter of the sets defined by the return map decrease. We first define the compact sets $U_k$ by $\mathcal{P}$.

$$U_k := \mathcal{P}^k(U_0) \text{ for all } k \in \mathbb{N}. \quad (8)$$

---

2Correction map lemma in companion paper of this journal.
Lemma 1. Let the compact sets $U_k \subset H$ be defined for all $k \in \mathbb{N}$ by the return map $\mathcal{P}^k$ in definition (8). Moreover, define for all $k \in \mathbb{N}$ the points $x_k$ by

$$x_k := \mathcal{P}^k(U_0) \subset U_k.$$  

Then the following properties hold for all $k \in \mathbb{N}$

$$U_k \subset U_{k-1} \quad (9)$$

$$\operatorname{diam} U_k \leq \frac{\delta_3}{2^{k-1}}. \quad (10)$$

Proof. We first show statement (9). For $k = 1$ we have

$$U_1 = \mathcal{P}(U_0) \subset U_0.$$  

For $k \geq 2$ we have

$$\mathcal{P}^k(U_0) = \mathcal{P}^{k-1}\mathcal{P}(U_0) \subset \mathcal{P}^{k-1}(U_0).$$  

Since $\mathcal{P}$ is continuous, it follows that since $U_k$ are images of $U_{k-1}$ under $\mathcal{P}$ that $U_k$ are compact sets by induction. We now show statement (10). We remarked that $\mathcal{P}$ does not necessarily have to be a first return map. We now take this into consideration and show that we reach the same points no matter whether we apply $\pi$ after each return or only once at the end. We provide a characterization of $U_k$. The points $p_k$ belong to the forward trajectory through the point $x$. Hence we define $T_k$ such that

$$x_k = \mathcal{P}^{T_k}(x) = S_T x_k \quad \text{for all } k \in \mathbb{N}.$$  

Also $x_0 := p$. Moreover, we know that $T_k \geq \frac{\ln|2C(k_0 + 1)|}{\nu} > 0$. We now set for all $k \in \mathbb{N}$

$$V_k := \left\{ S_T q : q \in U_0 \right\}$$

and

$$q_k := S_T q$$

and claim that $\mathcal{P}^k(q) = \pi(q_k)$ holds for all $q \in U_0$ and all $k \in \mathbb{N}$. In particular we have the characterization $U_k = \pi(V_k)$. Now, pick any $k \in \mathbb{N}$. Then we already know that $\pi(x_k) = x_k = \mathcal{P}^k(x)$ and $x_k \in U_k \cap \pi(V_k)$. This is the claim for $q = x$. Moreover, we have that $U_k, \pi(V_k) \subset H$, and all points of both $U_k$ and $\pi(V_k)$ can be written as $S_{\tau_i(q)}q$ with $q \in U_0$, where $\tau_i$ are continuous functions. This is used to prove that $\mathcal{P}^k(q) = \pi(q_k)$ holds for all $q \in U_0$ and all $k \in \mathbb{N}$.

Let’s consider

$$Q(\tau, q) = \langle x_k - p, F(0) \rangle \quad \text{for all } q \in U_0.$$  

By equations (11) and (15) of the companion paper we have that $Q(\sum_{i=1}^k T_{i,x}) = \langle x_k - x, F(0) \rangle = 0$ and $\partial_r Q(\tau, q) = \langle f(S_r q), F(0) \rangle \geq \frac{\alpha}{2} > 0$ for all $S_r q \in B_{\delta_1}$.  

In particular \(\partial_x Q(\sum_{i=1}^k T_i x) \neq 0\). Hence by the implicit function theorem there is a unique continuous function \(\tau(q)\) near \(x\) such that \(Q(\tau(q), q) = 0\). This is equivalent to \(S_{\tau(q)} q \in H\). Since both, \(\tau_1\) and \(\tau_2\) are such functions, they have to coincide near the point \(x\). Hence, by a prolongation we obtain \(\tau_1 = \tau_2\) on \(U_0 \subset B_{\delta_1(x)}\). Thus for \(q_k = S_{T_2^{(\sum_{i=1}^k T_i)}} q\) we have that \(\mathcal{P}^k(q) = \pi(q_k)\) as we wanted to show. We now want to prove statement (10). Hence we consider a point \(q \in U_0\). Thus for \(q_k = S_{T_2^{(\sum_{i=1}^k T_i)}} q\) we have that \(\mathcal{P}^k(q) = \pi(q_k)\) as we have shown above. From condition (6) of theorem 2 we obtain that

\[
\| q_k - x_k \| \leq C e^{-\nu \sum_{i=1}^k} \| q - x \| \leq (k_0 + 1) C \frac{\delta_3}{[2C(k_0 + 1)]^k}.
\]

Note that both \(C\) and \((k_0 + 1) \geq 1\). By condition (25) of the correction mapping lemma this yields

\[
\| \mathcal{P}^k(q) - x_k \| = \| \pi(q_k) - x_k \| \leq \frac{\delta_3}{2^k(k_0 + 1)} \text{ for all } k \in \mathbb{N}.
\]

As

\[
diam U_k = \max_{q', q'' \in U_0} \| \mathcal{P}^k(q') - \mathcal{P}^k(q'') \| 
\leq \max_{q' \in U_0} \| \mathcal{P}^k(q') - x_k \| + \max_{q'' \in U_0} \| x_k - \mathcal{P}^k(q'') \| 
\leq \frac{2 \delta_3}{2^k}.
\]

This completes the proof. \(\square\)

### 3 Existence of period orbit \(\Omega\) and its stability

By construction of a sequence of compact sets \(U_k\) with decreasing diameter, we have shown in lemma 1 that for all \(k \in \mathbb{N}_0\) there is one and only one point \(\tilde{x}\) which lies in all sets. We know that \(\tilde{x}\) is a fixed point of \(\mathcal{P}\) since \(\mathcal{P}(\tilde{x})\) lies in all compact sets \(U_k\) as well. Given a fixed point, there is a shortest time \(T > 0\) so that \(S_T \tilde{x} = \tilde{x}\) and thus \(\tilde{x}\) is a point of the periodic orbit \(\Omega\). Since \(\tilde{x} = x_\eta\) with \(\eta \perp F(0)\) and \(\| \eta \| \leq \delta\) we know by condition (6) of theorem 2 that both points \(x\) and \(\tilde{x}\) have the same \(\omega\)-limit sets. Thus \(x \in \omega(x) = \omega(\tilde{x}) = \Omega\) and \(x\) is a point of the periodic orbit \(\Omega\). Now we prove the following result.

**Proposition 1.** Define \(\delta_m := \min_{q \in H, \| q \| = \delta, t \in [0, T]} \| S_{T_k} q - S_t p \| > 0\). Then there are constants \(\delta'_2, t'_0 \geq 0\) such that for each \(q \in \mathbb{R}^2\) with \(\text{dist}(q, \Omega) \leq \delta'_2\) there is a \(t\) with \(\| t' \| \leq t'_0\) such that \(S_t q = S_{\theta} x + \eta\) with \(\theta \in [0, T], \| \eta \| \leq \delta_m\) and \(\langle \eta, F(0) \rangle = 0\).
Proof. Similar to the early part of the proof of proposition 2 we define new coordinates. The main difference here is that we consider coordinates for all points $S_\theta p$ with $\theta \in [0, T]$. Hence by the coordinate system introduced in the companion paper we obtain

\[
x_\theta(q) := S_\theta q - S_\theta x - y_\theta(q) F(\theta) \in F(\theta)^\perp
\]

\[
y_\theta(q) := \langle S_\theta q - S_\theta x, F(\theta) \rangle \in \mathbb{R}
\]

\[
\lambda_\theta(q) := \langle f(S_\theta q) - f(S_\theta x), F(\theta) \rangle \in \mathbb{R}
\]

\[
u_\theta(q) := f(S_\theta q) - f(S_\theta x) - \lambda_\theta(q) F(\theta) \in F(\theta)^\perp
\]

Since $[0, T]$ is a compact set the constants

\[
f_M := \max_{\theta \in [0, T]} \| f(S_\theta x) \|
\]

\[
\alpha_m := \min_{\theta \in [0, T]} \langle F(\theta), f(S_\theta x) \rangle > 0
\]

\[
\alpha_M := \max_{\theta \in [0, T]} \langle F(\theta), f(S_\theta x) \rangle > 0
\]

exist. We can choose $\delta_1' > 0$ such that for all $q$

\[
|y_\theta(q)| \leq \frac{1}{2} \alpha_m
\]

\[
\| u_\theta(q) \| \leq f_M
\]

there is a $\theta \in [0, T]$ with $\| S_\theta q - S_\theta x \| \leq \delta_1'$. We define

\[
k_0' := \frac{4}{\alpha_m} f_M
\]

\[
\epsilon_0' := 2f_M + \frac{1}{2} \alpha_m
\]

\[
\delta_2' := \min \left( \frac{1}{2} \frac{\delta_1'}{\alpha_m}, \frac{\delta_m}{k_0' + 1} \right)
\]

\[
t_0' := \frac{2\delta_2'}{\alpha_m}.
\]

Proposition 2 (Generalized lemma). Let $S_t q \in B_{\delta_1'}(S_\theta x)$ hold for all $t \in [0, \tilde{\tau}]$ with $\tilde{\tau} > 0$. Then for all $t \in [0, \tilde{\tau}]$ and all $\tau_1 \leq \tau_2 \leq \tilde{\tau}$ the following bounds hold:

\[
\frac{1}{2} \alpha_m \leq \frac{d}{dt} y_\theta(S_t q) \leq \frac{1}{2} \alpha_m + \alpha_M
\]

\[
\frac{1}{2} \alpha_m (\tau_2 - \tau_1) \leq y_\theta(S_{\tau_2} q) - y_\theta(S_{\tau_1} q) \leq \left( \frac{1}{2} \alpha_m + \alpha_M \right) (\tau_2 - \tau_1)
\]

and

\[
\| x_\theta(S_{\tau_2} q) - x_\theta(S_{\tau_1} q) \| \leq k_0' (y_\theta(S_{\tau_2} q) - y_\theta(S_{\tau_1} q))
\]
Proof. We first show inequality (17). By equation (11) we have
\[ S_t q = S_\theta x + y_\theta(S_t q)F(\theta) + x_\theta(S_t q). \]
Hence by differentiation we conclude that
\[
\frac{d}{dt} S_t q = \frac{d}{dt} y_\theta(S_t q) + \frac{d}{dt} x_\theta(S_t q).
\] (20)
By equation (11) we have \( x(S_t q) \perp F(\theta) \) for all \( t \in [0, \tilde{\tau}] \), we conclude that \( \frac{d}{dt} x_\theta(S_t q) \perp F(\theta) \) holds too. Moreover from equation (11) we obtain
\[ S_\theta q = S_\theta x + y_\theta(q)F(\theta) + x_\theta(q). \] (21)
By (21) we have
\[
f(S_t q) = f(S_\theta x) + \lambda_\theta(S_t q)F(\theta) + u_\theta(S_t q)
\]
Using (13) and (14) yields
\[
f(S_t q) = f(S_\theta x) + \langle f(S_t q) - f(S_\theta p), F(\theta) \rangle + f(S_t q) - f(S_\theta x) - \lambda_\theta(S_t q)F(\theta)
\]
which by little algebraic manipulation and using \( \frac{d}{dt} y_\theta(S_t q) = f(S_t q) \) yields
\[
0 = \frac{d}{dt} y_\theta(S_t q)F(\theta) - f(S_\theta x) + f(x)F(\theta) - \lambda_\theta(S_t q).
\]
Using \( \alpha_0 := \langle F(0), f(x) \rangle > 0 \) and rearranging yields
\[
\frac{d}{dt} y_\theta(S_t q)F(\theta) = \alpha_0 + \lambda_\theta(S_t q). \]
(22)
Equation (22) with bound (15) yields condition (17) as required. Since we consider the time interval \( t \in [0, \tilde{\tau}] \) with \( 0 \leq \tau_1 \leq \tau_2 \leq \tilde{\tau} \) condition (18) follows from
\[
\int_{\tau_1}^{\tau_2} \frac{d}{dt} y_\theta(S_t q) dt = y_\theta(S_{\tau_2} q) - y_\theta(S_{\tau_1} q)
\]
and bounds of condition (17). Now, we multiply (22) by \( \frac{d}{dt} x_\theta(S_t q) \) and with \( \frac{d}{dt} x_\theta(S_t q) \perp F(\theta) \) we obtain
\[
\| \frac{d}{dt} x_\theta(S_t q) \|^2 = \langle f(S_t q), \frac{d}{dt} x_\theta(S_t q) \rangle
\]
which by (21) becomes
\[
\langle f(S_\theta x) + u_\theta(S_t q), \frac{d}{dt}x_\theta(S_t q) \rangle
\]

\[
\| \frac{d}{dt}x_\theta(S_t q) \| \leq \langle \| f(S_\theta x) \| + \| u_\theta(S_t q) \| \rangle. \tag{23}
\]

Hence
\[
\| x_\theta(S_{\tau_2}) - x_\theta(S_{\tau_1}) \| = \int_{\tau_1}^{\tau_2} \| \frac{d}{dt}x_\theta(S_t q) \| dt \leq \int_{\tau_1}^{\tau_2} (\| f(x) \| + \| u_\theta(S_t q) \|) dt \text{ by (23)}
\]
\[
\leq 2(\tau_2 - \tau_1)\| f(x) \| \text{ by (15)}
\]
\[
\leq k_0(y(S_{\tau_2} q) - y(S_{\tau_1} q)) \text{ by (18)}.
\]

which proves condition (19). This concludes the prove of proposition 2. \(\square\)

**Lemma 2** (generalized lemma). Let
\[
\pi_\theta' : \left\{ \begin{array}{rl}
B_{\delta_2}(S_{\theta p}) & \rightarrow H' := S_{\theta x} + F(\theta)^{-1} \\
q & \mapsto \pi_\theta(q)
\end{array} \right.
\]  \(\tag{24}\)

be a continuous map defined by
\[
\pi_\theta'(q) = S_{t'(q)} q,
\]
where \(t'(q)\) is a continuous function satisfying \(|t'(q)| \leq \frac{2\alpha_0}{\alpha_0} =: t'_0\) for all \(q \in B_{\delta_2}(S_{\theta p})\). Then for \(x' H' \cap B_{\delta_2}(S_{\theta x})\) we have
\[
\| \pi_\theta'(q) - S_{\theta x^*} \| \leq (k'_0 + 1) \| S_{\theta q} - S_{\theta x'} \|. \tag{25}
\]

**Proof.** We only consider the case \(y_\theta(q) \leq 0\) we have that as long as \(S_{\tau} q \in B_{\delta_2}(S_{\theta x})\) with \(\tau \geq 0\) we have that by condition (18) of proposition 2 \(y_\theta(S_{\tau} q) \geq y_\theta(q) + \frac{\tau}{2}\alpha_0\). For \(\tilde{\tau} = -\frac{2}{\alpha_0} y_\theta(q) \geq 0\) we have \(y_\theta(S_{\tilde{\tau}} q) \geq 0\). Observe that \(|\tilde{\tau}| \leq \frac{2}{\alpha_0} \delta_2 = t'_0\). The existence of a time \(t' \in [0, \tilde{\tau}]\) such that \(y_\theta(S_{t'} q) = 0\) is satisfied is implied by the intermediate vale theorem. Uniqueness of \(t'\) follows from proposition 2 as \(y_\theta(S_{\tau} q)\) is monotonously increasing in \(\tau\). Now, by the implicit function theorem we can define the continuous function \(t'(q)\) by \(y_\theta(S_{t'} q) = 0\). Since \(y_\theta\) and \(S_t\) are continuous functions, it follows that \(t'\) is continuous. That proves that the projection mapping \(\pi_\theta\) is also continuous as required.
Next, we show by contradiction that $\tilde{\tau}$ is close enough to zero so that a trajectory $S_\tau q$ remains in $B_{\delta'}(S_\theta x)$ for all time $\tau \in [0, \tilde{\tau}]$. Assume the contrary. Let there be a $\tau_0 \in [0, \tilde{\tau}]$ with $\| S_{\tau_0} q - S_\theta p \| = \delta'_1$ and $\| S_\tau q - S_\theta p \| < \delta'_1$ for all $\tau \in [0, \tau_0]$. Then by
\[
f(S_\tau q) = f(S_\tau x) + \lambda_\theta(S_\tau q) F(\theta) + u_\theta(S_\tau q)
\]
and bounds (15) we have
\[
\| f(S_\tau q) \| \leq 2 \| f(S_\theta x) \| + \frac{1}{2} \alpha_0
\]
for all $q \in B_{\delta'_1}(S_\theta x)$. This yields
\[
\delta'_1 = \| S_{\tau_0} q - S_\theta x \| \\
\leq \| \int_0^{\tau_0} f(S_\tau q) d\tau \| + \| S_\tau q - S_\theta x \| \\
\leq |\tilde{\tau}| \epsilon_0 + \delta_2 \\
\leq \delta_2 \left( \frac{2\epsilon_0}{\alpha_0} + 1 \right) = \frac{\delta'_1}{2}\).
\]
Hence a contradiction. In the final step of the proof we need to show property 25. By (12) we have
\[
y_\theta(q) = \langle S_\tau q - S_\theta x, F(\theta) \rangle \\
= \langle S_\tau q - S_\theta x', F(\theta) \rangle + \langle S_\theta x' - S_\theta x, F(\theta) \rangle
\]
with $\langle S_\theta x' - S_\theta x, F(\theta) \rangle = 0$. Hence $|y_\theta(q)| \leq \| S_\tau q - S_\theta x \|$. Condition (19) of proposition 2 implies that
\[
\| x_\theta(\pi_\theta(q)) - x_\theta(q) \| \leq k'_0 \| t(q) \| \leq k'_0 \| S_\tau q - S_\theta x \|.
\]
We conclude the following
\[
\| \pi_\theta(q) - S_\theta x' \| = \| x_\theta(\pi_\theta(q)) - x_\theta(S_\theta x') \| \\
\leq \| x_\theta(\pi_\theta(q)) - x_\theta(q) \| + \| x_\theta(q) - x_\theta(S_\theta x') \| \\
\leq (k'_0 + 1) \| S_\tau q - S_\theta x \|.
\]
This concludes the proof of lemma 2.

Thus by above lemma we have
\[
\pi'_\theta : B_{\delta'_1}(S_\theta x) \to S_\theta x + F(\theta)\perp.
\]
Hence, we can write $\pi'_\theta(q) = S_{t'(q)} q = S_\theta x + \eta$ with $\eta \perp F(\theta)$ and $|t'(q)| \leq t'_0$. Now, by condition (equivalent of 41) we have that
\[
\| \eta \| \leq (k'_0 + 1) \delta'_2 \leq \delta_m.
\]
This concludes the proof of lemma 2.
4 Conclusion

Economists frequently find their models to be systems of differential equations with right-hand side discontinuities. The research initiated in [11] aims at enhancing the theory of switching economic regimes towards a theory of existence, uniqueness, and exponentially asymptotically stability of non-smooth periodic orbits. The theory proposed here also provides a formula for the basin of attraction. The main advantage of our theory is that it does not require the calculation of explicit solutions. This is a desirable result since solutions to typically complex economic models may not be found. While our theory is sufficiently rich to study a large class of macroeconomic models, its main disadvantage, however, is that it only provides conditions for planar systems. Moreover, it requires to find a Lyapunov function for which there is no construction proof. The expansion of our theory to \( n > 2 \) dimensions would be a natural development. This, however, is a nontrivial undertaking.

References


*Received: May 7, 2020; Published: May 28, 2020*