Economic Stability of Non-Smooth Periodic Orbits in the Plane: Omega Limit Sets Part I

Pascal Stiefenhofer

Department of Economics, University of Exeter, UK
&
Department of Mathematics, University of Sussex, UK

Abstract

Stiefenhofer and Giesl establish conditions for exponentially asymptotically stability of non-smooth economic periodic orbits derived from a dynamical system defined by a set of autonomous ordinary differential equations with discontinuous right-hand side [11]. This paper expands the omega limit set for all points in a neighborhood of a point belonging to an exponentially asymptotically periodic orbit. This requires a characterization of solutions in a neighborhood of a point belonging to an exponentially asymptotically stable orbit. The paper introduces a correction mapping and shows a contraction property from which the stability argument follows.

Mathematics Subject Classification: 34D20

Keywords: Omega Limit Sets, Filippov Differential Equations, Exponential Asymptotic Stability

1 Introduction

Filippov introduces a solution concept for differential equations with discontinuous right-hand side [2]. Such equations increasingly appear in economic

---

1EPSRC Research Grant (Engineering and Physical Science Research Council, 2011-2016), 1091684, Stability in Nonsmooth Systems with Applications to Biomechanics.
modeling [1][6][7][8] and [9]. These models are important because they provide the basis for economic policy analysis when smoothness of trajectories is not satisfied. However, the slow theoretical progress of Filippov models in economics, which rarely explore properties beyond existence and uniqueness, prevents economists from rigorous analysis and more effective policy design. A step forward is the theory introduced in [11] which provides conditions for exponentially asymptotically stability of non-smooth periodic orbits. We attempt to enhance this theory in this paper. Applications of this local theory are found in [13] and [12].

This paper first recalls the conditions of exponentially asymptotically stability of non-smooth periodic orbits introduced in Stiefenhofer and Giesl ([11], Theorem 2, p.11) summarized in theorem 2 below. Completing the theory outlined in [11] requires to prove theorem 3 introduced in section 2 of this paper. The proof requires to (i) characterize solutions near a point \(x \in K\), where \(K\) is a compact set. We achieve this by an appropriate coordinate transformation. Then (ii) it requires the introduction of a correction mapping \(\pi\) satisfying some desirable conditions such as continuity and boundedness. A full proof also requires to (iii) define a Poincaré-like map \(P\) and (iv) to show existence and stability of the period orbit \(\Omega\). Section three shows (i), section four shows (ii), and the remaining properties are studied in a companion paper with the same title (Part II) of this journal. A conclusion is provided at the end of this paper.

Let’s consider a nonsmooth dynamical system defined by an autonomous ordinary differential equation

\[
\dot{x} = f(x)
\]

where \(f\) is a discontinuous function at \(x_2 = 0\) and \(x \in \mathbb{R}^2\) (Stiefenhofer and Giesl 2019a)[11]. The discontinuity of \(f \in C^1(\mathbb{R}^2 \setminus (\mathbb{R} \times \{0\}), \mathbb{R}^2)\) implies that the phase space \(\mathbb{X} = \mathbb{R}^2\) is divided into subspaces \(\mathbb{X} = \mathbb{X}^+ \cup \mathbb{X}^0 \cup \mathbb{X}^\cdot\), where \(\mathbb{X}^+ = \{x \in \mathbb{R}^2 : x_2 > 0\}\), \(\mathbb{X}^- = \{x \in \mathbb{R}^2 : x_2 < 0\}\), and \(\mathbb{X}^0 = \{x \in \mathbb{R}^2 : x_2 = 0\}\). By defining \(f := f^\pm\) where \(f(x) = f^+(x)\) if \(x \in \mathbb{X}^+\), and \(f(x) = f^-(x)\) if \(x \in \mathbb{X}^-\), we have

\[
\dot{x} = f(x) = \begin{cases} 
  f^+(x) & \text{if } x \in \mathbb{X}^+ \\
  f^-(x) & \text{if } x \in \mathbb{X}^-.
\end{cases}
\]  

An initial value condition for equation (1) at time \(t = 0\) is given by \(x(0) = x_0 \in \mathbb{X}\). We restrict ourselves to a set of assumptions which according to results by [2] guarantees global existence, uniqueness, and continuous dependence on the initial condition of solutions of the differential equation (1).

**Assumption 1.** Consider equation (1). We assume \(f \in C^1(\mathbb{R}^2 \setminus (\mathbb{R} \times \{0\}), \mathbb{X})\). Each function \(f^\pm(x)\) with \(x \in \mathbb{X}^+\) or \(x \in \mathbb{X}^-\) can be extended to a continuous function up to \(x \in \mathbb{X}^0\). Each function \(Df^\pm(x)\) with \(x \in \mathbb{X}^+\) or \(x \in \mathbb{X}^-\) can
be extended to a continuous function up to $x \in \mathbb{X}^0$. $f^+(x_1, 0) - f^-(x_1, 0)$ is a $C^1$-function with respect to $x_1$. For all $(x_1, 0) \in \mathbb{X}^0$ it holds that $f^+_2(x_1, 0) \cdot f^-_2(x_1, 0) > 0$.

The assumption $f^+_2(x_1, 0) \cdot f^-_2(x_1, 0) > 0$ for all $(x_1, 0) \in \mathbb{X}^0$ states that $f^+_2, f^-_2 < 0$, or $f^+_2, f^-_2 > 0$. This assumption excludes all sliding phenomena on the $\mathbb{X}^0$ manifold and shall be relaxed in future work. It implies a discontinuity in $+/-$ direction if both $f^+_2, f^-_2 < 0$ or a discontinuity in $−/+ $ direction if both $f^+_2, f^-_2 > 0$. Let the flow of the system given by (1) be defined by $S_t(x_0) := (x_1(t), x_2(t)) \in \mathbb{X}$, where $(x_1(t), x_2(t)) \in \mathbb{X}$ is its solution with initial value $((x_1(0), x_2(0)) = x_0$. Hence the flow $S_t x_0$ maps the initial point $x_0$ to a point $x(t)$ at time $t = 0$ to a point $x(t)$ at time $t \geq 0$. An adjacent trajectory is defined by $S_\theta(x_0 + \eta) := (y_1(\theta), y_2(\theta)) \in \mathbb{X}$, where $(y_1(\theta), y_2(\theta)) \in \mathbb{X}$, and $\|\eta\| > 0$ and time $\theta \geq 0$. Let $K \subseteq \mathbb{X}$ be positively invariant if $S_t x_0 \in K$ for all $t \geq 0$ and all $x_0 \in K$. A periodic orbit $\Omega$ of the system (1) is a set defined by $\Omega := \{S_t(x_0) : t \in [0, T]\}$, such that $S_T(x_0) = x_0 \} \subset \mathbb{X}$, with minimal period $T > 0$. Let $K \subseteq \mathbb{X}$ and $K \neq \emptyset$ be a compact, connected and positively invariant set which contains no equilibria. Moreover, set $K^+ := K \cap \{x \in \mathbb{R}^2 : x_2 > 0\}$ and $K^- := K \cap \{x \in \mathbb{R}^2 : x_2 < 0\}$. We now define a neighborhood $A(\Omega)$ of $\Omega$ consisting of a set of points $x_0 \in \mathbb{X}$ such that the distance between $S_t(x_0)$ and $\Omega$ vanishes as $t \to \infty$. The basin of attraction $A(\Omega)$ of an exponentially asymptotically stable orbit $\Omega$ is the set defined by $A(\Omega) := \{x_0 \in \mathbb{X} : \text{dist}(S_t x_0, \Omega) \xrightarrow{t \to \infty} 0\}$.

**Theorem 2.** Let assumption 1 hold, and let $\emptyset \neq K \subset \mathbb{R}^2$ be a compact, connected and positively invariant set with $f^\pm(x) \neq 0$ for all $x \in K^\pm$. Moreover, assume that $W^\pm : \mathbb{X}^\pm \to \mathbb{R}$ are continuous functions and let the orbital derivatives $(W^\pm)'$ exist and be continuous functions in $\mathbb{X}^\pm$ and continuously extendable up to $\mathbb{X}_0$. We set $K^0 := \{x \in K : x_2 = 0\}$. Let the following conditions hold:

1. $L_{W^\pm}(x) := \max_{\|v^\pm\| = e^{-W^\pm(x), v^\pm \perp f^\pm(x)}} L_{W^\pm}(x, v^\pm) \leq -\nu < 0$

   
   $L_{W^\pm}(x, v^\pm) := e^{2W^\pm(x)} \{h^\pm(x) + (Df^\pm(x)) v^\pm + \langle \nabla W^\pm(x), f^\pm(x) \rangle \cdot \|v^\pm\|^2\}$

   for all $x \in K^\pm$.

2. $f_2^\pm(x) \cdot \sqrt{(f_1^\pm(x))^2 + (f_2^\pm(x))^2} e^{W^\pm(x) - W^\pm(x)} < 1$

   for all $x \in K^0$ with $f_2^\pm(x) < 0$, $f_2^\pm(x) < 0$.

Then there is one and only one periodic orbit $\Omega \subset K$. Moreover, $\Omega$ is exponentially asymptotically stable with exponent $-\nu < 0$ and for its basin of attraction the inclusion $K \subset A(\Omega)$ holds.
2 Omega limit set for all points in a neighborhood

The next result generalizes theorem 3 in Stiefenhofer and Giesl ([14], p.516) to a full neighborhood of $x$. It provides the conditions for a point $x$ to belong to an exponentially asymptotically stable periodic orbit. A good theory of $\omega$–limit sets for smooth dynamical systems is provided by [3] and [5].

The long run is a case which is associated with time structures where neither of the solutions changes phase space. During this time segment, solutions are smooth and the conditions of theorem 2 state that the distance between two adjacent smooth solutions decreases. Stability analysis of such solutions relies on [4]. Short run cases are cases associated with short time intervals where the right-hand side of the differential equation (1) is discontinuous. Hence short run time intervals are associated with jumps. Stiefenhofer and Giesl [11] show that in the short run, the distance between to adjacent solutions decreases in forward time. We now provide the notation and definitions of the various times structures. Local contraction conditions in each case, i.e, jumps in $+/-$ and $-/+\,)$ direction are given in [11] and [14]. In this paper we patch the solutions together and begin showing show global contraction of adjacent solutions $S_t(x_0)$ and $S_t(x_0 + \eta)$.

**Definition 1** (Long run time structure of $S_t$).

\[ G^- := \left\{ \bigcup_j (t_{j-1}^+, t_j^-) : j \in \{2n - 1\} \text{ and } n \in \mathbb{N}_0 \right\} \]

\[ G^+ := \left\{ \bigcup_j (t_{j-1}^+, t_j^-) : j \in \{2n\} \text{ and } n \in \mathbb{N}_0 \right\} \]

$\forall t \in G^\pm$ solutions $S_t^\pm$ are smooth.

**Definition 2** (Short run time structure of $S_t$).

\[ J^- := \left\{ \bigcup_j [t_j^-, t_{j+1}^-) : j \in \{2n - 1\} \text{ and } n \in \mathbb{N}_0 \right\} \]

\[ J^+ := \left\{ \bigcup_j [t_j^-, t_{j+1}^-) : j \in \{2n\} \text{ and } n \in \mathbb{N}_0 \right\} \]

$\forall t \in J^\mp$ solutions $S_t^-$ jump to $S_t^+$. $\forall t \in J^\mp$ solutions $S_t^-$ jump to $S_t^+$.

\[ I^- := \left\{ \bigcup_j [t_j^-, t_{j+1}^-) : j \in \{2n - 1\} \text{ and } n \in \mathbb{N}_0 \right\} \]

\[ I^+ := \left\{ \bigcup_j [t_j^-, t_{j+1}^-) : j \in \{2n\} \text{ and } n \in \mathbb{N}_0 \right\} \]

$\forall t \in I^\pm$ solutions $S_t^-$ jump to $S_t^+$. $\forall t \in I^\pm$ solutions $S_t^-$ jump to $S_t^+$. 
Short run intervals are time intervals \( t \in J^\pm \) or \( t \in I^\pm \) where equation (1) is associated with a right-hand side discontinuity. We observe that \( I \) structures are time structures at which time is frozen at \( t^-_j = t^+_j \), \( F \) structures are time structure where an adjacent solution is frozen. Hence, we next consider the time structure of an adjacent solution. Case \( S^-_t x_0 \) switches to \( S^+_t x_0 \) first: We consider the case where \( S^-_t x_0 \) switches to \( S^+_t x_0 \) before an adjacent trajectory switches from \( S^-_t (x_0 + \eta) \) to \( S^+_t (x_0 + \eta) \). Hence for all \( t \in I^\pm \) there is a \( \theta \in \bigcup_j [\theta^-_j, \theta^+_j] : \theta^-_j \neq \theta^+_j, j \in \{2n - 1\} \) and \( n \in \mathbb{N}_0 \). Case \( S^+_t x_0 \) switches to \( S^-_t x_0 \) first: We consider the case where \( S^+_t x_0 \) switches to \( S^-_t x_0 \) before an adjacent trajectory switches from \( S^-_t (x_0 + \eta) \) to \( S^+_t (x_0 + \eta) \). Hence for all \( t \in I^\pm \) there is a \( \theta \in \bigcup_j [\theta^-_j, \theta^+_j] : \theta^-_j \neq \theta^+_j, j \in \{2n\} \) and \( n \in \mathbb{N}_0 \). Case \( S^+_t x_0 \) and \( S^+_t (x_0 + \eta) \) smooth: We consider the smooth case when \( S^+_t x_0 \in X^+ \) for all \( t \in \mathcal{G}^+ \). Then there is \( \theta \in \bigcup_j (\theta^-_j, \theta^+_j) : j \in \{2n\} \) and \( n \in \mathbb{N}_0 \) such that \( S^+_\theta (x_0 + \eta) \in X^+ \). Case \( S^-_t x_0 \) and \( S^-_t (x_0 + \eta) \) smooth: We consider the smooth case when \( S^-_t x_0 \in X^- \) for all \( t \in \mathcal{G}^- \). Then there is \( \theta \in \bigcup_j (\theta^-_j - 1, \theta^-_j) : j \in \{2n - 1\} \) and \( n \in \mathbb{N}_0 \) such that \( S^\theta (x_0 + \eta) \in X^- \). It remains to formalize the time structure of an adjacent solution of \( S^+_t x_0 \) denoted by \( S^\theta (x_0 + \eta) \) with initial condition \( ||\eta|| \leq \frac{\delta}{2} \). We want to synchronize the time of two nearby solutions such that \( \left( S^+_t x_{\theta + \eta_n} (x + \eta) - S^+_t x \right)^T f^+(S^+_t x) = 0 \) holds. This requires to define a multi valued mapping \( T \) where \( T : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ \) with \( \theta = T(t) \). Stiefenhofer and Giesl [14] show that such a mapping exists. We now define the complete time structure in terms of \( T \).

**Definition 3** (Synchronized (LR,SR) time).

\[
\theta_j = \begin{cases} 
T^\pm(t) & \text{if } t \in \mathcal{G}^+ \\
T^J(t) & \text{if } t \in J^\pm \\
T^I(t) & \text{if } t \in I^\pm 
\end{cases}
\]

with \( T^J(t) := T^J(t^+_j), \forall t \in J^\pm \)

\( T^I(t) := \lim_{t \to t^+_j} T^I(t), \forall t \in I^\pm \)  

(2)

for all \( j \in \{2n\} \) and \( n \in \mathbb{N}_0 \). For the opposite jump direction we define \( \theta_{j-1} \) similarly.

\[
\theta_j = \begin{cases} 
T^\pm(t) & \text{if } t \in \mathcal{G}^- \\
T^J(t) & \text{if } t \in J^\pm \\
T^I(t) & \text{if } t \in I^\pm 
\end{cases}
\]

with \( T^J(t) := T^J(t^-_j), \forall t \in J^\pm \)

\( T^I(t) := \lim_{t \to t^-_j} T^I(t), \forall t \in I^\pm \)  

(3)
for all \( j \in \{2n - 1\} \) and \( n \in \mathbb{N}_0 \).

We define \( F(t) \) as a smooth function in a direction which is not perpendicular to \( f(S_t x) \).

\[
F(t) := \frac{f(S_t x)}{\|f(S_t x)\|}.
\]

**Theorem 3.** Let assumptions of theorem 2 hold. Let \( x \in K^\pm \) satisfy \( x \in \omega(x) \). Assume there is a continuous map \( F : \mathbb{R}^+ \to \mathbb{R}^2 \) with \( \|F(t)\| = 1 \) and \( \langle F(t), f(S_t x) \rangle > 0 \) for all \( t \geq 0 \). Furthermore, assume that there are constants \( \delta, \nu > 0 \) and \( C \geq 1 \) such that for all \( \eta \in \mathbb{R}^2 \) with \( \eta \perp F(0) \) and \( \| \eta \| \leq \delta \) there is a piecewise multi valued mapping \( T_{x+\eta}^x : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( T_{x+\eta}^x(t) \) depends continuously on \( \eta \) and satisfies

\[
\frac{2}{3} \cdot t \leq T(t_i) \leq \frac{4}{3} \cdot t \forall t_i \in G^\pm.
\]

\[
\frac{1}{3} \cdot t \leq T(t_i) \leq \frac{2}{3} \cdot t \forall t_i \in I^\pm.
\]

\[
\frac{2}{3} \left( \frac{1}{2 + \frac{\nu}{\delta^2}} \right) \cdot t \leq T(t_i) \leq \frac{4}{3} \left( \frac{1}{2 + \frac{\nu}{\delta^2}} \right) \cdot t \forall t_i \in J^\pm.
\]

\[
\langle S_{T_{x+\eta}^x(t)}(x + \eta) - S_t x, F(t) \rangle = 0
\]

and

\[
\|S_{T_{x+\eta}^x(t)}(x + \eta) - S_t x\| \leq C e^{-\mu t} \| \eta \|
\]

for all \( t \geq 0 \). Then \( x \) is a point of an exponentially asymptotically stable period orbit \( \Omega \).

The proof of theorem (3) consists of four main steps: (i) Characterization of solutions near a point \( x \in K \). (ii) Definition of a correction mapping \( \pi \). (iii) Definition of a Poincaré-like map \( \mathcal{P} \). (iv) Existence of period orbit \( \Omega \) and its stability. We begin our discussion with (i) characterizing solutions near a point \( x \in K \) and then (ii) show a boundedness and continuity of the correction mapping \( \pi \).

### 3 Characterization of solutions near \( x \in K \)

We first introduce a new coordinate system and then characterize the behavior of \( x \in K \) in terms of \( f(x) \). The next lemma shows that trajectories starting within a neighborhood can only move within a cone. We center the new coordinate system at the point \( x \in K \).

- Define by \( y(q) \) the scalar amount in \( F(0) \)-direction.
• Define by $x(q)$ the vectorial amount in $F(0)\perp$-direction.

We now define a Hyperplane which consists of the points $q \in \mathbb{R}^2$ where $y(q) = 0$ holds.

$$H := \{ q \in \mathbb{R}^2 : x + F(0)\perp, \text{ and } y(q) = 0 \}.$$  

For arbitrary $q \in \mathbb{R}^2$, we define

$$y(q) := \langle q - x, F(0) \rangle \in \mathbb{R} \quad (9)$$

$$x(q) := q - x - y(q)F(0) \in F(0)\perp \quad (10)$$

By equation (10) we can express

$$q = x + y(q)F(0) + x(q) \quad (11)$$

and $\|q - x\|^2 = |y(q)|^2 + \|x(q)\|^2$. We also define $f(q)$ for all $q \in \mathbb{R}^2$. Hence we define

$$\lambda(q) := \langle f(q) - f(x), F(0) \rangle \in \mathbb{R} \quad (12)$$

$$u(q) := f(q) - f(x) - \lambda(q)F(0) \in g(0)\perp \quad (13)$$

Thus by equation (13) be obtain

$$f(q) = f(x) + \lambda(q)F(0) + u(q). \quad (14)$$

We have the following upper bounds on $\lambda(q)$ and $u(q)$. By assumptions of theorem 2 we know that $f$ is continuous in $x$. Hence there is a $\delta_1$ with $0 < \delta_1 \leq \delta$ such that for all $q \in B_{\delta_1}$ the following bounds hold:

$$|\lambda(q)| \leq \frac{1}{2} \alpha_0 \quad (15)$$

$$\|u(q)\| \leq \|f(x)\|, \quad (16)$$

with

$$\alpha_0 := \langle F(0), f(x) \rangle > 0 \quad (17)$$

We can finally characterize solutions for all $q \in B_{\delta_1}$. We show that adjacent trajectories can only move within a cone.

**Lemma 1.** Let $S_t q \in B_{\delta_1}$ hold for all $t \in [0, \bar{\tau}]$ with $\bar{\tau} > 0$. Then for all $t \in [0, \bar{\tau}]$ and all $\tau_1 \leq \tau_2 \leq \bar{\tau}$ the following bounds hold:

$$\frac{1}{2} \alpha_0 \leq \frac{d}{dt} y(S_t q) \leq \frac{3}{2} \alpha_0 \quad (18)$$

$$\frac{1}{2} \alpha_0 (\tau_2 - \tau_1) \leq y(S_{\tau_2} q) - y(S_{\tau_1} q) \leq \frac{3}{2} \alpha_0 (\tau_2 - \tau_1) \quad (19)$$

and

$$\|x(S_{\tau_2} q) - x(S_{\tau_1} q)\| \leq k_0 (y(S_{\tau_2} q) - y(S_{\tau_1} q)), \quad (20)$$

where $k_0 := 4\frac{\|f(p)\|}{\alpha_0}$.
Proof. We first show inequality (18). By equation (11) we have
\[ S_t q = x + y(S_t q) F(0) + x(S_t q). \]
Hence by differentiation we conclude that
\[
\begin{align*}
f(S_t q) &= \frac{d}{dt} S_t q \\
&= \frac{d}{dt} y(S_t q) + \frac{d}{dt} x(S_t q).
\end{align*}
\] (21)
By equation (10) we have \( x(S_t q) \perp F(0) \) for all \( t \in [0, \tilde{\tau}] \), we conclude that \( \frac{d}{dt} x(S_t q) \perp F(0) \) holds too. By (11) we have
\[
f(S_t q) = f(x) + \lambda(S_t q) F(0) + u(S_t q)
\]
Using (12) and (13) yields
\[
f(S_t q) = f(x) + (f(S_t q) - f(x), F(0)) + f(S_t q) - f(x) - \lambda(S_t q) F(0)
\]
which by little algebraic manipulation and using \( \frac{d}{dt} y(S_t q) = f(S_t q) \) yields
\[
0 = \frac{d}{dt} y(S_t q) F(0) - f(x) + f(x) F(0) - \lambda(S_t q).
\]
Using (17) and rearranging yields
\[
\frac{d}{dt} y(S_t q) F(0) = \alpha_0 + \lambda(S_t q).
\] (22)
Equation (22) with bound (15) yields condition (18) as required. Since we consider the time interval \( t \in [0, \tilde{\tau}] \) with \( 0 \leq \tau_1 \leq \tau_2 \leq \tilde{\tau} \) condition (19) follows from
\[
\int_{\tau_1}^{\tau_2} \frac{d}{dt} y(S_t q) dt = y(S_{\tau_2} q) - y(S_{\tau_1} q)
\]
and bounds of condition (18). Now, we multiply (22) by \( \frac{d}{dt} x(S_t q) \) and with \( \frac{d}{dt} x(S_t q) \perp F(0) \) we obtain
\[
\| \frac{d}{dt} x(S_t q) \|^2 = \langle f(S_t q), \frac{d}{dt} x(S_t q) \rangle
\]
which by (11) becomes
\[
= \langle f(x) + u(S_t q), \frac{d}{dt} x(S_t q) \rangle
\]
\[
\| \frac{d}{dt} x(S_t q) \| \leq \| f(p) \| + \| u(S_t q) \|. \tag{23}
\]

Hence
\[
\| x(S_{\tau_2}) - x(S_{\tau_1}) \| = \left\| \int_{\tau_1}^{\tau_2} \frac{d}{dt} x(S_t q) dt \right\|
\leq \int_{\tau_1}^{\tau_2} \| \frac{d}{dt} x(S_t q) dt \|
\leq \int_{\tau_1}^{\tau_2} (\| f(x) \| + \| u(S_t q) \|) dt \text{ by (23)}
\leq 2(\tau_2 - \tau_1) \| f(x) \| \text{ by (16)}
\leq k_0(y(S_{\tau_2} q) - y(S_{\tau_1} q)) \text{ by (19)}.
\]

which proves condition (20). This concludes the prove of lemma 1.

\[\square\]

4 The correction mapping \( \pi \)

We define the operator \( \pi \) which maps nearby points to the hyperplane \( H \) along trajectories. We show that there is some short time interval such that trajectories through points in a small ball around \( x \in K \) intersect the hyperplane \( H := x + F(0)^\perp \).

Lemma 2. Let
\[
\pi : \begin{cases}
B_{\delta_2}(p) & \rightarrow & H := x + F(0)^\perp \\
q & \mapsto & \pi(q)
\end{cases}
\tag{24}
\]

be a continuous map defined by
\[
\pi(q) = S_{t^*(q)} q,
\]
where \( t^*(q) \) is a continuous function satisfying \( |t^*(q)| \leq \frac{2\delta_2}{\alpha_0} =: t_0 \) for all \( q \in B_{\delta_2}(p) \). Then for \( x^* \in H \cap B_{\delta_2}(p) \) we have
\[
\| \pi(q) - x^* \| \leq (k_0 + 1) \| q - x^* \|. \tag{25}
\]

Proof. We only consider the case \( y(q) \leq 0 \). We consider the continuous function \( y(S_\tau q) \) and show that for \( q \in B_{\delta_2}(x) \) \( y(S_\tau q) \) vanishes for some time \( \tau = t^* \). We show this by condition (19) of lemma 1. Then we show by contradiction that \( t^* \) is close enough to zero so that a trajectory \( S_\tau q \) remains in \( B_{\delta_1}(x) \) for all time \( \tau \in [0, t^*] \). Finally, we show condition (25) using condition (20) of lemma 1. Since we consider the case \( y(q) \leq 0 \) we have that as long as \( S_\tau q \in B_{\delta_1}(x) \) with \( \tau \geq 0 \) we have that by condition (19) of lemma 1 \( y(S_\tau q) \geq y(q) + \frac{\tau}{2} \alpha_0 \). For \( \bar{\tau} = -\frac{2}{\alpha_0} y(q) \geq 0 \) we have \( y(S_{\bar{\tau}} q) \geq 0 \). Observe that \( |\bar{\tau}| \leq \frac{2}{\alpha_0} \delta_2 = t_0 \).
The existence of a time $t^* \in [0, \tilde{\tau}]$ such that $y(S_\tau q) = 0$ is satisfied is implied by the intermediate value theorem. Uniqueness of $t^*$ follows from lemma 1 as $y(S_\tau q)$ is monotonously increasing in $\tau$. Now, by the implicit function theorem we can define the continuous function $t^*(q)$ by $y(S_\tau q) = 0$. Since $y$ and $S_t$ are continuous functions, it follows that $t^*$ is continuous. That proves that the projection mapping $\pi$ is also continuous as required. Next, we show by contradiction that $\tilde{\tau}$ is close enough to zero so that a trajectory $S_\tau q$ remains in $B_\delta_1(x)$ for all time $\tau \in [0, \tilde{\tau}]$. Assume the contrary. Let there be a $\tau_0 \in [0, \tilde{\tau}]$ with $\| S_{\tau_0} q - x \| = \delta_1$ and $\| S_\tau q - x \| < \delta_1$ for all $\tau \in [0, \tau_0]$. Then by (14), (15), and (16) we have

$$\| f(q) \| \leq 2 \| f(x) \| + \frac{1}{2} \alpha_0$$

for all $q \in B_\delta_1(x)$. This yields

$$\delta_1 = \| S_{\tau_0} q - x \|
\leq \| \int_0^{\tau_0} f(S_\tau q) d\tau \| + \| q - x \|
\leq \tilde{\tau} \| \alpha_0 + \delta_2
\leq \delta_2 \left( \frac{2 \epsilon_0}{\alpha_0} + 1 \right) = \frac{\delta_1}{2}.$$  

Hence a contradiction. In the final step of the proof we need to show property 25. By (9) we have

$$y(q) = \langle q - x, F(0) \rangle
= \langle q - x^*, F(0) \rangle + \langle x^* - x, F(0) \rangle$$

with $\langle x^* - x, F(0) \rangle = 0$. Hence $|y(q)| \leq \| q - x^* \|$. Condition (20) of lemma 1 implies that

$$\| x(\pi(q)) - x(q) \| \leq k_0 | t(q) | \leq k_0 \| q - x^* \| .$$

We conclude the following

$$\| \pi(q) - x^* \| = \| x(\pi(q)) - x^* \|
\leq \| x(\pi(q)) - x(q) \| + \| x(q) - x^* \|
\leq (k_0 + 1) \| q - x^* \| .$$

This concludes the proof of lemma 2. $\square$
5 Conclusion

Stiefenhofer and Giesl [14] study exponentially asymptotically stability of non-smooth economic periodic orbits in a model of switching economic regimes. In this paper, we enhance the theory by expanding their local contraction criterion over the non-smooth time segment of a periodic orbit to all full neighborhood of $x$. We show that non-smooth trajectories, starting within a neighborhood of $x$ can only move within a cone. We also provide some conditions in preparation of a complete stability proof by introducing a correction mapping on which we will base a Poincaré mapping in a companion paper of this journal in order to complete the proof.

References


Received: May 7, 2020; Published: May 28, 2020