A Reliable Modification Method for Chen-Lee-Liu Equation with Different Optical Solitons

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Abstract
The rationale behind this study is to derive a rapid converging numerical scheme for the Chen-Lee-Liu (CLL) equation. In doing so, we employ the reliable Modification of the Adomian Decomposition Method (MADM) for solving functional equations. Additionally, certain exact soliton solutions will be considered as benchmark solutions for numerical implementations. In the end, the efficacy of the method will be proven by examining various absolute error results that will be demonstrated in tables and figures.

Keywords: Chen-Lee-Liu equation, Solitons, A reliable Modification of the Adomian Decomposition Method

1 Introduction
The nonlinear Schrödinger equations with the self-steeping effect are called derivative nonlinear Schrödinger equations. One of these equations is the Chen-Lee-Liu (CLL) equation that describes the propagation of a wave solitary in single-mode fiber in optical fibers [6]. This equation has different types of solitons such as bright, dark, and singular to mention a few. Also, the CLL equation was studied using certain numerical methods in [2,3,13]. Moreover, the MADM was introduced
by Wazwaz in 1999 as a powerful modification of the ADM that can solve different types of linear and nonlinear differential equations [4]. This modified method has been demonstrated to be computationally efficient in various models in mathematical physics and engineering. In this paper, we shall numerically examine the CLL equation with different types of solitons with the help of MADM. The CLL equation is an integrable equation that was pioneered in 1979 with various applications, including ultrashort pulse propagation and modeling optical and photonic crystal fibers to mention a few. First, consider the dimensionless form of the CLL equation that reads

$$iu_t + au_{xx} + b|u|^2u_x = 0 \quad (1.1)$$

where $u = u(x,t)$ is a complex-valued function for the wave propagation in $x$ (distance) and $t$ (time) variables; $a$ represents the coefficient of group velocity dispersion and $b$ represents the coefficient of self-steeping effect. Additionally, there have been several analytical approaches the yield various sets of valid exact optical soliton solutions for the CLL equation [9,10]. Thus, some of these important exact solutions together with their constraints conditions are given by:

(1) Bright solitons

The first type: the first type of bright soliton takes the following form [8]:

$$u(x,t) = \frac{A \text{sech} (\eta s)}{\sqrt{1 + B \text{sech}^2 (\eta s)}} e^{i[-kx + \omega t + \theta(s)]}, \quad s = x - vt, \quad (1.2)$$

where

$$A = \sqrt{-\frac{2\delta (2B + 1)}{\delta}}, \quad \eta = \sqrt{-\delta}, \quad \text{and} \quad 2B + 1 = \left(1 - \frac{16\gamma \delta}{3\sigma^2}\right)^{-1/2}; \forall \delta < 0, \sigma > 0, \gamma < \frac{3\sigma^2}{16\delta}.$$  

The second type: the second type of bright soliton takes the following form [8]:

$$u(x,t) = P \sqrt{1 + \text{sech}(\mu s)} e^{i[-kx + \omega t + \theta(s)]}, \quad s = x - vt, \quad (1.3)$$

where

$$P = \sqrt{-\frac{8\delta}{5\sigma}} \quad \text{and} \quad \mu = \frac{4\delta}{5}; \quad \forall \delta > 0, \sigma < 0, \gamma = \frac{15\sigma^2}{64\delta}.$$  

The third type: the third type of bright soliton takes the following form [8]:

$$u(x,t) = \frac{P}{\sqrt{1 + r \cosh(\rho s) + \lambda \sinh(\rho s)}} e^{i[-kx + \omega t + \theta(s)]}, \quad s = x - vt, \quad (1.4)$$
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where
\[ p = \sqrt{-\frac{4\delta}{\sigma}}, \quad \rho = \sqrt{-4\delta}, \quad \text{and} \quad r = \sqrt{1 + \frac{\lambda^2}{16\delta} \frac{\sigma^2 - 1}{3\sigma}}, \quad \forall \delta < 0, \ \sigma > 0, \ \gamma < \frac{3\sigma^2(1+\lambda^2)}{16\delta}. \]

(2) Dark solitons

The first type: the first type of dark soliton takes the following form [9]:
\[ u(x, t) = p \sqrt{1 - \text{sech}(\mu s)} e^{i[-kx + \omega t + \theta(s)]}, \quad s = x - vt, \quad (1.5) \]
where
\[ p^2 = -\frac{8\delta}{5\sigma}, \quad \mu^2 = \frac{4\delta}{5}, \quad \gamma = \frac{15\sigma^2}{64\delta}, \quad \forall \delta > 0, \ \sigma < 0. \]

The second type: the second type of dark soliton takes the following form [9]:
\[ u(x, t) = \frac{A \tanh(\eta s)}{\sqrt{3 + \tanh^2(\eta s)}} e^{i[-kx + \omega t + \theta(s)]}, \quad s = x - vt, \quad (1.6) \]
where
\[ A = \sqrt{-\frac{\sigma}{\gamma}}, \quad \eta = \sqrt{\frac{\sigma^2}{16\gamma}}, \quad \gamma = \frac{3\sigma^2}{16\delta}; \quad \forall \gamma > 0, \ \sigma < 0. \]

The third type: the third type of dark soliton takes the following form [9]:
\[ u(x, t) = \frac{\lambda \cosh(\mu s)}{\sqrt{\varepsilon + \cosh^2(\mu s)}} e^{i[-kx + \omega t + \theta(s)]}, \quad s = x - vt, \quad (1.7) \]
where
\[ \delta = -\frac{\mu^2 (\varepsilon + 3)}{\varepsilon}, \quad \sigma = \frac{2\mu^2 (2\varepsilon + 3)}{\lambda^2 \varepsilon}, \quad \gamma = -\frac{3\mu^2 (\varepsilon + 1)}{\lambda^4 \varepsilon}; \quad \forall \delta < 0, \sigma > 0. \]

(3) Singular solitons

The first type: the first type of singular soliton takes the following form [10]:
\[ u(x, t) = p \sqrt{1 + \coth(\mu s)} e^{i[-kx + \omega t + \theta(s)]}, \quad s = x - vt, \quad (1.8) \]
where
\[ p = \sqrt{-\frac{2\delta}{\sigma}}, \quad \mu = \sqrt{-\delta}; \quad \forall \delta < 0, \sigma > 0, \quad \gamma = \frac{3\sigma^2}{16\delta}. \]

The second type: the second type of singular soliton takes the following form [10]:

\[ u(x, t) = \frac{A}{\sqrt{1 + r \sinh(\mu s)}} e^{i[-kx + \omega t + \theta(s)]}, \quad s = x - vt, \quad (1.9) \]

where

\[ A = \sqrt{-\frac{4\delta}{\sigma}}, \quad \mu = \sqrt{-4\delta}, \quad r = \sqrt{\frac{16\delta \gamma}{3\sigma^2} - 1}; \quad \forall \delta < 0, \sigma > 0, \gamma > \left| \frac{3\sigma^2}{16\delta} \right|. \]

The third type: the third type of singular soliton takes the following form [10]:

\[ u(x, t) = \frac{p \csc(qs)}{\sqrt{1 - R \coth^2(qs)}} e^{i[-kx + \omega t + \theta(s)]}, \quad s = x - vt, \quad (1.10) \]

where

\[ p = \sqrt{\frac{2\delta(1+R)}{\sigma}}, \quad q = \sqrt{-\delta}, \quad \gamma = \frac{3\sigma^2 R}{4\delta(1+R)}; \quad \forall \delta < 0, \sigma < 0, \gamma > 0, R < -1. \]

The fourth type: the fourth type of singular soliton takes the following form [10]:

\[ u(x, t) = \left( \frac{A}{m + \sinh^2(\mu s)} \right)^{1/2} e^{i[-kx + \omega t + \theta(s)]}, \quad s = x - vt, \quad (1.11) \]

where

\[ A = \sqrt{-\frac{2\delta(2m-1)}{\sigma}}, \quad \mu = \sqrt{-\delta}, \quad m = \frac{1}{2} \left( 1 + \left[ 1 - \frac{16\delta \gamma}{3\sigma^2} \right]^{\frac{1}{2}} \right); \quad \forall \delta < 0, \gamma < \left| \frac{3\sigma^2}{16\delta} \right|. \]

Thus, we shall use the aforementioned exact soliton solutions as benchmark solutions for establishing a comparative study with the numerical solutions given by the MADM; see also [1,5,7,11,12,14] for more related Adomian-based decomposition methods for solving partial differential equations. The paper is structured as follows: The recursive scheme for the CLL equation is derived in Section 2, using the MADM. The results provided by the method are shown and discussed in section 3; while section 4 provides some concluding remarks.

2 The Solution algorithm

This section will describe the MADM for the CLL equation. First, we consider the
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operator notation by letting \( L = \frac{\partial}{\partial t} \) and its corresponding inverse operator \( L^{-1} = \int_{0}^{t} \cdot \, dt \). Employing this inverse \( L^{-1} \) on Eq. (1.1), we obtain

\[
\begin{align*}
    u(x, t) &= f(x) + ai \int_{0}^{t} u_{xx} dt - b \int_{0}^{t} A dt,
\end{align*}
\]  

where \( f(x) = u(x, 0) \) and \( A = |u|^2 u_x \). Further, the classical Adomian method proposed that the solution \( u(x, t) \) be decomposed into a summation of an infinite series of the form

\[
    u(x, t) = \sum_{n=0}^{\infty} u_n(x, t).
\]  

The nonlinear term \( A \) is also decomposed into a summation of an infinite series of the form

\[
    A = \sum_{n=0}^{\infty} A_n (u_0, u_1, ..., u_n),
\]  

where \( A_n \)'s are the Adomian polynomials for determining nonlinear terms using the following formula

\[
    A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} N \left( \sum_{i=0}^{\infty} \lambda^i u_i \right), \quad n = 0, 1, 2, ...
\]  

Now, the MADM starts off by the assumption that the function \( f(x) \) be divided into two parts, that is

\[
    f(x) = f_0(x) + f_1(x).
\]  

Consequently, substituting Eq. (2.5) into Eq. (2.1) alongside utilizing the decompositions in Eqs. (2.2)-(2.3) and on using the formula in Eq. (2.4) for the nonlinear terms, we obtain the recursive relation in the MADM sense follows

\[
\begin{align*}
    u_0(x, t) &= f_0(x), \\
    u_1(x, t) &= f_1(x) + ai \int_{0}^{t} u_{0xx} dt - b \int_{0}^{t} A_0 dt, \\
    u_{k+1}(x, t) &= ai \int_{0}^{t} u_{kxx} dt - b \int_{0}^{t} A_k dt, \quad k \geq 1,
\end{align*}
\]  

where \( A_0 \) and \( A_k \) (\( k \geq 1 \)) are the Adomian polynomial for the nonlinear terms to be determined recurrently based on the generalized formula expressed in Eq. (2.4). Therefore, the aiming general recursive scheme by the MADM is determined in Eq. (2.6); this scheme will be simulated alongside some exact soliton solutions in the next section.

3 Results and discussions

This section presents the obtained numerical results using the said method and sets
out some comparative analysis. Considering bright, dark and singular soliton solutions, we are able to numerically simulate the derived recursive scheme with the help of the Maple software and present the corresponding absolute error analysis in Tables 1-3 and their respective graphical representations in Figures 1-3. Thus, the present method can be relied upon after examining the revealed minimal error between the numerical solutions and the benchmark exact solutions.

### Bright solitons

<table>
<thead>
<tr>
<th>$x$</th>
<th>$b = -0.1$</th>
<th>$b = 0.1$</th>
<th>$b = -0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1.482728939 \times 10^{-7}$</td>
<td>$5.292009153 \times 10^{-4}$</td>
<td>$1.200102906 \times 10^{-5}$</td>
</tr>
<tr>
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<td>$1.482687222 \times 10^{-7}$</td>
<td>$5.314397219 \times 10^{-4}$</td>
<td>$1.216403697 \times 10^{-5}$</td>
</tr>
<tr>
<td>2</td>
<td>$1.482563350 \times 10^{-7}$</td>
<td>$5.373923508 \times 10^{-4}$</td>
<td>$1.252791545 \times 10^{-5}$</td>
</tr>
<tr>
<td>3</td>
<td>$1.482356028 \times 10^{-7}$</td>
<td>$5.468030214 \times 10^{-4}$</td>
<td>$1.286303737 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

**Table 1.** The absolute error of the MADM for types of bright solitons when $t = 0.3$

### Dark solitons

<table>
<thead>
<tr>
<th>$x$</th>
<th>$b = 0.1$</th>
<th>$b = 0.1$</th>
<th>$\lambda = 0.001$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1.501942185 \times 10^{-7}$</td>
<td>$6.588540778 \times 10^{-9}$</td>
<td>$2.972736098 \times 10^{-7}$</td>
</tr>
<tr>
<td>2</td>
<td>$1.259391637 \times 10^{-7}$</td>
<td>$9.375214451 \times 10^{-8}$</td>
<td>$2.976125904 \times 10^{-7}$</td>
</tr>
<tr>
<td>3</td>
<td>$1.022479234 \times 10^{-7}$</td>
<td>$1.696263867 \times 10^{-7}$</td>
<td>$2.981180150 \times 10^{-7}$</td>
</tr>
<tr>
<td>4</td>
<td>$8.587071350 \times 10^{-8}$</td>
<td>$2.054223572 \times 10^{-7}$</td>
<td>$2.987169809 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

**Table 2.** The absolute error of the MADM for types of dark solitons when $t = 0.3$
Singular solitons

<table>
<thead>
<tr>
<th>Type</th>
<th>First Type</th>
<th>Second Type</th>
<th>Third Type</th>
<th>Fourth Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$b = -0.1$</td>
<td>$b = -0.1$</td>
<td>$b = 0.1$</td>
<td>$b = -0.1$</td>
</tr>
<tr>
<td>$t = 0.3$</td>
<td>$4.21722771 \times 10^{-9}$</td>
<td>$3.43197232 \times 10^{-9}$</td>
<td>$1.08808366 \times 10^{-6}$</td>
<td>$1.28381341 \times 10^{-8}$</td>
</tr>
<tr>
<td>$t = 1$</td>
<td>$2.29085108 \times 10^{-9}$</td>
<td>$5.98101162 \times 10^{-10}$</td>
<td>$6.08802472 \times 10^{-8}$</td>
<td>$7.99039261 \times 10^{-10}$</td>
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<tr>
<td>$t = 3$</td>
<td>$2.31198658 \times 10^{-9}$</td>
<td>$1.94045459 \times 10^{-10}$</td>
<td>$2.68273237 \times 10^{-9}$</td>
<td>$2.36478171 \times 10^{-10}$</td>
</tr>
<tr>
<td>$t = 4$</td>
<td>$2.32154741 \times 10^{-9}$</td>
<td>$7.73921185 \times 10^{-11}$</td>
<td>$1.18162786 \times 10^{-10}$</td>
<td>$9.37883694 \times 10^{-11}$</td>
</tr>
</tbody>
</table>

Table 3. The absolute error of the MADM for types of singular solitons when $t = 0.3$

Fig. 1. Comparison between exact and MADM solutions for different types of bright solitons.

Fig. 2. Comparison between exact and MADM solutions for different types of dark solitons.
First Type

Second Type

Third Type

Fourth Type

Fig. 3. Comparison between exact and MADM solutions for different types of singular solitons.

4 Conclusion

In conclusion, a rapid converging numerical scheme for the CLL equation was proposed via the application of the MADM. Certain exact optical solutions in optical fibers consisting of bright, dark, and singular solitons have been sought for the validation of the numerical results. The simulated results confirmed that the method indeed possessed a high level of accuracy as shown in the presented tables and figures. We thus remark that the MADM can be relied upon while treating various forms of Schrödinger equations.

References


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