Some New Fixed Point Theorems for New Nonlinear Conditions with $MT$-Functions

Ing-Jer Lin $^1$ and Po-Chun Hsieh

Department of Mathematics
National Kaohsiung Normal University
Kaohsiung 82444, Taiwan

This article is distributed under the Creative Commons by-nc-nd Attribution License. Copyright © 2020 Hikari Ltd.

Abstract

In this paper, we establish some new fixed point theorems by applying Lin-Wu’s convergence theorems.

Mathematics Subject Classification: 47H10, 54H25

Keywords: $MT$-function ($R$-function), fixed point theorem, Lin-Wu’s convergence theorem, cyclic CWLD mapping, best proximity point, Banach contraction principle

1. Introduction

It is well-known that fixed point theory has a wide range of applications in many different fields of mathematics. The famous Banach contraction principle [1, 4] has played a significant role in nonlinear analysis and applied mathematical analysis. Let $(X, d)$ be a metric space. A point $v$ in $X$ is a fixed point of a mapping $T : X \to X$ if $Tv = v$. The set of fixed points of $T$ is denoted by $\mathcal{F}(T)$.

Theorem 1.1 (Banach contraction principle [1, 4]). Let $(X, d)$ be a complete metric space and $T : X \to X$ be a selfmapping. Suppose that there exists a nonnegative number $\lambda < 1$ such that

$$d(Tx, Ty) \leq \lambda d(x, y) \quad \text{for all } x, y \in X.$$
Then $T$ has a unique fixed point in $X$.

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$, the set of positive integers and real numbers, respectively. Let $f$ be a real-valued function defined on $\mathbb{R}$. For $c \in \mathbb{R}$, we recall that

$$
\limsup_{x \to c^+} f(x) = \inf_{\varepsilon > 0} \sup_{0 < x - c < \varepsilon} f(x).
$$

**Definition 1.2** [2]. A function $\varphi : [0, \infty) \to [0, 1)$ is said to be an $MT$-function (or $R$-function) if

$$
\limsup_{s \to t^+} \varphi(s) < 1 \text{ for all } t \in [0, \infty).
$$

It is obvious that if $\varphi : [0, \infty) \to [0, 1]$ is a nondecreasing function or a nonincreasing function, then $\varphi$ is an $MT$-function. Hence the set of $MT$-functions is a rich class.

In 2012, Du [2] established the following characterizations of $MT$-functions.

**Theorem 1.3** [2]. Let $\varphi : [0, \infty) \to [0, 1)$ be a function. Then the following statements are equivalent.

(a) $\varphi$ is an $MT$-function.

(b) For each $t \in [0, \infty)$, there exist $r_{t}^{(1)} \in [0, 1)$ and $\varepsilon_{t}^{(1)} > 0$ such that $\varphi(s) \leq r_{t}^{(1)}$ for all $s \in (t, t + \varepsilon_{t}^{(1)})$.

(c) For each $t \in [0, \infty)$, there exist $r_{t}^{(2)} \in [0, 1)$ and $\varepsilon_{t}^{(2)} > 0$ such that $\varphi(s) \leq r_{t}^{(2)}$ for all $s \in [t, t + \varepsilon_{t}^{(2)}]$.

(d) For each $t \in [0, \infty)$, there exist $r_{t}^{(3)} \in [0, 1)$ and $\varepsilon_{t}^{(3)} > 0$ such that $\varphi(s) \leq r_{t}^{(3)}$ for all $s \in (t, t + \varepsilon_{t}^{(3)})$.

(e) For each $t \in [0, \infty)$, there exist $r_{t}^{(4)} \in [0, 1)$ and $\varepsilon_{t}^{(4)} > 0$ such that $\varphi(s) \leq r_{t}^{(4)}$ for all $s \in [t, t + \varepsilon_{t}^{(4)}]$.

(f) For any nonincreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1$. 


(g) \( \varphi \) is a function of contractive factor; that is, for any strictly decreasing sequence \( \{ x_n \}_{n \in \mathbb{N}} \) in \([0, \infty)\), we have \( 0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1 \).

Let \( A \) and \( B \) be nonempty subsets of a metric space \((X, d)\). A mapping \( T : A \cup B \to A \cup B \) is called cyclic if

\[
T(A) \subset B \quad \text{and} \quad T(B) \subset A.
\]

For any nonempty subsets \( A \) and \( B \) of \( X \), denote

\[
\text{dist}(A, B) = \inf \{d(x, y) : x \in A, y \in B\}.
\]

A point \( x \in A \cup B \) is called to be a best proximity point for a mapping \( T : A \cup B \to A \cup B \) if

\[
d(x, Tx) = \text{dist}(A, B).
\]

**Definition 1.4** [3, Definition 3.1]. Let \( A \) and \( B \) be nonempty subsets of a metric space \((X, d)\). A mapping \( T : A \cup B \to A \cup B \) is called a cyclic weak light deliver mapping (abbreviated, CWLD mapping) if it is cyclic and there exists an \( \mathcal{MT} \)-function \( \varphi \) such that

\[
d(Tx, Ty) \leq \varphi(d(x, y)) \max \left\{d(x, y), \frac{1}{4}[d(Tx, x) + d(Ty, y) + d(x, Ty) + d(y, Tx)]\right\} + (1 - \varphi(d(x, y))) \text{dist}(A, B)
\]

for all \( x \in A \) and \( y \in B \).

In 2017, Lin and Wu [3] proved the following convergence theorem.

**Theorem 1.5** [3, Theorem 3.2]. Let \( A \) and \( B \) be nonempty subsets of a metric space \((X, d)\) and \( T : A \cup B \to A \cup B \) be a cyclic CWLD mapping. Then there exists a sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \( A \cup B \), defined by \( x_{n+1} = Tx_n \) for all \( n \in \mathbb{N} \), such that the following statements hold:

(a) \( d(x_{n+1}, x_{n+2}) \leq \varphi(d(x_n, x_{n+1}))d(x_n, x_{n+1}) + (1 - \varphi(d(x_n, x_{n+1}))) \text{dist}(A, B) \)

for all \( n \in \mathbb{N} \),

(b) \( \lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) = \text{dist}(A, B) \).

**Remark 1.6.** In fact, the conclusion of Lin-Wu’s convergence theorem (i.e. Theorem 1.5) was obtained as follows:

“there exists a sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \( A \cup B \) such that

\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) = \text{dist}(A, B),
\]
However, one can obtain more conclusions from the proof of [3, Theorem 3.2].

In this paper, we establish some new fixed point theorems by applying Lin-Wu’s convergence theorem.

2. New fixed point theorems

In this section, we first establish the following new fixed point theorem by applying Lin-Wu’s convergence theorem.

**Theorem 2.1.** Let \((X, d)\) be a complete metric space and \(T : X \rightarrow X\) be a selfmapping. Suppose that there exists an \(\mathcal{MT}\)-function \(\mu : [0, \infty) \rightarrow [0, 1)\) such that

\[
d(Tx, Ty) \leq \mu(d(x, y)) \max \left\{ d(x, y), \frac{1}{4} [d(Tx, x) + d(Ty, y) + d(x, Ty) + d(y, Tx)] \right\}
\]

for all \(x, y \in X\). Then \(T\) admits a unique fixed point in \(X\).

**Proof.** Let \(A = B = X\). Then \(A \cup B = X\) and \(T\) is a selfmapping on \(A \cup B\). Clearly, \(\text{dist}(A, B) = 0\) and (2.1) deduces

\[
d(Tx, Ty) \leq \mu(d(x, y)) \max \left\{ d(x, y), \frac{1}{4} [d(Tx, x) + d(Ty, y) + d(x, Ty) + d(y, Tx)] \right\} + (1 - \mu(d(x, y)))\text{dist}(A, B)
\]

for all \(x \in A\) and \(y \in B\). So \(T\) is a cyclic CWLD mapping. By applying Lin-Wu’s convergence theorem (i.e. Theorem 1.5), there exists a sequence \(\{x_n\}_{n \in \mathbb{N}}\) in \(A \cup B\), defined by \(x_{n+1} = Tx_n\) for all \(n \in \mathbb{N}\), such that the following statements hold:

(i) \(d(x_{n+1}, x_{n+2}) \leq \mu(d(x_n, x_{n+1}))d(x_n, x_{n+1}) + (1 - \mu(d(x_n, x_{n+1})))\text{dist}(A, B)\) for all \(n \in \mathbb{N}\).

(ii) \(\lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) = \text{dist}(A, B)\).

By (i), we obtain

\[
d(x_{n+1}, x_{n+2}) \leq \mu(d(x_n, x_{n+1}))d(x_n, x_{n+1}) + (1 - \mu(d(x_n, x_{n+1})))\text{dist}(A, B) = \mu(d(x_n, x_{n+1}))d(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N}.
\]

(2.2)

Since \(\mu(t) < 1\) for all \(t \in [0, \infty)\), (2.2) shows that the sequence \(\{d(x_n, x_{n+1})\}_{n \in \mathbb{N}}\) is strictly decreasing in \([0, \infty)\). Applying Theorem 1.3, we obtain

\[
0 \leq \sup_{n \in \mathbb{N}} \mu(d(x_{n+1}, x_n)) < 1.
\]
Let 
\[ \gamma := \sup_{n \in \mathbb{N}} \mu(d(x_{n+1}, x_n)). \]

So \( \gamma \in [0, 1) \). For any \( n \in \mathbb{N} \), by (2.2) again, we have
\[ d(x_{n+1}, x_{n+2}) \leq \mu(d(x_n, x_{n+1}))d(x_n, x_{n+1}) \leq \gamma d(x_n, x_{n+1}). \tag{2.3} \]

Hence, by (2.3), we get
\[ d(x_{n+1}, x_{n+2}) \leq \gamma d(x_n, x_{n+1}) \leq \cdots \leq \gamma^n d(x_1, x_2) \quad \text{for each} \quad n \in \mathbb{N}. \tag{2.4} \]

For \( m, n \in \mathbb{N} \) with \( m > n \), we get from (2.4) that
\[
d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\
\leq \gamma^{m-2} d(x_1, x_2) + \gamma^{m-3} d(x_1, x_2) + \cdots + \gamma^{n-1} d(x_1, x_2) \\
\leq \gamma^{n-1} d(x_1, x_2) + \gamma^n d(x_1, x_2) + \cdots \tag{2.5} \\
= \frac{\gamma^{n-1} d(x_1, x_2)}{1 - \gamma}.
\]

Since \( \gamma \in [0, 1) \), we know that
\[
\lim_{n \to \infty} \frac{\gamma^{n-1} d(x_1, x_2)}{1 - \gamma} = 0. \tag{2.6}
\]

By (2.5) and (2.6), we obtain
\[
\lim_{n \to \infty} \sup\{d(x_m, x_n) : m > n\} = 0,
\]
which proves that \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( X \). By the completeness of \( X \), there exists \( v \in X \) such that \( x_n \to v \) as \( n \to \infty \). Now, we want to show \( v \in \mathcal{F}(T) \). By (2.1), we get
\[
d(Tv, x_{n+1}) = d(Tv, Tx_n) \\
\leq \mu(d(v, x_n)) \max\left\{d(v, x_n), \frac{1}{4}[d(Tv, v) + d(Tx_n, x_n) + d(v, Tx_n) + d(x_n, Tv)]\right\} \\
= \mu(d(v, x_n)) \max\left\{d(v, x_n), \frac{1}{4}[d(Tv, v) + d(x_{n+1}, x_n) + d(v, x_{n+1}) + d(x_n, Tv)]\right\} \\
< \max\left\{d(v, x_n), \frac{1}{4}[d(Tv, v) + d(x_{n+1}, x_n) + d(v, x_{n+1}) + d(x_n, Tv)]\right\}
\]
for all \( n \in \mathbb{N} \). By taking the limit as \( n \to \infty \) on both sides of the last inequality, we obtain
\[
d(Tv, v) \leq \max\left\{0, \frac{1}{4}[d(Tv, v) + d(v, Tv)]\right\} = \frac{1}{2} d(Tv, v),
\]
which implies $d(v, Tv) = 0$. Hence $v \in \mathcal{F}(T)$. Finally, we show that $\mathcal{F}(T)$ is a singleton set. Assume that there exist $u, v \in \mathcal{F}(T)$ with $u \neq v$. By (2.1) again, we obtain

$$d(u, v) = d(Tu, Tv) \leq \mu(d(u, v)) \max \left\{ d(u, v), \frac{1}{4}[d(Tu, u) + d(Tv, v) + d(u, Tv) + d(v, Tu)] \right\}$$

$$= \mu(d(u, v)) \max \left\{ d(u, v), \frac{1}{2}d(u, v) \right\}$$

$$= \mu(d(u, v))d(u, v)$$

$$< d(u, v),$$

which leads to a contradiction. So $\mathcal{F}(T)$ must be a singleton set. Therefore $T$ has a unique fixed point in $X$. The proof is completed. □

Applying Theorem 2.1, we obtain some new fixed point theorems.

**Corollary 2.1.** Let $(X, d)$ be a complete metric space and $T : X \to X$ be a selfmapping. Suppose that there exists an $\mathcal{MT}$-function $\mu : [0, \infty) \to [0, 1)$ such that

$$d(Tx, Ty) \leq \mu(d(x, y))d(x, y) \quad \text{for all } x, y \in X. \quad (2.7)$$

Then $T$ admits a unique fixed point in $X$.

**Proof.** Clearly, (2.7) implies (2.1). Hence, the conclusion is immediate from Theorem 2.1. □

**Corollary 2.2.** Let $(X, d)$ be a complete metric space and $T : X \to X$ be a selfmapping. Suppose that there exists an $\mathcal{MT}$-function $\mu : [0, \infty) \to [0, 1)$ such that

$$d(Tx, Ty) \leq \frac{1}{4}\mu(d(x, y))[d(Tx, x) + d(Ty, y) + d(x, Ty) + d(y, Tx)] \quad (2.8)$$

for all $x, y \in X$. Then $T$ admits a unique fixed point in $X$.

**Proof.** Clearly, (2.8) implies (2.1). So, the conclusion is immediate from Theorem 2.1. □

**Theorem 2.2.** Let $(X, d)$ be a complete metric space and $T : X \to X$ be a selfmapping. Suppose that there exists a nondecreasing function $\alpha : [0, \infty) \to [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha(d(x, y)) \max \left\{ d(x, y), \frac{1}{4}[d(Tx, x) + d(Ty, y) + d(x, Ty) + d(y, Tx)] \right\}$$
for all $x, y \in X$. Then $T$ admits a unique fixed point in $X$.

**Proof.** Since $\alpha$ is a nondecreasing function, $\alpha$ is an $MT$-function. So, the conclusion is immediate from Theorem 2.1. □

The following results are immediate from Theorem 2.2.

**Corollary 2.3.** Let $(X, d)$ be a complete metric space and $T : X \to X$ be a selfmapping. Suppose that there exists a nondecreasing function $\alpha : [0, \infty) \to [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y) \quad \text{for all } x, y \in X.$$  

Then $T$ admits a unique fixed point in $X$.

**Corollary 2.4.** Let $(X, d)$ be a complete metric space and $T : X \to X$ be a selfmapping. Suppose that there exists a nondecreasing function $\alpha : [0, \infty) \to [0, 1)$ such that

$$d(Tx, Ty) \leq \frac{1}{4}\alpha(d(x, y))[d(Tx, x) + d(Ty, y) + d(x, Ty) + d(y, Tx)]$$  

for all $x, y \in X$. Then $T$ admits a unique fixed point in $X$.

**Theorem 2.3.** Let $(X, d)$ be a complete metric space and $T : X \to X$ be a selfmapping. Suppose that there exists a nonincreasing function $\beta : [0, \infty) \to [0, 1)$ such that

$$d(Tx, Ty) \leq \beta(d(x, y)) \max\left\{d(x, y), \frac{1}{4}[d(Tx, x) + d(Ty, y) + d(x, Ty) + d(y, Tx)]\right\}$$  

for all $x, y \in X$. Then $T$ admits a unique fixed point in $X$.

**Proof.** Since $\beta$ is a nonincreasing function, $\beta$ is an $MT$-function. Hence, the conclusion is immediate from Theorem 2.1. □

The following fixed point theorems are immediate from Theorem 2.3.

**Corollary 2.5.** Let $(X, d)$ be a complete metric space and $T : X \to X$ be a selfmapping. Suppose that there exists a nonincreasing function $\beta : [0, \infty) \to [0, 1)$ such that

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y) \quad \text{for all } x, y \in X.$$  

Then $T$ admits a unique fixed point in $X$. 
Corollary 2.6. Let $(X, d)$ be a complete metric space and $T : X \to X$ be a selfmapping. Suppose that there exists a nonincreasing function $\beta : [0, \infty) \to [0, 1)$ such that

$$d(Tx, Ty) \leq \frac{1}{4}\beta(d(x, y))[d(Tx, x) + d(Ty, y) + d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$. Then $T$ admits a unique fixed point in $X$.

Corollary 2.7. Let $(X, d)$ be a complete metric space and $T : X \to X$ be a selfmapping. Suppose that there exists $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leq \frac{\lambda}{4}[d(Tx, x) + d(Ty, y) + d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$. Then $T$ admits a unique fixed point in $X$.

Remark 2.1. Theorem 2.1, Theorem 2.2, Theorem 2.3, Corollary 2.1, Corollary 2.3 and Corollary 2.5 all improve and generalize the famous Banach contraction principle (i.e. Theorem 1.1).

Acknowledgements. The authors wish to express their thanks to Professor Wei-Shih Du for his useful suggestions and comments.

References


Received: September 29, 2020; Published: October 11, 2020