Interval Oscillation Criteria for Second-Order Nonlinear Delay Differential Equations with Damping

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Abstract
This paper is concerned with the oscillation of a class of second-order nonlinear functional differential equations with damping.

\[ x''(t) + p(t)x'(t) + f(t, x[\tau_1(t)], ..., x[\tau_m(t)], x'[\sigma_1(t)], ..., x'[\sigma_m(t)]) = 0, \quad t \geq t_0. \]

Several new oscillation criteria are established by using generalized Riccati transformations. An example is also considered to illustrate the main results.

Keywords: Oscillation; Damping; Delay; Nonlinear

1 Introduction

In this paper, we consider the oscillation of second-order nonlinear functional differential equations with damping

\[ x''(t) + p(t)x'(t) + f(t, x[\tau_1(t)], ..., x[\tau_m(t)], x'[\sigma_1(t)], ..., x'[\sigma_m(t)]) = 0, \quad t \geq t_0. \] (1.1)

Throughout the paper, we assume that solutions of (1.1) exist for any \( t \geq t_0 \). A solution of (1.1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory.

Our goal is to derive for Eq.(1.1), efficient condition for the oscillations which use information on the behavior of coefficients of equation on an infinite sequence of intervals rather than on the entire positive semi-axis.

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The oscillation problem for nonlinear delay equation such as
\[
x''(t) + q(t)f(x(\tau(t)))g(x'(t)) = 0, \quad t \geq t_0,
\]
\[
x''(t) + q(t)f(x(t))g(x'(t)) = 0, \quad t \geq t_0,
\]
\[
x''(t) + q(t)x(t) = 0, \quad t \geq t_0,
\]
have been studied by many authors with different methods, and has obtained a lot of results. Some results can be found in [1-4] and references therein.

In the presence of damping, a number of oscillation criteria [5-7] have been obtained for the following second order nonlinear differential equation
\[
(x'(t))^2 + p(t)x'(t) + q(t)f(x(t)) = 0, \quad t \geq t_0.
\]

In this paper, by using generalized Riccati transformations, we obtain several new interval criteria for oscillation, given by the behavior of equation (1.1) (or of \(p(t), f\)) only on a sequence of subinterval of \([t_0, \infty)\). Finally, an example is considered to test the efficiency of new result.

Before giving the main results, we introduce some denotations.

Let \(D = \{(t, s) : -\infty < s \leq t < \infty\}\). Then function \(H(t, s) \in C(D, R)\) is said to belong to the class \(X\), if
\[
(A_1) \quad H(t, t) = 0, \quad H(t, s) > 0 \quad \text{for } t > s; \\
(A_2) \quad H \text{ has partial derivatives } \frac{\partial H(t, s)}{\partial t} \text{ and } \frac{\partial H(t, s)}{\partial s} \text{ on } D \text{ such that }
\]
\[
\frac{\partial H(t, s)}{\partial t} = h_1(t, s) \sqrt{H(t, s)}, \quad \frac{\partial H(t, s)}{\partial s} = -h_2(t, s) \sqrt{H(t, s)},
\]
where \(h_1, h_2 \in L_{loc}(D, R^+).\)

As is well known, The following two lemmas are useful in working with nonlinear differential equations. The first can be proved similarly to [8, Lemma 3] and [9, Lemma 5.1]. The second can be proved similarly to [10, Lemma 2.1].

**Lemma 1.1.** Assume that
\[
(B_1) \quad p \in C([t_0, \infty), [0, \infty)), \lim_{t \to \infty} \int_{t_0}^{t} \exp \left(-\int_{t_0}^{s} p(\tau) \, d\tau \right) \, ds = \infty, \text{ for every } t \geq t_0; \\
(B_2) \quad \tau_i(t), \sigma_i(t) \in C([t_0, \infty), R), \text{ for } t \geq t_0, \text{ and } \lim_{t \to \infty} \tau_i(t) = \infty, \quad i = 1, 2, ..., m. \\
(B_3) \quad f \in C([t_0, \infty) \times R^{2m}, R) \text{ satisfies the one-side estimate }
\]
\[
f(t, x_1, ..., x_m, x_1', ..., x_m') \text{sign} x_1 \geq q(t) \sum_{i=1}^{m} \mu_i |x_i|, \quad x_i x_1 \geq 0 (i = 1, 2, ..., m),
\]
where \(\mu_i (i = 1, 2, ..., m)\) are nonnegative constants and \(\sum_{i=1}^{m} \mu_i > 0, q \in C([t_0, \infty), [0, \infty))\) and \(q(t)\) is not identically zero on any ray \([t^*, \infty) \subset [t_0, \infty)\). Then if \(x(t)\) is a nonoscillatory solution of Eq.(1.1), we have
\[
x(t)x'(t) > 0, \quad \text{for all large } t.
\]

**Lemma 1.2.** Assume that \(x(t) \in C^2([t_0, \infty), R)\) satisfies
\[
x(t) > 0, \quad x'(t) > 0, \quad x''(t) \leq 0, \quad t \geq t_0,
\]
then for each \(0 < k_i < 1\), there exists a \(T \geq t_0\), such that
\[
x(\tau_i(t)) \geq k_i x(t) \frac{\tau_i(t)}{t}, \quad t \geq T, \quad i = 1, 2, ..., m.
\]


2 Main Results

**Theorem 2.1.** In Eq.(1.1), suppose that conditions $(B_1)-(B_3)$ hold. If, for each sufficiently large $T \geq t_0$, there exist $g(t) \in C^1([t_0, \infty), R)$, $H \in X$ and $a, b, c \in R$ such that $T \leq a < c < b$ and

\[
\frac{1}{H(b, c)} \int_c^b H(b, s)\psi(s) - \frac{1}{4}a(s) \left[ h_2(b, s) + \sqrt{H(b, s)p(s)} \right]^2 \, ds
\]

\[
+ \frac{1}{H(c, a)} \int_a^c H(s, a)\psi(s) - \frac{1}{4}a(s) \left[ h_1(s, a) - \sqrt{H(s, a)p(s)} \right]^2 \, ds > 0
\]

(2.1)

where

\[
a(t) = \exp \left( -2 \int^t g(s)ds \right), \psi(t) = a(t) \left( g^2(t) + q(t)\sum_{i=1}^m k_i \mu_i \frac{\tau_i(t)}{t} - p(t)g(t) - g'(t) \right),
\]

(2.2)

then Eq.(1.1) is oscillatory.

**Proof.** Otherwise, let $x(t)$ be a non-oscillatory solution of Eq.(1.1). Without loss of generality, we may assume that $x(t) > 0$ on $[T_0, \infty)$ for $T_0 \geq t_0$. As $\lim_{t \to \infty} \tau_i(t) = \infty$, there exists $T' \geq T_0$ such that $\tau_i(t) \geq T_0$, $t \geq T'$, $i = 1, 2, ..., m$. Hence $x(\tau_i(t)) > 0$, $t \geq T'$, $i = 1, 2, ..., m$. By Lemma 1.1, there exists $T'' \geq T'$ such that $x'(t) > 0$ for $t \geq T''$. From Eq.(1.1), we can obtain $x''(t) \leq 0$ for $t \geq T''$. Define

\[
w(t) = a(t) \left( \frac{x'(t)}{x(t)} + g(t) \right), \quad t \geq T''.
\]

(2.3)

From (1.1), we have

\[
w'(t) = -2g(t)w(t) + a(t) \left\{ -p(t)\frac{x'(t)}{x(t)} - f(t, x(\tau_1(t)), ..., x(\tau_m(t)), x'(\sigma_1(t)), ..., x'(\sigma_m(t))) \right\}
\]

\[
+ a(t) \left\{ g'(t) - \frac{[x'(t)]^2}{x^2(t)} \right\}
\]

\[
\leq (-2g(t) + p(t))w(t) + a(t) \left\{ -p(t)\left[ \frac{w(t)}{a(t)} - g(t) \right] - q(t)\sum_{i=1}^m \mu_i x(\tau_i(t)) \right\}
\]

\[
+ a(t) \left\{ g'(t) - \left( \frac{w(t)}{a(t)} - g(t) \right)^2 \right\}.
\]

(2.4)

From Lemma 1.2, there exists $T''' \geq T''$, making latter inequality yields

\[
w'(t) \leq -\psi(t) - P(t)w(t) - \frac{w^2(t)}{a(t)}, \quad t \geq T''',
\]

(2.5)

where $\psi(t)$ is defined by (2.2).

Setting $T \geq T'''$, multiplying (2.5) by $H(t, s)$ integrating it with respect to $s$ from $c$ to $t$ for
$t \in [c, b)$, using $(A_1)$ and $(A_2)$, we get
\[
\int_c^t H(t, s)\psi(s)ds \leq -\int_c^t H(t, s)w'(s)ds - \int_c^t H(t, s)p(s)w(s)ds - \int_c^t H(t, s)\frac{w^2(s)}{a(s)}ds
\]
\[
\begin{align*}
&= H(t, c)\psi(c) - \int_c^t \left[ \frac{\partial H}{\partial s}(t, s)w(s) + H(t, s)p(s)w(s) + H(t, s)\frac{w^2(s)}{a(s)} \right] ds \\
&= H(t, c)w(c) - \int_c^t \left( h(t, s)\sqrt{H(t, s)}w(s) + H(t, s)p(s)w(s) + H(t, s)\frac{w^2(s)}{a(s)} \right) ds \\
&= H(t, c)w(c) - \int_c^t \left\{ H(t, s)w(s) + \frac{\sqrt{a(s)}}{2} \left[ h_2(t, s) + \sqrt{H(t, s)p(s)} \right] \right\} ds \\
&\quad + \frac{1}{4} \int_c^t a(s) \left[ h_2(t, s) + \sqrt{H(t, s)p(s)} \right]^2 ds \\
&\leq H(t, c)w(c) + \frac{1}{4} \int_c^t a(s) \left[ h_2(t, s) + \sqrt{H(t, s)p(s)} \right]^2 ds.
\end{align*}
\]
(2.6)

Let $t \to b^-$ in the above, diving both sides by $H(b, c)$, we obtain
\[
\frac{1}{H(b, c)} \int_c^b H(b, s)\psi(s) - \frac{1}{4} a(s) \left[ h_2(b, s) + \sqrt{H(b, s)p(s)} \right]^2 ds \leq w(c).
\]
(2.7)

Similarly, multiply (2.5) by $H(s, t)$, integrate it with respect to $s$ from $t$ to $c$ for $t \in (a, c]$ and use $(A_1)$ and $(A_2)$, then we get
\[
\int_t^c H(s, t)\psi(s)ds \leq -\int_t^c H(s, t)w'(s)ds - \int_t^c H(s, t)p(s)w(s)ds - \int_t^c H(s, t)\frac{w^2(s)}{a(s)}ds
\]
\[
\begin{align*}
&= -H(c, t)w(c) - \int_t^c \left[ \frac{\partial H}{\partial s}(s, t)w(s) + H(s, t)p(s)w(s) + H(s, t)\frac{w^2(s)}{a(s)} \right] ds \\
&= -H(c, t)w(c) + \int_t^c \left( h_1(s, t)\sqrt{H(s, t)} - H(s, t)p(s) \right)w(s) - H(s, t)\frac{w^2(s)}{a(s)} ds \\
&= -H(c, t)w(c) - \int_t^c \left\{ H(s, t)w(s) - \frac{\sqrt{a(s)}}{2} \left[ h_1(s, t) - \sqrt{H(s, t)p(s)} \right] \right\} ds \\
&\quad + \frac{1}{4} \int_t^c a(s) \left[ h_1(s, t) - \sqrt{H(s, t)p(s)} \right]^2 ds \\
&\leq -H(c, t)w(c) + \frac{1}{4} \int_t^c a(s) \left[ h_1(s, t) - \sqrt{H(s, t)p(s)} \right]^2 ds.
\end{align*}
\]
(2.8)

Let $t \to a^+$ in the above, diving both sides by $H(c, a)$, we obtain
\[
\frac{1}{H(c, a)} \int_a^c H(s, a)\psi(s) - \frac{1}{4} a(s) \left[ h_1(s, a) - \sqrt{H(s, a)p(s)} \right]^2 ds \leq -w(c).
\]
(2.9)

Now we assert that $x(t)$ has at least one zero in $(a, b)$. Otherwise adding (2.7) and (2.9) would yield an inequality which contradicts the assumption (2.1). Pick a sequence, $T \leq \tau_1 < \tau_2 < \ldots$, satisfying $\tau_n \to \infty$ as $n \to \infty$. For each $n \in N$, there exist $a_n, c_n, b_n \in R$ such that $\tau_n \leq a_n < c_n < b_n$ and
\[
\begin{align*}
&\frac{1}{H(b_n, c_n)} \int_{c_n}^{b_n} H(b_n, s)\psi(s) - \frac{1}{4} a(s) \left[ h_2(b_n, s) + \sqrt{H(b_n, s)p(s)} \right]^2 ds \\
&\quad + \frac{1}{H(c_n, a_n)} \int_{a_n}^{c_n} H(s, a_n)\psi(s) - \frac{1}{4} a(s) \left[ h_1(s, a_n) - \sqrt{H(s, a_n)p(s)} \right]^2 ds > 0,
\end{align*}
\]
(2.10)
According to the above proof, any solution \( x(t) \) of Eq. (1.1) has at least one zero in \((a_n, b_n)\). Taking into account that \( a_n \to +\infty \) and \( b_n \to +\infty \) as \( n \to +\infty \), we see that every solution has arbitrary large zeros. Thus, every solution of Eq. (1.1) is oscillatory. The proof is complete.

**Theorem 2.2.** Assume \((B_1)-(B_3)\) hold. Suppose for each sufficiently large \( l \geq t_0 \), there exist \( H \in X \) and \( g(t) \in C^1([t_0, \infty), R) \) such that

\[
\limsup_{t \to \infty} \int_{t}^{l} H(s, l) \psi(s) - \frac{1}{4} a(s) \left[ h_1(s, l) - \sqrt{H(s, l)p(s)} \right]^2 ds > 0 \tag{2.11}
\]

and

\[
\limsup_{t \to \infty} \int_{t}^{l} H(t, s) \psi(s) - \frac{1}{4} a(s) \left[ h_2(t, s) + \sqrt{H(t, s)p(s)} \right]^2 ds > 0, \tag{2.12}
\]

where \( a(t), \psi(t) \) are defined as in Theorem 2.1, then every solution of Eq. (1.1) is oscillatory.

**Proof.** For any sufficient large \( T \geq t_0 \), let \( a = T \). In (2.11), we choose \( l = a \), then there exists \( c > a \) such that

\[
\int_{a}^{c} H(s, a) \psi(s) - \frac{1}{4} a(s) \left[ h_1(s, a) - \sqrt{H(s, a)p(s)} \right]^2 ds > 0. \tag{2.13}
\]

Setting \( l = c \) in (2.12). Then there exists \( b > c \) such that

\[
\int_{c}^{b} H(b, s) \psi(s) - \frac{1}{4} a(s) \left[ h_2(b, s) + \sqrt{H(b, s)p(s)} \right]^2 ds > 0. \tag{2.14}
\]

Combing (2.13) and (2.14), we obtain (2.1), The conclusion thus comes from Theorem 2.1. The proof is complete.

For the case where \( H := H(t - s) \in C(D, R) \in X \), we have that \( h_1(t - s) = h_2(t - s) \) and denote them by \( h(t - s) \). The subclass of \( X \) containing such \( H(t - s) \) is denoted by \( X_0 \). Applying Theorem 2.1 to \( X_0 \), we obtain the following Theorem.

**Theorem 2.3.** Assume that \((B_1)-(B_3)\) hold and for sufficiently large \( T \geq t_0 \) there exist a function \( g(t) \in C^1([t_0, \infty), R) \), \( H \in X_0 \), and \( a, c \in R \) such that \( T \leq a < c \) and

\[
\int_{a}^{c} H(s - a) \psi(s) - \frac{1}{4} a(s) \left[ h(s - a) - \sqrt{H(s - a)p(s)} \right]^2 ds + \int_{a}^{c} H(s - a) \psi(2c - s) - \frac{1}{4} a(2c - s) \left[ h(s - a) + \sqrt{H(s - a)p(2c - s)} \right]^2 ds > 0, \tag{2.15}
\]

where \( a(t), \psi(t) \) is defined as in Theorem 2.1, then Eq. (1.1) is oscillatory.

**Proof.** Let \( b = 2c - a \). Then \( H(b - c) = H(c - a) = H(\frac{b - a}{2}) \), and for any \( w \in L[a, b] \), we have

\[
\int_{c}^{b} w(s)ds = \int_{a}^{c} w(2c - s)ds.
\]

Hence

\[
\int_{c}^{b} H(b - s)\psi(s)ds = \int_{a}^{c} H(s - a)\psi(2c - s)ds,
\]
Thus (2.15) holds implies that (2.1) holds for $H \in X_0$. Therefore every solution of Eq.(1.1) is oscillatory by Theorem 2.1.

From the above oscillation criteria, we can obtain different sufficient conditions for oscillation of all solutions of Eq.(2.1) by different choices of $H(t, s)$.

Let $H(t, s) = (t - s)^\lambda$, $t \geq s \geq t_0$, $\lambda > 1$ is a constant, $h(t - s) = \lambda(t - s)^{\frac{\lambda}{2} - 1}$. Based on Theorem 2.2, we obtain the following corollary.

**Corollary 2.4.** Let (B1)-(B3) hold. Suppose that for each sufficiently large $l \geq t_0$ and some $\lambda > 1$ there exist $H \in X$ and $g(t) \in C^1([t_0, \infty), \mathbb{R})$ such that

$$
\lim_{l \to \infty} \sup \frac{1}{l^{\lambda-1}} \int_l^t (s - l)^\lambda \left\{ \psi(s) - \frac{1}{4} a(s) \left[ \frac{\lambda}{s - l} - p(s) \right]^2 \right\} ds > 0, \quad (2.17)
$$

and

$$
\lim_{l \to \infty} \sup \frac{1}{l^{\lambda-1}} \int_l^t (t - s)^\lambda \left\{ \psi(s) - \frac{1}{4} a(s) \left[ \frac{\lambda}{t - s} + p(s) \right]^2 \right\} ds > 0, \quad (2.18)
$$

where $a(t)$ and $\psi(t)$ are defined as in Theorem 2.1. Then Eq.(1.1) is oscillatory.

## 3 Examples

In the following, we give an example to show the sharpness of our results.

$$
\frac{x''(t)}{t} + \frac{m}{t} x' + \frac{r}{t} x(\sigma t) \frac{1 + e^t}{(3 + \cos^2 t)(n - \sin^2 t)} \sqrt{1 + x'^2(t - \xi)} \left( \frac{1}{3} + \frac{1}{2 + \cos^2(t - 2\pi)} \right) = 0 \quad (3.1)
$$

where $t \geq t_0 > 2\pi$ and $r > 0$, $\sigma > 0$, $\xi > 0$, $m > 0$ are constants. It is easy to see that conditions (B1) - (B3) hold. Let us apply with $H(t, s) = (t - s)^2$, $g(t) = 0$, so that $\psi(t) = \frac{kr\sigma(1 + e^t)}{t(3 + \cos^2 t)(n - \sin^2 t)}$.

A straightforward computation yields

$$
\lim_{l \to \infty} \sup \int_l^t H(s, l)\psi(s) - \frac{1}{4} a(s) \left[ h_1(s, l) - \sqrt{H(s, l)p(s)} \right]^2 ds
$$

$$
= \lim_{l \to \infty} \sup \int_l^t \frac{kr\sigma(s - l)^2}{4n} \frac{1 + e^t}{(3 + \cos^2 t)(n - \sin^2 t)} \left( \frac{1}{3} + \frac{1}{2 + \cos^2(t - 2\pi)} \right) ds
$$

$$
> \lim_{l \to \infty} \int_l^t \frac{kr\sigma(s - l)^2}{4n} \frac{1 - \frac{1}{4} \left( 2 - \frac{m(s - l)}{s} \right)^2}{(2 - m)s + ml} ds = \infty \quad (3.2)
$$

and
\[
\limsup_{t \to \infty} \int_t^l H(t,s)\psi(s) - \frac{1}{4}a(s) \left[ h_2(t,s) + \sqrt{H(t,s)p(s)} \right]^2 ds
\]
\[
= \limsup_{t \to \infty} \int_t^l kr\sigma(t-s)\frac{1+e^s}{(3+\cos^2 s)(n-\sin^2 s)} - \frac{1}{4} \left[ 2 + \frac{m(t-s)}{s} \right]^2
\]
\[
> \limsup_{t \to \infty} \int_t^l \frac{kr\sigma(s-l)}{4n} - \frac{1}{4} \left[ 2 + \frac{m(t-s)}{s} \right]^2
\]
\[
= \limsup_{t \to \infty} \int_t^l \frac{kr\sigma(t-s)}{4n} - \frac{(2-m)s+mt}{4s^2} = \infty
\]

This means (2.11) and (2.12) hold. By Theorem 2.2, Eq.(3.1) is oscillatory.

References


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