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Solution of Impulsive Hamilton-Jacobi Equation and Its Applications

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Abstract

In this paper, we formulate a corresponding impulsive Hamilton-Jacobi equation and obtained its solution representation following the method of characteristics. The solution obtained is applied to traffic flow problem, and the result shows that an impulse is observed when a jam density is reached. At this point the flux is zero (and the flow is the function of the impulsive term alone).

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1 Intorduction

Many dynamical processes in real life are proned to abrupt changes and its natural to assume that such perturbations act in most cases instantaneously in the form of impulses. This is also particularly when the duration of the perturbations are negligible compared to the duration of the whole process of the system or phenomena. Impulsive differential equations or differential equations with impulse effects appears as a natural description of observed evolutional phenomena of several real world problems [12]. Many ideas are involved regarding impulses within a system thereby informing the model. This

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may be either an impulsive equation that models an impulsive jump defined by a jump function at the instance where the impulses occur or a continuous-time differential equation which governs the state of the phenomena between impulses. Hamilton- Jacobi equation is one of the most widely used equations to model and solve problems that deals with dynamic network flow, or to state that there exist many mathematical models meant to deal with road traffic in particular including Hamilton- Jacobi equation. The lighthill- whitham [8] and Richard [11] presentations and equations were used to model the same physical phenomenon and those formulations were equivalently applied on the highway transportation problems. However, the Hamilton- Jacobi theory has been mainly followed and exploited till now in the frame of an infinite one dimensional road problems [2, 3]. Imbert et al [6] introduced Hamilton- Jacobi equations for modeling junction problems to traffic flow and its application. Several works [1, 4, 5] had separately introduced an Hamilton-Jacobi formulation for networks of which they need to deal with tedious coupling conditions at each junctions. Given that impulsive differential equation is appearing new in research and is a very important concept to deal with discontinuities, also the context of traffic flow is of macroscopic nature based on partial differential equations for the traffic flow density, and had not been looked into to the best of our knowledge. We followed the work of Zhang et al [13] and [6] to formulate and obtained the solution to impulsive Hamilton-Jacobi equation, applied to traffic flow and also give illustration.

2 Preliminaries

In this section, some basic definitions and auxiliary results are presented.

Definition 2.1 *Hamiltonian- let X be a banach space and, Ω a subset of X then a function $H : \omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is called Hamiltonian provided it is the total energy of the system.*

Definition 2.2 *Hamilton-Jacobi equation associated with the Hamiltonian H is of the form*

$$u_t + H(t, x, u, Du) = 0 \quad on \quad (0, \infty) \times \mathbb{R}^n \tag{2.1}$$

$$u(0, x) = g(x) \quad on \quad (t = 0) \times \mathbb{R}^n \tag{2.2}$$

where $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the unknown function, $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the initial value $t = 0$. H, the Hamiltonian possessing various expressions depending on the phenomena. One of the Legendre transform for the Hamiltonian is

$$H = L^*(p) = p.q(p) - L(q(p)) \tag{2.3}$$

$$for \quad L^*(p) = \sup[p.q - L(q)], \quad (p \in \mathbb{R}^n) \tag{2.4}$$

where P is the generalized momentum corresponding to the position x, also the variable for which we substitute the gradient Du and q is the flux function [7]

Definition 2.3 *The function $u(x,t) \in C((0,T) \times \mathbb{R}^n, \mathbb{R})$ satisfying equation 21-22 is called the solution of the Hamilton-Jacobi equation with the representation.*

$$u(t,x) = g(x) + \int_0^t p.D_pH(x,p)ds \tag{2.5}$$

Let the Banach space X be given an infinite norm

$$\|.\| = \sup\{|u(t,x) : (t,x) \in (0,T) \times \Omega\} \tag{2.6}$$

and u let undergo discontinuity at $t = t_k$ k= 1,2,——m, then by Rankine-Hugoniot condition [9] we have

$$[u](t_k) = u^r(t_k) - u^l(t_k) \tag{2.7}$$

Define $u(t_k^+) = \lim_{t>t_k \to t_k} u(t)$ and $u(t_k^-) = \lim_{t<t_k \to t_k} u(t)$ where $0 < t_1 < t_2 \ldots < t_m = T$ corresponding to u^r and u^l respectively. Then at the point of discontinuities $t = t_k$ as fixed moment we have,

$$\Delta u(t_k,x) = u(t_k^+,x) - u(t_k^-,x)[10] \tag{2.8}$$

we now formulate the impulsive Hamilton-Jacobi equation and obtain its solution

3 Main Result

Let the unknown function u of system ((2.1)-(2.2)) undergo abrupt changes at designate position $t = t_k k = 1, 2...m$, we define thus:

$$\begin{aligned} u_t(t,x) + H(t,x,u,Du(t,x)) &= 0 \text{ on } \mathbb{R}^n \setminus C(t_k, t_{k+1}) & (3.1)\\ t &\in [0,T]\setminus[t_1,...,t_k],\ t \neq t_k, \\ u(0,x) &= g(x) \text{ on } [t=0] \times \mathbb{R}^n & (3.2)\\ \triangle u(t_k,x) &= I_k(u(t_k,x)),\ t = t_k & (3.3) \end{aligned}$$

where $u : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$, $\triangle u = D_x u$ and $I_k(u(t_k,x)) \in C(\mathbb{R},\mathbb{R})$ is the impulse effect. Equations (3.1 - 3.3) is called Impulsive Hamilton-Jacobi equation. By the method of characteristics, let $z(t) = u(t,x)$ solves equation 3.1-3.3 using the method of characteristics, and defining $z(t) = u(t,x)$ while x (t) and p(t) are solutions as defined of the equations

$$\begin{aligned} \dot{x}(t) &= D_p[u_t + H(x,p)] = D_pH(x,p) & (3.4)\\ \dot{p}(t) &= -D_z[u_t + H(x,p)].p - D_x[u_t + H(x,p)] = -D_xH(x,p) & (3.5)\\ \dot{z}(t) &= \frac{\partial u}{\partial x_i}\frac{dx_i}{dt} = p.D_p[u_t + H(x,p)] = p.D_pH(x,p). & (3.6) \end{aligned}$$

Integrating equation 3-6 with respect to t yields

$$\begin{aligned}\int_0^t \dot{z}(s)ds &= \int_0^{t_1} \dot{z}(s)ds + \int_{t_1}^{t_2} \dot{z}(s)ds + \int_{t_2}^{t_3} \dot{z}(s)ds + \ldots + \int_{t_{k-1}}^{t_m} \dot{z}(s)ds \\ &= \int_0^{t_m} p.D_pH(x,p)ds. \\ \Longrightarrow u(t,x) &= g(x) + \int_0^{t_m} pD_pH(x,p)ds + \sum_{k=1}^{m} I_k(u(t_k,x))\end{aligned}$$

Equation (3.7) is the solution representation for impulsive Hamilton- Jacobi equation. Now for application to traffic flow, we define some terms

Definition 3.1 *Traffic flow : This is the study of interactions between means of transportation and infrastructures with the aim of understanding on optional road network, the efficient movement of traffic and minimal traffic congestions.*

Definition 3.2 *Lanes: These are parallel corridors of traffic where speed are defined based on distances covered per unit time. The boundaries represent density function of time and position which has initial value at position* $x \in \Omega$ *i.e* $g(t) = u(t,o)$ *point of consideration.*

Definition 3.3 *Junction: Let* J_i *be lanes, then* $J_i \ \ i = 1, 2...N$ *is called the junction such that x=0 at that point. The following assumptions hold*

A1 that the proportion of cars going and coming from branch i =1, 2——N is a fixed number $\gamma_i > 0$

A2 let $\rho(t,x) > 0$ *be the car density at time t and at position x on branch k, k=1, 2,—–N. then car densities are given to be solutions of non- linear equations of the form*

$$\rho_t^i(t,x) + (q^i(\rho^i(t,x)))_x = 0 \quad i = 1, 2..N \tag{3.7}$$

representing in and out going of cars on the lanes [6], where q is the flux function $q^i : \mathbb{R} \longrightarrow \mathbb{R}$ *The assumption A1 indicates that for purpose of complexity* γ_i *represent car density within the given interval in the i-lane.*

Let the solution representation, equation 3.7 be considered to be

$$u^j(t,x) = g(t) + \frac{1}{\gamma_j}\int_0^x \rho^j(t,y)dy + \sum_{k=1}^{m} I_k(u(t_k,x))$$

and differentiating with respect to t and x yield

$$\begin{aligned} u_t^j(t,x) &= g'(t) + \frac{1}{\gamma_j}\int_0^x \rho_t^j(t,y)dy + \sum_{k=1}^{m} I_k(u_t(t_k,x)) \\ &= g'(t) - \frac{1}{\gamma_j}\int_0^x (q^j(\rho^j(t,y)))_x dy + \sum_{k=1}^{m} I_k(u_t(t_k,x)) \\ &= g'(t) - \frac{1}{\gamma_j}q^j(\rho^j(t,x)) + \frac{1}{\gamma_j}q^j(\rho^j(t,0^+)) + \sum_{k=1}^{m} I_k(u_t(t_k,x)) \end{aligned}$$

This shows that for $j = m+1, m+2, \ldots, m+n$

$$u_t^j(t,x) + \frac{1}{\gamma_j}q^j(\rho^j(t,x)) = h^j(t), \tag{3.8}$$

whence $h^j(t) = g'(t) + \frac{1}{\gamma_j}q^j(\rho^j(t,0^+)) + \sum\limits_{k=1}^{m} I_k(u_t(t_k,x))$.
Comparing with equation(3.8), if $h^j(t) \equiv 0$, we determine the value of $g(t)$ from

$$g(t) = -\frac{1}{\gamma_j}\int_0^x q^j(\rho^i(t,0))dy - \sum_{k=1}^{m} I_k(u(t_k,x))$$

Hence the instantaneous flux passing through the junction would be given by

$$u_t^i(t,0) = g(t) = -\int_0^m \sum_{i=0}^{m}(\rho^i(s,0))ds + \sum_{k=1}^{m} I_k^*(u(t_k,x)) \tag{3.9}$$

For illustration, we noted that time and location are the factors for the variation of flow and density. When there are no vehicles on the lane, the density is zero. If vehicles get increasing, the maximum density within an interval may be reached called jam density then the flow is zero.

Consider the problem

$$\begin{aligned} u_t(t,x) &+ H(t,x,u,Du(t,x)) = 0 \text{ on } \mathbb{R}^n \setminus C(t_k,t_{k+1}) \\ t &\in [0,T]\backslash[t_1,...,t_k],\ t \neq t_k, \\ u(0,x) &= g(x) = \begin{cases} x & x>0 \\ -x & x<0 \end{cases} \text{ on } [t=0]\times\mathbb{R}^n \\ \triangle u(t_k,x) &= I_k(u(t_k,x)) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases} \ t=t_k, \end{aligned}$$

where $H(t,x,u,Du(t,x)) = [H(u)]_x = DuDx - u^3 - u^2$

The solution is as shown below using matlab.

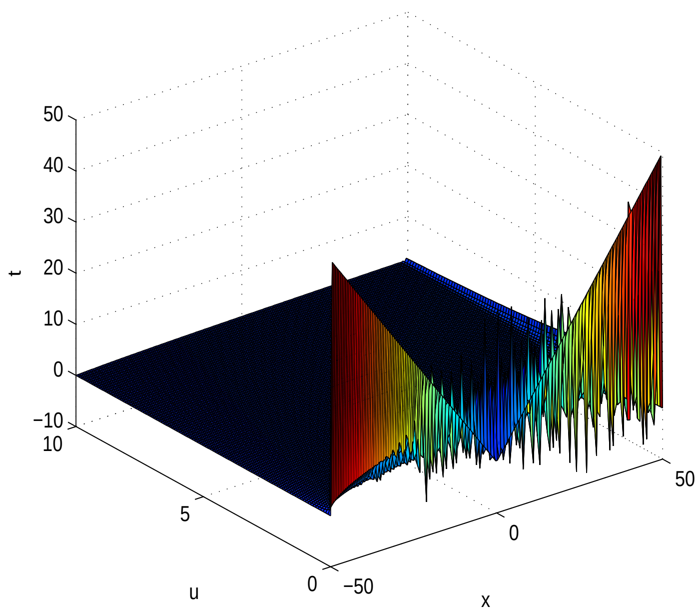

Figure 1: Surf(x,t,u) of example 1

The solution of problem 3.11-3.13 as shown in the figure expressed the flow $u(t,x)$ as a function of time t and density at any given time.The graph indicate that the flux begins with a steady flow, which then decreases as density increases with some negligible impulses experiences at points t_k, until a jam density is reached (at the junction). The flux at this point is zero (and the flow function $u(t,x) = I_k(u_k,(t_k,x))$). The flow then increases with some negligible impulses experienced, until it assumed a steady flow. This phenomenon is liken to the traffic flow from different lane towards a round about.

4 Conclusion and Recommendation

This paper has helped to indicate solutions of Hamilton- Jacobi equations at various points of impulses time t_k. It is necessary and desirable to explore on this impulsive Hamiltonian- Jacobi equations varying the Hamiltonian. It is therefore recommended for further research on phenomenon like gas pipelines, blood vessels and internet communications wireless networks for which their processes may often be characterized by abrupt changes.

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