The Log-Conditions for
the Variable Exponent Hardy Inequality

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Abstract

In this note we discuss a logarithmic regularity condition in neighborhood of the origin on the exponent function \( p(x) \) in dependence of the weights \( v, \omega \) for the variable exponent Hardy inequality

\[
\left\| v(x)^{\frac{1}{p(x)}} \int_0^x f(t) \, dt \right\|_{L^{p(x)}(0,1)} \leq C \left\| \omega(x)^{\frac{1}{p(x)}} f(x) \right\|_{L^{p(x)}(0,1)}
\]

to hold.

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1. Introduction

There are several papers devoted to the validity of the variable exponent Hardy inequality (see, the references below)

\[
\left\| v^{\frac{1}{p(x)}} \int_0^x f(t) \, dt \right\|_{L^{p(x)}(0,1)} \leq C \left\| \omega^{\frac{1}{p(x)}} f(\cdot) \right\|_{L^{p(x)}(0,1)} \quad (1.1)
\]
Let \( v, \omega : (0, l) \to [0, \infty) \) be positive measurable functions and \( p^- = \inf\{p(x) : x \in (0, l)\} > 1 \), \( p^+ = \sup\{p(x) : x \in (0, l)\} < \infty \); then the inequality (1.1) holds under certain conditions on the weight functions \( v, w \) and the exponents \( p(x), q(x) \) near the origin. Two type conditions arise here: a balance condition on the weights and a regularity condition on the exponents (see, below). Necessary and sufficient conditions for the validity of general inequality (1.1) were found in [9] for the case of \( p(x) \leq q(x) \), in [11] for cases \( p(0) \leq q(0) \), in [10] for cases \( p(0) > q(0) \), in [1] for mixed cases. We refer also to the papers [2], [3], [4], [5], [16], [15], [12], [13], [14] on the variable exponent Hardy inequality, where obtained essential results.

The inequality

\[
\| x^{\beta(x)-1} H f(\cdot) \|_{L^p(0,l)} \leq C \| x^{\beta(x)} f(\cdot) \|_{L^p(0,l)}
\]  
(1.2)

is a particular case of (1.1) when \( p(x) = q(x) \), \( v(x) = x^{\beta(x)-1} \) and \( \omega(x) = x^{\beta(x)} \), where \( Hf(x) = \int_0^x f(t)dt \) is the Hardy operator. For the constant exponents and \( p > 1, \beta \in \mathbb{R} \) this inequality holds if \( \beta < 1 - \frac{1}{p} \) (see, e.g. the monograph [8]).

To state some known results, we introduce the following classes of measurable functions. We say, \( s(x) : (0, \infty) \to \mathbb{R} \) is in the class \( \Lambda_0 \) if there exist constants \( C_1, \delta > 0 \) such that

\[
|s(x) - s(0)| \ln \frac{1}{x} \leq C_1, \quad x \in (0, \delta) \tag{1.3}
\]

and is in \( \Lambda_1 \) if there exist constants \( C_2, \delta > 0 \) such that

\[
\left| s(x) - s\left(\frac{x}{2}\right) \right| \ln \frac{1}{x} \leq C_2, \quad x \in (0, \delta). \tag{1.4}
\]

Necessary and sufficient conditions on the \( p(x) \) and \( \beta(x) \) for the validity of the inequality (1.2) with a constant \( C > 0 \) not depending \( f(x) \) is \( \beta(0) < 1 - \frac{1}{p(0)} \) provided the exponents \( p, \beta \) are continuous near the origin and the conditions \( p, \beta \in \Lambda_0 \) are satisfied (see, e.g. [2], [3], [4], [16]). In [5] (see, also [15, 6]) it was proved that the condition \( p, \beta \in \Lambda_1 \) is necessary for the inequality (1.2) to hold the \( \beta \) is a constant. Also it was proved in [15] that the condition \( \Lambda_1 \) is a sufficient one for the inequality (1.2) to hold if the constant \( C_2 \) in the condition \( \Lambda_1 \) for \( p \) is sufficiently small near the origin. Notice the condition \( \Lambda_1 \) is weaker then \( \Lambda_0 \). Indeed, the function \( p(x) = 2 + \frac{C}{\sqrt{\ln x}}, \quad 0 < x < l \) satisfies the condition \( \Lambda_1 \) but not \( \Lambda_0 \).
In this note, we shall focus on the results of sufficiency and necessity of log regularity conditions for the inequalities (1.1) and (1.2) to hold in the finite interval (0, l).

The space of functions $L^{p(\cdot)}(0, l)$ is introduced as the class of measurable functions $f(x)$ in $(0, l)$, which have a finite $I_p(f) := \int_0^1 |f(x)|^{p(x)} \, dx$ modular. A norm in $L^{p(\cdot)}(0, l)$ is given in the form

$$\|f\|_{L^{p(\cdot)}(0, l)} = \inf \left\{ \lambda > 0, I_p \left( f, \frac{f}{\lambda} \right) \leq 1 \right\}$$

We use the letters $C, C_1, C_2, \ldots$ to denote different positive constants, of which value is not essential for our purposes.

As to the basic properties of spaces $L^{p(\cdot)}$, we refer to [7].

2. Main Results

We shall state some sufficiency and necessity assertions concerning the inequalities (1.1) and (1.2). Set $W(x) = \int_0^x \sigma(t) \, dt$, where $\sigma(x) = \omega(x)^{-\frac{p(x)}{p}} - 1$. For a fixed $x \in (0, l)$ determine a number $0 < x_1 < x$ such that $W(x_1) = \frac{1}{2} W(x)$. We say, the function $s(x) : (0, l) \rightarrow \mathbb{R}$ is in the class of $\Lambda_0$ if there exist $C_3, \delta > 0$ such that

$$|s(x) - s(0)| \ln \frac{1}{W(x)} \leq C_3, \quad 0 < x < \delta \quad (2.1)$$

and is in the class of $\Lambda_1$ if there exist $C_4, \delta > 0$ such that

$$|s(x) - s(x_1)| \ln \frac{1}{W(x)} \leq C_4, \quad 0 < x < \delta. \quad (2.2)$$

**Theorem 1.** [4],[5] Let $\beta \in \mathbb{R}$ and $p : (0, l) \rightarrow [1, \infty)$ be an increasing function on $(0, \varepsilon)$ such that $p(x)$ is continuous at $x = 0$ and $\beta < 1 - \frac{1}{p(0)}$, $p^- > 1$; then for the inequality (1.2) to hold it is necessary that $p(\cdot) \in \Lambda_1$.

**Theorem 2.** [4],[5] Let $p \in \mathbb{R}$ and $\beta : (0, l) \rightarrow \mathbb{R}$ be a decreasing function on $(0, \varepsilon)$ such that $\beta(x)$ is continuous at $x = 0$ and $\beta(0) < 1 - \frac{1}{p(0)}$, $p^- > 1$; then for the inequality (1.2) to hold it is necessary that $\beta(\cdot) \in \Lambda_1$.

**Theorem 3.** [4],[5] Let $p, \beta : (0, l) \rightarrow \mathbb{R}$ be measurable functions such that $1 < \beta^-, \beta^+ < \infty$, $1 < p^- < p^+ < \infty$; then for the inequality (1.2) to hold in the interval $(0, l)$ it is sufficient that $\beta, p \in \Lambda_0$, whenever $\beta(0) < 1 - \frac{1}{p(0)}$. 
Theorem 4. Let \( p, \sigma : (0, l) \to \mathbb{R} \) be measurable functions such that \( 1 < p^-, \ p^+ < \infty \); then for the inequality

\[
\left\| W(\cdot)^{- \frac{1}{p^-}} \int_0^x f(t) \, dt \right\|_{L^p(0, l)} \leq C \left\| \omega(\cdot)^{\frac{1}{p^-}} f(\cdot) \right\|_{L^p(0, l)}
\]  

(2.3)

to hold it is necessary that \( p \in \bar{\Lambda}_1 \) and

\[
\sup_{t \in (0, l)} V(t)^{\frac{1}{p^-}} W(t)^{\frac{1}{p^+}} < \infty,
\]  

(2.4)

where \( \sigma(x) = \omega^{- \frac{1}{p(x)-1}}, \ W(x) = \int_0^x \sigma(t) \, dt \) and \( V(x) = \int_x^l W(t)^{-p(t)} \sigma(t) \, dt \).

Theorem 5. \([1], [11]\) Let \( p : (0, l) \to \mathbb{R} \) be measurable function such that \( 1 < p^-, \ p^+ < \infty \) then for (2.3) to hold in the interval \((0, l)\) it is sufficient that the condition (2.4) and \( p \in \bar{\Lambda}_0 \) is satisfied.

3. Proof of main results

For the proof of Theorem 1, 2 and 3 we refer to [4]. Other proves of these theorems are given in [15]. Here we derive an alternative proof of that theorem using the general results of [1] and [11].

We use the following elementary Lemma.

Lemma 1. Suppose \( s : (0, l) \to \mathbb{R} \) be a measurable function such that \( s \in \Lambda_0 \) and \( 0 < s^-, s^+ < \infty \); then it holds the estimate

\[
C^{-1} x^{s(0)} \leq x^{s(x)} \leq C x^{s(0)}, \quad 0 < x < \delta, \quad 0 < x < \delta.
\]  

(3.1)

Proof of Lemma 1.

Proof. Let \( 0 < x \leq \delta \) and \( s(x) \geq s(0) \) then we have \( x^{s(x)-s(0)} \leq 1 + \delta^{s(x)-s(0)} \), where \( \delta \) is a certain number from the interval \((0, 1)\). If \( 0 < x \leq \delta \) and \( s(x) < s(0) \) then by condition \( s \in \Lambda_0 \) we have

\[
x^{s(x)-s(0)} = \left( \frac{\delta}{x} \right)^{s(0)-s(x)} \delta^{s(x)-s(0)} 
\]

\[
\leq \left( \frac{\delta}{x} \right)^{c_1/\ln(\frac{1}{\delta})} \delta^{s(x)-s(0)} \leq C
\]

Therefore, for \( 0 < x \leq \delta \) we have the estimation

\[
x^{s(x)-s(0)} \leq C,
\]
where the positive constant $C$ depends on $s^-, s^+, s(0), \delta$. Same inequality holds for the function $x^{s(0) - s(x)}$. By using of these inequalities and by the representation

$$x^{s(x)} = x^{s(0)} x^{s(x) - s(0)}$$

we have the estimate (3.1).

To complete the proof of Lemma 1, note that the condition $s \in \Lambda_0$ is equivalent to

$$-C \leq |s(x) - s(0)| \ln \frac{1}{x} \leq C.$$

Proof of Theorem 3.

Proof. To prove Theorem 3 we can also apply the results of [1], [11], where it was proved that the condition (2.4) is necessary and sufficient for the inequality (1.1) to hold if $p(0) \leq q(0)$ and the regularity condition (2.1) is satisfied for both functions $p(x)$ and $q(x)$.

To apply the cited result, we accept $p(x) = q(x)$ and $V(x) := \int_x^\infty t^{(\beta(t) - 1)p(t)} dt$, $W(x) := \int_0^x t^{-\frac{\beta(t)p(t)}{p(t) - 1}} dt$. The conditions $p, \beta \in \Lambda_0$ imply $-\frac{\beta(t)p(t)}{p(t) - 1} \in \Lambda_0$. Therefore, it follows from Lemma 1 that

$$C_4^{-1} x^{\beta(0)p'(0)} \leq x^{-\beta(x)p'(x)} \leq C_4 x^{-\beta(0)p'(0)}$$

for $0 < x < 1$ by some $C_4 > 0$. Integrating this inequality over the intervals $(x, \infty)$ and $(0, x)$ respectively, we get

$$C_5^{-1} x^{1-\beta(0)p'(0)} \leq W(x) \leq C_5 x^{1-\beta(0)p'(0)}, \ 0 < x < 1. \quad (3.2)$$

To complete the proof of Theorem 3 it suffices to apply the estimates (3.2) to verify the conditions (2.4) and (2.1). Now, theorem 3 follows from the upper referred results of the works [1] and [11].

Proof of Theorem 4

Proof. For the number $a > 0$ let $a^{(1)}, a^{(2)}$ such that $W(a^{(1)}) = \frac{1}{2} W(a)$. Set $a^{(1)}, a^{(2)}$ such that $W(a^{(1)}) = 2W(a)$ and $W(a^{(2)}) = 4W(a)$, consider a test function $f_0(x) = \chi_{(a^{(1)}, a)}(x) - \frac{\sigma_{a^{(1)}, a}^x}{W(a)^{\frac{1}{p(0)}}}$, where $\chi_{(a^{(1)}, a)}(x)$ is a characteristic function of the interval $(a^{(1)}, a)$. 

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\[ \text{where the positive constant C depends on s-, s+, s(0), \delta. Same inequality holds for the function x^{s(0)-s(x)}. By using of these inequalities and by the representation} \]

\[ x^{s(x)} = x^{s(0)} x^{s(x) - s(0)} \]

\[ \text{we have the estimate (3.1).} \]

\[ \text{To complete the proof of Lemma 1, note that the condition s \in \Lambda_0 is equivalent to} \]

\[ -C \leq |s(x) - s(0)| \ln \frac{1}{x} \leq C. \]

\[ \text{□} \]

\[ \text{Proof of Theorem 3.} \]

\[ \text{Proof. To prove Theorem 3 we can also apply the results of [1], [11], where it was proved that the condition (2.4) is necessary and sufficient for the inequality (1.1) to hold if p(0) \leq q(0) and the regularity condition (2.1) is satisfied for both functions p(x) and q(x).} \]

\[ \text{To apply the cited result, we accept p(x) = q(x) and V(x) :=} \]

\[ \int_x^\infty t^{(\beta(t) - 1)p(t)} dt, \]

\[ W(x) := \int_0^x t^{-\frac{\beta(t)p(t)}{p(t) - 1}} dt. \] \[ \text{The conditions p, \beta \in \Lambda_0 imply} -\frac{\beta(t)p(t)}{p(t) - 1} \in \Lambda_0. \] \[ \text{Therefore, it follows from Lemma 1 that} \]

\[ C_4^{-1} x^{\beta(0)p'(0)} \leq x^{-\beta(x)p'(x)} \leq C_4 x^{-\beta(0)p'(0)} \]

\[ \text{for 0 < x < 1 by some C_4 > 0. Integrating this inequality over the intervals (x, \infty) and (0, x) respectively, we get} \]

\[ C_5^{-1} x^{1-\beta(0)p'(0)} \leq W(x) \leq C_5 x^{1-\beta(0)p'(0)}, \ 0 < x < 1. \] \[ (3.2) \]

\[ \text{To complete the proof of Theorem 3 it suffices to apply the estimates (3.2) to verify the conditions (2.4) and (2.1). Now, theorem 3 follows from the upper referred results of the works [1] and [11].} \]

\[ \text{□} \]

\[ \text{Proof of Theorem 4} \]

\[ \text{Proof. For the number a > 0 let a^{(1)}, a^{(2)} such that W(a^{(1)}) = \frac{1}{2} W(a). Set a^{(1)}, a^{(2)} such that W(a^{(1)}) = 2W(a) and W(a^{(2)}) = 4W(a), consider a test function f_0(x) = \chi_{(a^{(1)}, a)}(x) - \frac{\sigma_{a^{(1)}, a}^x}{W(a)^{\frac{1}{p(0)}}}, where} \]

\[ \chi_{(a^{(1)}, a)}(x) \text{ is a characteristic function of the interval (a^{(1)}, a).} \]
It follows from the definition that,

\[
I_{p(\cdot)} \left( \omega_{\frac{1}{p(\cdot)}} f_0 (\cdot) \right) = \int_{a(1)}^a \frac{\sigma(x)^{p(x)}}{W(x)} \omega(x) \, dx \\
= \int_{\frac{1}{2}W(a)}^{W(a)} \frac{dW}{W} = \ln 2 < 1
\]

This and 2.1 imply \( \| \omega (\cdot)^{\frac{1}{p(\cdot)}} \|_{p(\cdot)} \leq 1 \).

Inserting the function \( f_0 \) into the inequality 1.1, it follows that

\[
\left\| W(\cdot)^{-1} \sigma (\cdot)^{\frac{1}{p(\cdot)}} Hf_0 (\cdot) \right\|_{L^{p(\cdot)}} \leq C
\]

since \( \| \omega (x)^{\frac{1}{p(\cdot)}} f (\cdot) \|_{p(\cdot)} \leq 1 \).

This implies

\[
I_{p(\cdot)} \left( W^{-1} \sigma^{\frac{1}{p(\cdot)}} Hf_0 \right) \leq C_1
\]

Therefore,

\[
C_1 \geq \int_0^t W^{-p(x)} \sigma(x) \left( \int_0^x \chi(a(1),a) (t) \frac{\sigma(t)}{W(t)^{\frac{1}{p(\cdot)}}} \, dt \right)^{p(x)} \, dx \\
\geq \int_{a(1)}^{a(2)} W^{-p(x)} \sigma(x) \left( \int_{a(1)}^a \chi(a(1),a) (t) \frac{\sigma(t)}{W(t)^{\frac{1}{p(\cdot)}}} \, dt \right)^{p(x)} \, dx
\]  \( (3.3) \)

for \( t \in (a(1), a) \). By using increasing of the function \( p (\cdot) \) we have

\[
W(t)^{\frac{1}{p(t)}} \leq \left( \frac{W(a)}{W(l)} \right)^{\frac{1}{p(l)}} W(l)^{\frac{1}{p(l)}} \\
\leq W(a)^{\frac{1}{p(l)}} W(l)^{\frac{1}{p(l)} - \frac{1}{p(a)}} = C_2 W(a)^{\frac{1}{p(a)}}
\]

Therefore,

\[
\int_{a(1)}^a \frac{\sigma(t)}{W(t)^{\frac{1}{p(t)}}} \, dt \geq \frac{1}{2C_2} W(a)^{\frac{1}{p(a)}}.
\]
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From the last inequality and 3.3 it follows that

\begin{equation}
C_1 \geq \frac{1}{2C_2} \int_{a^{(1)}}^{a^{(2)}} W(a)^{\frac{p(x)}{p'(x)}} W(x)^{-p(x)} \sigma(x) \, dx \\
\geq \frac{1}{2C_2} \int_{a^{(1)}}^{a^{(2)}} W(a)^{1-\frac{p(x)}{p'(x)}} \sigma(x) \, dx
\end{equation}

(3.4)

for \( x \in (a^{(1)}, a^{(2)}) \) we have

\[
\left( \frac{1}{W(a)} \right)^{\frac{p(x)}{p'(x)}-1} \geq \left( \frac{1}{W(a)} \right)^{\frac{p(a^{(1)})}{p'(a)}-1}
\]

Therefore, it follows from 3.4 that

\[
C_1 \geq \frac{\ln 2}{2C_2} \left( \frac{1}{W(a)} \right)^{\frac{p(a^{(1)})}{p'(a)}-1}
\]

i.e.

\[
[p(a^{(1)}) - p(a)] \ln \frac{1}{W(x)} \leq C_3
\]

This implies (2.2) and thereby completes the proof of Theorem 4. \( \square \)

Proof of Theorem 5

Proof. To prove the theorem we use the upper cited result [1] and [11] in the case \( p(x) = q(x), V(x) = W(x)^{-p(x)} \sigma(x) \) and \( \sigma(x) = \omega(x)^{-\frac{1}{p'(x)-1}} \). Verify the condition (2.4) under these settings. It follows from the condition (2.1) that \( \int_t^l W^{-p(s)} \sigma(s) \, ds \sim \int_t^l W^{-p(0)} dW(s) \sim W(t)^{1-p(0)}, 0 < t < l \). Therefore, it is nothing to check the condition (2.4). This completes the proof of Theorem 5. \( \square \)

References


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