A New Numerical Method for Parabolic Fractional Equation with Purely Nonlocal Conditions

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Abstract

The aims of this paper are to present a numerical technique for solving the one-dimensional parabolic fractional differential equation that combine classical and integral conditions:

\[
\begin{align*}
\frac{\partial^\alpha}{\partial t^\alpha} u - \frac{\partial^2 u}{\partial x^2} + u(x,t) &= q(x,t) \quad 0 \leq x \leq 1, \quad 0 < t \leq T \\
u(x,0) &= f(x), \quad 0 \leq x \leq 1 \\
\int_0^1 a(x)u(x,t) \, dx &= \varphi(t), \quad 0 < t \leq T \\
\int_0^1 u(x,t) \, dx &= \psi(t), \quad 0 < t \leq T
\end{align*}
\]

A Laplace transform technique is introduced for solving considered equation, definite integrals are approximated by high-precision quadrature schemes. To invert the equation numerically back into the time domain, we apply the Stehfest inversion algorithm.

1 Introduction

For any positive integer \(0 < \alpha < 1\), the gamma function \(\Gamma\) and the left Caputo derivative \(\frac{\partial^\alpha}{\partial t^\alpha}\), are respectively, defined as
This paper is focused on the numerical of the following parabolic fractional equation

$$
\frac{\partial}{\partial t}^\alpha u(x,t) - \frac{\partial^2 u}{\partial x^2} + u(x,t) = q(x,t) \quad 0 \leq x \leq 1, \quad 0 < t \leq T,
$$

(1.1)

with the initial condition

$$
u(x,0) = f(x), \quad 0 \leq x \leq 1, \quad (1.2)
$$

and the integral conditions

$$
\int_0^1 a(x)u(x,t)\,dx = \varphi(t), \quad 0 < t \leq T, \quad (1.3)
$$

$$
\int_0^1 u(x,t)\,dx = \psi(t), \quad 0 < t \leq T, \quad (1.4)
$$

The Fractional Partial Equations (FDEs) are generalization of differential equations of integer order to arbitrary order. These generalizations play a crucial role in engineering, physics and applied mathematics. Therefore, they have generated a lot of interest from engineers and scientist in recent years. Since FDEs have memory, nonlocal relations in space and time, and complex phenomena can be modeled by using these equations. Indeed, we can find numerous applications in viscoelasticity, electrodynamics, signal processing, control theory, porous media, fluid flow, rheology, diffusive transport, electrical networks, electromagnetic theory, probability and many other physical processes.

The notion of nonlocal condition has been introduced to extend the study of the classical initial value problems and it is more precise for describing nature phenomena than the classical condition since more information is taken into account, thereby decreasing the negative effects incurred by a possibly erroneous single measurement taken at the initial value. The importance of nonlocal conditions in many applications is discussed in [14], [19].

Mathematical modelling by evolution problems with a nonlocal constraint of the form

$$
\frac{1}{1-\ell} \int_{\ell}^1 u(x,t)\,dx = \zeta(t)
$$
Numerical method for parabolic fractional equation

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is encountered in heat transmission theory, thermoelasticity, chemical engineering, underground water flow, and plasma physics.

The existence and uniqueness of solution to initial and boundary-valued problem for fractional differential equation has been extensively studied by many authors see for examples. Many methods were used to investigate the existence and uniqueness of the solution of mixed problems which combine classical and integral conditions. J. R. Cannon [9] used the potential method, combining a Dirichlet and an integral condition for a parabolic equation. L. A. Mouravey and V. Philinovoski [13] used the maximum principle, combining a Neumann and an integral condition for heat equation. M.Z. Djibibe and K. Tcharie [17], Ionkin [10] and L. Bougoffa [8] used the Fourier method for same purpose.

Recently, mixed problems with integral conditions for generalization of equation (1.1) have been treated using the energy-integral method. See M. Z. Djibibe and K. Tcharie [15], M.Z. Djibibe and K. Tcharie [16], M. Z. Djibibe el al. [17],[18], N. I. Yurchuk [19],[20], M. Mesloub, A. Bouziani and N. Kechkar [12].

Ang [22] has considered a one dimensional heat equation with nonlocal integral conditions. The author has taken the Laplace transform of the problem and then used a numerical technique for inverse Laplace transform to obtain the numerical solution. Ekolin [16] and Liu [21] have used the finite difference method to solve a similar type problem numerically. After, Shruti [17] studied Sobolev-Type differential equation subject to nonlocal initial boundary conditions by Laplace transform method. Merad [1] used the Adomian decomposition method to solve a similar type equation.

The purpose of the present article is to give a method of solution to problem (1.1), (1.2), (1.3) and (1.4) using Laplace transform technique. In recent years, Laplace transform method has been used to approximate the solution of different classes of linear partial differential equation. The main difficulty in using Laplace Transform transform method consist in funding its inverse. Numerical inversion methods are then used to overcome this difficulty. There are many numerical techniques available in literature to invert Laplace transform. In this paper we focus exclusively on the Stehfest inversion algorithm [7] in order to efficiently and accurately invert the Laplace transform. Motivated by this, we extend and generalize the study for partial differential equations with integral conditions to the study of fractional partial differential equations with integral conditions. Also we expand the works in classical problems of fractional partial differential equations to non standard problems. Also we extend the application of the numerical inversion method for obtained existence of solutions.
2 Method based on Laplace Transform

Suppose that $u(x,t)$ is defined and is of exponential order for $t \geq 0$, that is, there exists $M$, $\beta$ and $t_0 > 0$ such that $|u(x,t)|Me^{\beta t}$ for $t > t_0$. Then the Laplace Transform $U(x,s)$ exists and it is given by:

$$U(x,s) = \int_0^{+\infty} u(x,t)e^{-st}dt,$$

where $s$ is a positive real parameter.

First we take the Laplace transform on both sides of (1.1)-(1.2) with respect to $t$, and get

$$\int_0^{+\infty} e^{-st}c_0^\alpha u dt - \int_0^{+\infty} e^{-st}\frac{\partial^2 u}{\partial x^2} dt + \int_0^{+\infty} e^{-st}u(x,t) dt = \int_0^{+\infty} e^{-st}q(x,t) dt,$$

(2.1)

Integrating by parts the integrals of the left-hand side of (2.1), we obtain

$$\int_0^{+\infty} e^{-st}c_0^\alpha u dt = s^\alpha U(x,s) - s^{\alpha-1}f(x),$$

(2.2)

$$\int_0^{+\infty} \frac{\partial^2 u}{\partial x^2} e^{-st} dt = \frac{d^2 U(x,s)}{dx^2},$$

(2.3)

$$\int_0^{+\infty} u(x,t)e^{-st} dt = U(x,s),$$

(2.4)

$$\int_0^{+\infty} q(x,t)e^{-st} dt = Q(x,s).$$

(2.5)

Substuting (2.2), (2.3) and (2.5) into (2.1), we have

$$- \frac{d^2 U(x,s)}{dx^2} + (1 + s^\alpha)U(x,s) = Q(x,s) + s^{\alpha-1}f(x).$$

(2.6)

Taking account the purely nonlocal conditions (1.4) and (2.1), its follows that

$$\int_0^1 a(x)U(x,s) dx = \Phi(s),$$

(2.7)

$$\int_0^1 U(x,s) dx = \Psi(s),$$

(2.8)
where
\[ \Phi(s) = \int_0^{+\infty} \varphi(t) e^{-st} \, dt, \quad \Psi(s) = \int_0^{+\infty} \psi(t) e^{-st} \, dt. \]

Let \( \Delta(s) = 1 + s^\alpha > 0 \).

Using the method of variation of paramter we have the general solution of (2.6) as
\[
U(x, s) = C_1(s) e^{x \sqrt{1 + s^\alpha}} + C_2(s) e^{-x \sqrt{1 + s^\alpha}}
+ \frac{1}{\sqrt{1 + s^\alpha}} \int_0^x (Q(\tau, s) + s^{\alpha-1} f(\tau)) \sinh(\sqrt{1 + s^\alpha} (\tau - x)) \, d\tau. \tag{2.9}
\]

From the boundary conditions (2.7) and (2.8), we get
\[
C_1(s) \int_0^1 a(x) e^{x \sqrt{1 + s^\alpha}} \, dx + C_2(s) \int_0^1 a(x) e^{-x \sqrt{1 + s^\alpha}} \, dx = \Phi(s) +
\frac{1}{\sqrt{1 + s^\alpha}} \int_0^1 \left[ \int_0^x a(x)(Q(\tau, s) + s^{\alpha-1} f(\tau)) \sinh(\sqrt{1 + s^\alpha} (x - \tau)) \, d\tau \right] \, dx, \tag{2.10}
\]

\[
e^{x \sqrt{1 + s^\alpha}} C_1(s) + C_2(s) = \frac{\sqrt{1 + s^\alpha} e^{x \sqrt{1 + s^\alpha}}}{e^{\sqrt{1 + s^\alpha}} - 1} \Psi(s) +
\frac{e^{x \sqrt{1 + s^\alpha}}}{e^{\sqrt{1 + s^\alpha}} - 1} \int_0^1 \left[ \int_0^x (Q(\tau, s) + s^{\alpha-1} f(\tau)) \sinh(\sqrt{1 + s^\alpha} (x - \tau)) \, d\tau \right] \, dx. \tag{2.11}
\]

Thus, \( C_1(s) \) and \( C_2(s) \) are given by
\[
\begin{pmatrix} C_1(s) \\ C_2(s) \end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{pmatrix}^{-1} \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix}, \tag{2.12}
\]
and

\[
\begin{align*}
    a_{11}(s) &= \int_0^1 a(x)e^{x\sqrt{1+s^\alpha}}\,dx \\
    a_{12}(s) &= \int_0^1 a(x)e^{-x\sqrt{1+s^\alpha}}\,dx \\
    a_{21}(s) &= e^{\sqrt{1+s^\alpha}} \\
    a_{22}(s) &= 1 \\
    b_1(s) &= \Phi(s) + \\
    &\quad \frac{1}{\sqrt{1+s^\alpha}} \int_0^1 \left( (Q(\tau,s) + s^{\alpha-1}f(u)) \int_0^1 a(x) \sinh(\sqrt{1+s^\alpha}(x-\tau)) \,dx \right) \,d\tau. \\
    b_2(s) &= \frac{\sqrt{1+s^\alpha}e^{\sqrt{1+s^\alpha}}}{e^{\sqrt{1+s^\alpha}}-1} \Psi(s) + \\
    &\quad \frac{e^{\sqrt{1+s^\alpha}}}{\sqrt{1+s^\alpha}(e^{\sqrt{1+s^\alpha}}-1)} \int_0^1 (Q(\tau,s) + s^{\alpha-1}f(\tau)) \left( \cosh(\sqrt{1+s^\alpha}(1-\tau)) - 1 \right) \,d\tau \\
    &\quad (2.13)
\end{align*}
\]

Thus, to find a solution in Laplace domain, one has to evaluate all the integral appearing in (2.13). Using high-precision quadrature schemes, we have the following approximations of the above integrals

\[
\begin{align*}
    \int_0^1 a(x)e^{x\sqrt{1+s^\alpha}}\,dx &= \frac{1}{2} \int_{-1}^1 a \left( \frac{x+1}{2} \right) e^{\sqrt{1+s^\alpha}(x+1)/2} \,dx \\
    &\approx \frac{\pi h}{2} \sum_{j=-\infty}^{+\infty} w_j a \left( \frac{x_j+1}{2} \right) e^{\sqrt{1+s^\alpha}(x_j+1)/2} \\
    &\quad (2.14) \\
    \int_0^1 a(x)e^{-x\sqrt{1+s^\alpha}}\,dx &= \frac{1}{2} \int_{-1}^1 a \left( \frac{x+1}{2} \right) e^{-\sqrt{1+s^\alpha}(x+1)/2} \,dx \\
    &\approx \frac{\pi h}{2} \sum_{j=-\infty}^{+\infty} w_j a \left( \frac{x_j+1}{2} \right) e^{-\sqrt{1+s^\alpha}(x_j+1)/2} \\
    &\quad (2.15)
\end{align*}
\]
\[
\int_{0}^{1} \left( Q(u,s) + s^{\alpha-1}f(u) \right) \left( \cosh(\sqrt{1 + s^{\alpha}(1 - u)}) - 1 \right) \, du
\]
(2.16)

\[
= \frac{1}{2} \int_{-1}^{1} \left[ Q \left( \frac{1 + u}{2}, s \right) + s^{\alpha-1}f \left( \frac{1 + u}{2} \right) \right] \times \left[ \cosh \left( \sqrt{1 + s^{\alpha}(1 - u)} \right) - 1 \right] \, du
\]

\[
\simeq \frac{\pi h}{2} \sum_{j=-\infty}^{+\infty} w_j \left[ Q \left( \frac{1 + x_j}{2}, s \right) + s^{\alpha-1}f \left( \frac{1 + x_j}{2} \right) \right] \times \left[ \cosh \left( \sqrt{1 + s^{\alpha}(1 - x_j)} \right) - 1 \right],
\]
(2.17)

and

\[
\int_{0}^{1} \left[ (Q(u,s) + s^{\alpha-1}f(u)) \int_{u}^{1} a(x) \sinh(\sqrt{1 + s^{\alpha}(x - u)}) \, dx \right] \, du
\]

\[
= \frac{\pi h}{2} \sum_{j=-\infty}^{+\infty} w_j \left[ Q \left( \frac{1 + x_j}{2}, s \right) + s^{\alpha-1}f \left( \frac{1 + x_j}{2} \right) \right] \frac{1 - x_j}{2} \times
\]

\[
\sum_{i=-\infty}^{+\infty} w_i a \left( \frac{1 - \frac{1}{2}(x_j + 1)}{2} x_i + \frac{1 + \frac{1}{2}(x_j + 1)}{2} \right) \times
\]

\[
\sinh \left[ \sqrt{1 + s^{\alpha}} \left( \frac{1 - \frac{1}{2}(x_j + 1)}{2} x_i + \frac{1 + \frac{1}{2}(x_j + 1)}{2} - \frac{1}{2}(x_j + 1) \right) \right],
\]
(2.18)

where

\[
x_j = \tanh \left( \frac{\pi}{2} \sinh h_j \right), \quad w_j = \frac{\cosh h_j}{\cosh^2 \left( \frac{\pi}{2} \sinh h_j \right)}.
\]

3 Numerical inversion of Laplace Transform

We have a solution in Laplace domain. So we expect to obtain a solution of original problem by means of inverting the Laplace transform. Simple transforms can be inverted using readily available table. More complexe functions can be analytically inverted through the complex inversion formula

\[
f(t) = \frac{1}{2\pi j} \int_{\mu-j\infty}^{\mu+j\infty} F(s) e^{st} \, ds.
\]

Sometimes, an analytical inversion of a Laplace domain solution is difficult to obtain; therefore a numerical inversion method must be used. A nice comparison of four frequently used numerical Laplace inversion algorithms is given by Hassan Hassanzadeh, Mehran Poladi-Davish [6]. In this work we use the Stehfest’s algorithm [7] that is easy to implement. This numerical technique was
first introduced by Graver [4] and its algorithm then offered by [7]. Stehfest’s algorithm approximates the times domain solution as:

\[ f(t) = \frac{\ln 2}{t} \sum_{n=1}^{2N} \omega_n F\left(\frac{n \ln 2}{t}\right). \]

The \( \omega_n \) coefficients only depend on the number of expansion terms \( N \). They are

\[
\omega_n = (-1)^{n+N} \sum_{k=\left\lfloor \frac{n+1}{2} \right\rfloor}^{\min(n, N)} \frac{k^N (2k)!}{(N-k)!k!(k-1)!(n-k)!(2k-n)!}.
\]

The \( \omega_n \) coefficients become very and alternate in sign when \( n \) increases. The precision of the Stehfest inversion method depends on the Stehfest number \( N \). Indeed, one can see in equation (3.1) that the inversion is based on summation of \( N \) weighted values. The default Stehfest number is often chosen in the range \( 6 \leq N \leq 18 \).

Taking this into account, we can obtain a solution to problem (1.1)-(1.4) as

\[
u(x, t) = \frac{\ln 2}{t} \sum_{n=1}^{2N} \omega_n U\left(x, \frac{n \ln 2}{t}\right)
\]

\[
u(x, t) = \frac{\ln 2}{t} \sum_{n=1}^{2N} \omega_n \left[ C_1 \left(\frac{n \ln 2}{t}\right) e^{x \sqrt{1 + \frac{n^\alpha \ln^\alpha 2}{t^\alpha}}} + C_2 \left(\frac{n \ln 2}{t}\right) e^{-x \sqrt{1 + \frac{n^\alpha \ln^\alpha 2}{t^\alpha}}} \right]
\]

\[ - \frac{\sqrt{t^\alpha}}{n^\alpha \ln^\alpha 2 + t^\alpha} \int_0^x \left( Q\left(\frac{u, n \ln 2}{t}\right) + \frac{n^\alpha \ln^\alpha 2}{t^\alpha - 1} f(u) \right)
\times \sinh\left(\sqrt{1 + \frac{n^\alpha \ln^\alpha 2}{t^\alpha}(x - u)}\right) du
\]

where \( \omega_n \) is given by (3.2).

**References**


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