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A New Analytical Method for Solutions of Nonlinear Impulsive Volterra-Fredholm Integral Equations

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Abstract

The main purpose of this paper is to demonstrate the use of a new comparison result and Mönch fixed point theorem for solutions of nonlinear impulsive Volterra-Fredholm integral equations in Banach space. The benefit of this method is that it overcomes the difficulties caused by Fredholm operator without any restrictive conditions on estimate of noncompactness measure. The results obtained here improve and develop some previous works largely.

Mathematics Subject Classification: 45B05, 45D05, 45G15

Keywords: Volterra-Fredholm integral equations, Comparison result, Fixed point theorem

1 Introduction

One of the fundamental classes of equations is Volterra-Fredholm integral ones. Nonlinear Volterra-Fredholm integral equations arise in the fields of parabolic boundary value problems, the spatio-temporal development of an epidemic and various scientific problems. This type of equations has been applied to a wide range of science and engineering. For this reason, Volterra-Fredholm integral equations have been attracted more attention in recent decades. Maleknejad and Hadizadeh [8] investigated this type of systems by means of Adomian's decomposition method. Maleknejad and Fadaei Yami [9] extended the method used in [8] to solve nonlinear Volterra-Fredholm integral systems.

Wazwaz [1] provided a modified decomposition method for handing nonlinear Volterra-Fredholm integral equation. Yousefi and Razzaghi [15] presented a numerical method for solving nonlinear Volterra-Fredholm integral equations based on Legendre wavelets method. Babolian et al.[3] proposed a new direct method for numerical solutions of nonlinear Volterra-Fredholm integral and integro-differential equations. Mahmoudi [16] studied the nonlinear Volterra and Fredholm integral equation of the second kind using continues Legendre wavelets. Negarchi and Nouri [13] used a Müntz-Legendre collocation and the least squares approximation method to consider numerical behavior of a class of Volterra-Fredholm integral equation. Maleknejad and Dehbozorgi [10] established a new computational method for nonlinear Volterra integral equation of the second kind.

Impulsive equations describe processes which experience a sudden change of their state at certain moments. Because it can explain well various natural phenomenon, these problems have aroused people's widespread attention, see [2] and [7]. In recent decades, many authors have studied various kinds of differential, integral and integro-differential equations, particularly in the fields of existence and controllability results of many systems, see [4],[5],[6],[11],[12] and [14].

In [2], when the nonlinear term f does not include the first derivative term x' and integral term Sx, by monotone iterative technique, Guo investigated a class of initial value problems. Inspired by the above literatures, consider the following problem for nonlinear impulsive Volterra-Fredholm integral equations in Banach space

$$\begin{aligned} x(t) &= p(t) + \int_0^t (t-s) f(s, x(s), x'(s), (Tx)(s), (Sx)(s)) ds \\ &+ \sum_{0<t_k<t} [I_k(x(t_k), x'(t_k)) + q(t, t_k) \bar{I}_k(x(t_k), x'(t_k))], t \in J, \end{aligned} \tag{1}$$

where $p, q \in C^1[0, a]$, $f \in C(J \times X \times X \times X \times X, X)$, I_k, $\bar{I}_k \in C(X \times X, X)$,

$$(Tx)(t) = \int_0^t k(t,s)x(s)ds, (Sx)(t) = \int_0^a h(t,s)x(s)ds. \tag{2}$$

X is a real Banach space, $J = [0, a]$ $(a > 0)$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = a$. $F = \{(t, s) \in J \times J \mid t \geq s\}$. In (2), $k \in C(F, R^+)$, $h \in C(J \times J, R^+)$, R^+=[0, $+\infty$), R_+=(0, $+\infty$). Let $J_0 = [0, t_1], J_1 = (t_1, t_2], \cdots, J_{m-1} = (t_{m-1}, t_m], J_m = (t_m, a]$, $k^* = \max\{k(t, s) \mid (t, s) \in F\}$, $h^* = \max\{h(t, s) \mid (t, s) \in J \times J\}$, I_k and $\bar{I}_k$ denotes the jump of $x(t)$ at $t = t_k$.

Applying a new comparison result, the Mönch fixed theorem and estimate step by step, we obtain the existence of solutions for problem (1). Compared to previous results, this paper has three advantages: Firstly, we present a new analytical method which will be used to overcome difficulties when conditions of norm include the norm of Sx, and in this way we obtain the existence of solutions for nonlinear impulsive Volterra-Fredholm integral equations (1). Secondly, the restrictive conditions on estimate of noncompactness measure are omitted. Thirdly, the condition (H_2) in this paper is weaker than previous works in this fields.

2 Preliminaries

This section states some necessary definitions and preliminary results.

Lemma 2.1 *([11]) Let $D = \{x_n\} \subset L[J, X]$ and there exists $g \in L[J, R^+]$ such that $\|x_n(t)\| \leq g(t), a.e. t \in J$ for all $x_n \in D$, Then $\alpha(D(t)) \in L[J, R^+]$ and*

$$\alpha\left(\left\{\int_0^t x_n(s)ds | n \in N\right\}\right) \leq 2\int_0^t \alpha(D(t))ds, t \in J.$$

Lemma 2.2 *([11]) Let $B \subset C[J, X]$ is equicontinuous bounded set, then $\alpha(B(t)) \in C[J, R^+]$ and $\alpha\left(\int_0^a B(s)ds\right) \leq \int_0^a \alpha(B(s))ds$.*

Lemma 2.3 *([11]).(Mönch) Let E be a Banach space and $\Omega \subset E$ be a bounded open set, $\theta \in \Omega$, $A : \overline{\Omega} \to E$ is continuous and satisfies the following conditions*

(i) $x \neq \lambda Ax$, $\forall \lambda \in [0, 1]$, $x \in \partial\Omega$;

(ii)$D \subset \overline{\Omega}$ is countable and $D \subset \overline{co}(\{\theta\} \bigcup (AD))$ imply that D is a relatively compact set. Then A has at least one fixed point in Ω.

Lemma 2.4 *([6]) Suppose that u, f, $g \in C([t_0, T), R^+)$. Let $w \in C(R^+, R^+)$ be nondecreasing with $w(u) > 0$ for $u > 0$, and α, $\beta \in C^1([t_0, T), [t_0, T))$ be nondecreasing with $\alpha(t)$, $\beta(t) \leq t$ on $[t_0, T)$. If*

$$u(t) \leq M_1 + M_2\int_{\alpha(t_0)}^{\alpha(t)} f(s)w(u(s))ds + M_3\int_{\beta(t_0)}^{\beta(t)} g(s)w(u(s))ds + M_4\int_{t_0}^{T} [u(s)]^\lambda ds, \tag{3}$$

where M_i $(i = 1, 2, 3, 4)$ are nonnegative constants, $0 \leq \lambda < 1$ and $\int_1^\infty \frac{1}{w(s)}ds = \infty$. Then for $t_0 \leq t < T$, we have

$$u(t) \leq m^{-}(0) \tag{4}$$

where $m(s) = (2s - MM_1)^{1-\lambda} - s^{1-\lambda} - (1-\lambda)MM_4T$, M is a positive constant and $m^{-}(\cdot)$ represents the inverse of $m(\cdot)$.

3 Main Results

We are concerned with the existence of solutions of problem (1). Let $PC[J, X] := \{x : J \to X | x \in C[(t_k, t_{k+1}), X]$, and there exist $x(t_k^-)$ and $x(t_k^+)$, $k = 1, 2, \ldots, m$ with $x(t_k^-) = x(t_k)\}$. Endowed with the norm $||x||_{PC} = sup_{t\in J}||x(t)||$, $(PC[J, X], || \cdot ||_{PC})$ is a Banach space. $T_r = \{x \in X | ||x|| \le r\}$ and $B_r = \{x \in PC[J, X] | ||x||_{PC} \le r\}$. $\alpha(\cdot)$ represents noncompactness measure.

For this purpose, we also need the following assumptions hold

(H_1) For any $r > 0$, f is uniformly continuous on $J \times T_r \times T_r \times T_r \times T_r$, I_k and $\bar{I}_k$ $(k = 1, 2, \cdots, m)$ are bounded on $T_r \times T_r$.

(H_2) There exist $b_1,\ b_5 \in L[J, R^+]$ and $b_2,\ b_3,\ b_4 \in C[J, R^+]$, $0 \le \lambda_1 < 1$ such that

$$\|f(s, x, y, u, v)\| \le b_1(s) + b_2(s)\|x\| + b_3(s)\|y\| + b_4(s)\|u\| + b_5(s)\|v\|^{\lambda_1}, \quad (5)$$

$$s \in J, x, y, u, v \in X.$$

(H_3) Suppose $l_i \in C[J, R^+]$ $(i = 1, 2, 3, 4)$ are bounded, and for any equicontinuous set $D \subset B_r$, $(t, s) \in F$ and $0 \le \lambda_2 < 1$ such that

$$\alpha(f(t, s, D(s), (TD)(s), (SD)(s))$$

$$\le l_1(t)\alpha(D(s)) + l_2(t)\alpha(D'(s)) + l_3(t)\alpha((TD)(s)) + l_3(t)[\alpha((SD)(s))]^{\lambda_2}. \quad (6)$$

Theorem 3.1 *Assume that (H_1)-(H_3) are satisfied. Then the Problem (1) has at least one solution in $PC[J, X]$.*

Proof. $x \in PC[J, X]$ is one solution of equation (1) if and only if $x \in PC[J, E]$ is the solution of nonlinear impulsive integral equation

$$x(t) = Ax(t), \quad (7)$$

where

$$(Ax)(t) = p(t) + \int_0^t (t - s) f(s, x(s), x'(s), (Tx)(s), (Sx)(s))ds$$

$$+ \sum_{0<t_k<t} [I_k(x(t_k), x'(t_k)) + q(t, t_k)\bar{I}_k(x(t_k), x'(t_k))], t \in J. \quad (8)$$

Then

$$(Ax)'(t) = p'(t) + \int_0^t f(s, x(s), x'(s), (Tx)(s), (Sx)(s))ds$$

$$+ \sum_{0<t_k<t} q'(t, t_k)\bar{I}_k(x(t_k), x'(t_k)). \quad (9)$$

Let $\bar{x} \in \Omega_0$, where $\Omega_0 = \{x \in PC[J, X] \mid x = \lambda Ax, 0 \leq \lambda \leq 1\}$. Then there exists $\lambda_0 \in [0, 1]$ such that

$$\bar{x}(t) = \lambda_0 (A\bar{x})(t), \quad \bar{x}'(t) = \lambda_0 (A\bar{x})'(t), t \in J.$$

(i) $t \in J_0 = [0, t_1]$. From (8) and (9) we have

$$\bar{x}(t) = \lambda_0 \left(p(t) + \int_0^t (t-s) f(s, \bar{x}(s), \bar{x}'(s), (T\bar{x})(s), (S\bar{x})(s)) ds \right), \tag{10}$$

$$\bar{x}'(t) = \lambda_0 \left(p'(t) + \int_0^t f(s, \bar{x}(s), \bar{x}'(s), (T\bar{x})(s), (S\bar{x})(s)) ds \right). \tag{11}$$

Since $p \in C^1[0, a]$ and (H_2), there exists a constant M such that

$$\|\bar{x}(t)\|$$

$$\leq M + a \int_0^t \Big(b_1(s) + b_2(s) \|\bar{x}(s)\| + b_3(s) \|\bar{x}'(s)\| + b_4(s) \|(T\bar{x})(s)\| + b_5(s) \|(S\bar{x})(s)\|^{\lambda_1} \Big) ds,$$

$$\leq M + a \int_0^t b_1(s) ds + a \int_0^t \Big(b_2(s) + a k^* b_4(s) \Big) \|\bar{x}(s)\| ds + a \int_0^t b_3(s) \|\bar{x}'(s)\| ds \tag{12}$$
$$+ a (h^*)^{\lambda_1} \textstyle\int_0^t b_5(s) ds \int_0^a \|\bar{x}(s)\|^{\lambda_1} ds.$$

$$\|\bar{x}'(t)\| \leq M + \int_0^t b_1(s) ds + \int_0^t \Big(b_2(s) + a k^* b_4(s) \Big) \|\bar{x}(s)\| ds + \int_0^t b_3(s) \|\bar{x}'(s)\| ds$$
$$+ (h^*)^{\lambda_1} \int_0^t b_5(s) ds \int_0^a \|\bar{x}(s)\|^{\lambda_1} ds. \tag{13}$$

Set $u(t) = \|\bar{x}(t)\| + \|\bar{x}'(t)\|$, thus $u(t) \in C([0, t_1), R^+)$, Adding the above inequalities, we know

$$u(t) \leq 2M + (a+1) \int_0^t b_1(s) ds + (a+1) \int_0^t \Big(b_2(s) + a k^* b_4(s) \Big) \|u(s)\| ds$$

$$+ (a+1) \int_0^t b_3(s) \|u(s)\| ds + (a+1)(h^*)^{\lambda_1} \int_0^t b_5(s) ds \int_0^t \|u(s)\|^{\lambda_1} ds. \tag{14}$$

Then by Lemma 2.4 and continuity of $x(t)$ on J_0, there exists a constant C_0 independent of u such that $u(t) \leq C_0, t \in J_0$. The above inequality implies that $\|\bar{x}(t)\| + \|\bar{x}'(t)\| \leq C_0$. From ($H_1$),there also exists a constant $\beta > 0$ independent of $\bar{x}$ such that

$$\|f(s, x(s), x'(s), (Tx)(s), (Sx)(s))\| \leq \beta, \tag{15}$$

$$\|I_1(x, y)\| \leq \beta, \|\bar{I}_1(x, y)\| \leq \beta, \tag{16}$$

$$\forall\ t \in J_0, \|\bar{x}(t)\|, \|\bar{x}'(t)\| \le C_0. \tag{17}$$

Thus

$$\|\bar{x}(t_1^+)\| = \|\bar{x}(t_1) + I_1(\bar{x}(t_1), \bar{x}'(t_1))\| \le C_0 + \beta, \tag{18}$$

$$\|\bar{x}'(t_1^+)\| = \|\bar{x}'(t_1) + \bar{I}_1(\bar{x}(t_1), \bar{x}'(t_1))\| \le C_0 + \beta. \tag{19}$$

(ii) $t \in (t_1, t_2]$, Let

$$\bar{u}(t) = \bar{x}(t), t \in (t_1, t_2]; \bar{u}(t) = \bar{x}(t_1^+), t = t_1. \tag{20}$$

$$\bar{u}'(t) = \bar{x}'(t), t \in (t_1, t_2]; \bar{u}'(t) = \bar{x}'(t_1^+),\ t = t_1. \tag{21}$$

Then $\bar{u}(t), \bar{u}'(t) \in C([t_1, t_2], X)$ and

$$\bar{u}(t) = \lambda_0(x_0 + tx_1) + \lambda_0 \int_0^{t_1} (t-s) f(s, \bar{x}(s), \bar{x}'(s), (T\bar{x})(s)), (S\bar{x})(s))ds$$

$$+\lambda_0 I_1(\bar{x}(t_1), \bar{x}'(t_1)) + \lambda_0(t - t_1)\bar{I}_1(\bar{x}(t_1), \bar{x}'(t_1)), \tag{22}$$

$$\bar{u}'(t) = \lambda_0 x_1 + \lambda_0 \int_0^{t_1} f(s, \bar{x}(s), \bar{x}'(s), (T\bar{x})(s)), (S\bar{x})(s))ds$$

$$+\lambda_0 \bar{I}_1(\bar{x}(t_1), \bar{x}'(t_1)). \tag{23}$$

Consequently, similar to the proof of (i), we can know that there exists $C_1 > 0$ that does not depend on $\bar{x}$ such that $\|\bar{u}(t)\| + \|\bar{u}'(t)\| \le C_1, t \in [t_1, t_2]$. So $\|\bar{x}(t)\| \le C_1, \|\bar{x}'(t)\| \le C_1,\ t \in [t_1, t_2]$. By the same method as above, we can show that there exists a constant $C_m > 0$ that does not depend on $\bar{x}$ such that

$$\|\bar{x}(t)\| \le C_m, \|\bar{x}'(t)\| \le C_m, t \in J_m = [t_m, a].$$

Let $C = \max\{C_i | 0 \le i \le m\}$. Then $\|\bar{x}(t)\| \le C, \|\bar{x}'(t)\| \le C,\ t \in J$. Thus Ω_0 is a bounded set in $PC[J, X]$. Take $R > C$, let $\Omega = \{x \in PC[J, X] \mid \|x\|_{PC} < R\}$. Then Ω is a bounded open set in $PC[J, X]$ and $\theta \in \Omega$. From the chose of R we know that if $x \in \partial\Omega$ and $\lambda \in [0, 1]$, then $x \ne \lambda Ax, x' \ne \lambda(Ax)'$, That is, (i) of in Lemma 2.3 is satisfied.

Let $H \subset \overline{\Omega}$ be a countable set and $H \subset \overline{co}(\{\theta\} \bigcup A(H))$. By ($H_1$), it is easy to see that $A(H)$ is equicontinuous on each $J_k (k = 0, 1, 2, \ldots, m)$.

Since TH, SH are equicontinuous bounded sets on each $J_k (k = 0, 1, 2, \ldots, m)$, by Lemma 2.2 we get

$$\alpha((TH)(s)) \le \int_0^s \alpha(k(s,\tau)H(\tau)d\tau) \le k^* \int_0^s \alpha(H(\tau)d\tau), s \in J_0, \tag{24}$$

$$\alpha((SH)(s)) \le \int_0^a \alpha(h(s,\tau)H(\tau)d\tau) \le h^* \int_0^a \alpha(H(\tau)d\tau), s \in J_0. \tag{25}$$

Therefore we have

$$\alpha(H(t)) \leq \alpha((AH)(t))$$
$$= \alpha\left(\int_0^t (t-s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))ds \Big|\ x\in H\right)$$
$$\leq 2al_1\int_0^t \alpha(H(s))ds + 2al_2\int_0^t \alpha(H'(s))ds$$
$$+2a^2l_3k^*\int_0^t \alpha(H(s))ds + 2a^2l_4[h^*]^{\lambda_2}\int_0^a [\alpha(H(s))]^{\lambda_2}ds, \tag{26}$$
$$\alpha(H'(t)) \leq \alpha((AH)'(t))$$
$$= \alpha\left(\int_0^t f(s,x(s),x'(s),(Tx)(s),(Sx)(s))ds \Big| x\in H\right)$$
$$\leq 2l_1\int_0^t \alpha(H(s))ds + 2l_2\int_0^t \alpha(H'(s))ds$$
$$+2al_3k^*\int_0^t \alpha(H(s))ds + 2al_4[h^*]^{\lambda_2}\int_0^a [\alpha(H(s))]^{\lambda_2}ds. \tag{27}$$

where $l_i = \max\{l_i(t)\}(i=1,2,3,4)$.

Let $m(t) = \alpha(H(t)) + \alpha(H'(t)), t \in J_0$, then $m(t) \in C[J_0, R^+]$. Thus (26) and (27) imply

$$m(t) \leq 2(a+1)\left((l_1+l_2)\int_0^t m(s)ds + l_3k^*a\int_0^t m(s)ds + l_4[h^*]^{\lambda_2}a\int_0^a [m(s)]^{\lambda_2}ds\right)$$
$$\leq 2(a+1)(l_1+l_2+l_3k^*a)\int_0^t m(s)ds + 2(a+1)l_4[h^*]^{\lambda_2}a\int_0^a [m(s)]^{\lambda_2}ds \tag{28}$$

By Lemma 2.4, $m(t) \equiv 0$, $t \in J_0$. Therefore $\alpha(H(t)) = \alpha(H'(t)) = 0$, that is, H, H' is relatively compact sets in $PC[J_0, X]$, Thus $\alpha(H(t_1)) = \alpha(H'(t_1)) = 0$, Since $I_1, \bar{I}_1 \in C[X\times X, X]$, $\alpha(I_1(H(t_1), H'(t_1))) = \alpha(\bar{I}_1(H(t_1), H'(t_1))) = 0$. $H(t_1), H'(t_1)$ are relatively compact sets in X, For $t \in (t_1, t_2]$, we know

$$\alpha(H(t)) \leq 2a\left(l_1\int_0^t \alpha(H(s))ds + l_2\int_0^t \alpha(H'(s))ds + al_3k^*\int_0^t \alpha(H(s))ds\right.$$
$$\left.+al_4[h^*]^{\lambda_2}\int_0^a [\alpha(H(s))]^{\lambda_2}ds\right) + \alpha(I_1(H(t_1), H'(t_1))) + (t-t_1)\alpha(\bar{I}_1(H(t_1), H'(t_1)))$$
$$= 2a\left(l_1\int_0^t \alpha(H(s))ds + l_2\int_0^t \alpha(H'(s))ds\right.$$
$$\left.+al_3k^*\int_0^t \alpha(H(s))ds + al_4[h^*]^{\lambda_2}\int_0^a [\alpha(H(s))]^{\lambda_2}ds\right) \tag{29}$$
$$\alpha(H'(t)) \leq \alpha\left(\int_0^t f(s,x(s),x'(s),(Tx)(s),(Sx)(s))ds \Big|\ x\in H\right)$$

$$+\alpha\left((I_1(x(t_1)),x'(t_1)))\,\Big|\, x\in H\right)$$

$$\le 2\left(l_1\int_0^t \alpha(H(s))ds + l_2\int_0^t \alpha(H'(s))ds\right)$$

$$+al_3k^*\int_0^t \alpha(H(s))ds + al_4[h^*]^{\lambda_2}\int_0^a [\alpha(H(s))]^{\lambda_2}ds\Big) \qquad (30)$$

From (29) and (30) we obtain

$$m(t)\le 2(a+1)[l_1+l_2+al_3k^*]\int_0^t m(s)ds + 2al_4[h^*]^{\lambda_2}(a+1)\int_0^a m(s)ds. \quad (31)$$

From Lemma 2.4 we get $m(t)\equiv 0, t\in J_1$. Therefore $\alpha(H(t))=\alpha(H'(t))=0, t\in J_1$. Especially, $\alpha(H(t_2))=\alpha(H'(t_2))=0$, so H, H' are relatively compact sets in $C[J_1,E]$. Similarly, we can show that H,H' are relatively compact sets in $C[J_k,E](k=2,3,\cdots,m)$. thus H,H' are relatively compact sets in $PC[J,E]$, H is relatively compact set in $PC[J,E]$. The condition (ii) in Lemma 2.3 is satisfied, so A has at least one fixed point in Ω, that is, system (1) has at least one fixed point in $PC[J,E]$.

Remark 3.1: When conditions of norm include the norm of Sx, it is difficult to obtain the existence of solutions of system (1), thus the new comparison result we present in this paper is important and interesting.

Remark 3.2: Condition (H_2) require b_1 and b_5 are only Lebesgue integrable in this paper.

Remark 3.3: we do not need the restrictive conditions on estimate of noncompact measure in this paper.

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