The Study of Convergence Theorems for Nonlinear Cyclic Mappings

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Abstract

In this paper, we establish some new convergence theorems for new nonlinear mappings satisfying the following condition:

(S) there exists an \( MT\)-function \( \varphi : [0, \infty) \rightarrow [0, 1) \) such that

\[
d(Tx, Ty) \leq \varphi(d(x, y)) \max \left\{ d(x, y), \frac{1}{5}[d(Tx, x) + 2d(Ty, y) + d(y, Tx)] \right\} \\
+ (1 - \varphi(d(x, y))) \text{dist}(A, B)
\]

for all \( x \in A \) and \( y \in B \).

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1. Introduction and preliminaries

Let \( M \) be a nonempty subset of a metric space \((X, d)\). If the fixed point equation \(Tx = x\) has at least one solution, then we state that the map \( T : M \rightarrow X \) has a fixed point in \( M \) (that is, \( d(x, Tx) = 0 \)). By our experience, the

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equation $Tx = x$ does not necessarily have a solution. In this case, we turn to find an element $x \in M$ such that $d(x, Tx)$ is minimum and call the $x$ is the best approximation of the fixed point of $T$.

The cyclic mappings and the best proximity points were introduced by Kirk, Srinavasan and Veeramani [10] in 2003. Some new results on cyclic mappings have been established in the literature, see e.g. [1, 9, 11-12, 14].

**Definition 1.1** [10]. Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$. A mapping $S : A \cup B \to A \cup B$ is called a cyclic mapping if

$$S(A) \subseteq B \text{ and } S(B) \subseteq A. \quad (1.1)$$

**Definition 1.2** [9]. Let $A$ and $B$ be nonempty closed subsets of a complete metric space $(X, d)$. A cyclic mapping $T : A \cup B \to A \cup B$ is called a cyclic contraction if for some $\alpha \in (0, 1)$, the condition

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha) \text{dist}(A, B)$$

holds for all $x \in A$, $y \in B$, where $\text{dist}(A, B) = \inf\{d(x, y) : x \in A \text{ and } y \in B\}$.

**Theorem 1.3** [9]. Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$. $T : A \cup B \to A \cup B$ be a cyclic contraction mapping, $x_1 \in A$ and define $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. Suppose $\{x_{2n-1}\}$ has a convergent subsequence in $A$. Then there exists $x \in A$ such that $d(x, Tx) = \text{dist}(A, B)$.

**Theorem 1.4** [12, Proposition 6]. Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$. Suppose $T : A \cup B \to A \cup B$ is a generalized cyclic contraction mapping then starting with any $x_0 \in A \cup B$, we have $d(x_n, Tx_n) \to \text{dist}(A, B)$, where $Tx_n = x_{n+1}$, $n \in \mathbb{N} \cup \{0\}$,

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = \text{dist}(A, B).$$

All over this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let $f$ be a real-valued function defined on $\mathbb{R}$. For $c \in \mathbb{R}$, we recall that

$$\limsup_{x \to c} f(x) = \inf_{\varepsilon > 0} \sup_{0 < |x - c| < \varepsilon} f(x)$$

and

$$\limsup_{x \to c^+} f(x) = \inf_{\varepsilon > 0} \sup_{0 < x - c < \varepsilon} f(x).$$
Definition 1.5 [3-8, 14]. A function $\varphi : [0, \infty) \rightarrow [0, 1)$ is said to be an $\mathcal{MT}$-function (or $R$-function) if

$$\limsup_{s \to t^+} \varphi(s) < 1 \text{ for all } t \in [0, \infty)$$

Obviously, if $\varphi : [0, \infty) \rightarrow [0, 1)$ is a nondecreasing function or a nonincreasing function, then $\varphi$ is an $\mathcal{MT}$-function. Hence the set of $\mathcal{MT}$-functions is an important class.

In 2012, Du [5] first proved the following characterizations of $\mathcal{MT}$-functions.

Theorem 1.6 [5]. Let $\varphi : [0, \infty) \rightarrow [0, 1)$ be a function. Then the following statements are equivalent.

(a) $\varphi$ is an $\mathcal{MT}$-function.

(b) For each $t \in [0, \infty)$, there exist $r_t^{(1)} \in [0, 1)$ and $\varepsilon_t^{(1)} > 0$ such that $\varphi(s) \leq r_t^{(1)}$ for all $s \in (t, t + \varepsilon_t^{(1)})$.

(c) For each $t \in [0, \infty)$, there exist $r_t^{(2)} \in [0, 1)$ and $\varepsilon_t^{(2)} > 0$ such that $\varphi(s) \leq r_t^{(2)}$ for all $s \in [t, t + \varepsilon_t^{(2)}]$.

(d) For each $t \in [0, \infty)$, there exist $r_t^{(3)} \in [0, 1)$ and $\varepsilon_t^{(3)} > 0$ such that $\varphi(s) \leq r_t^{(3)}$ for all $s \in (t, t + \varepsilon_t^{(3)})$.

(e) For each $t \in [0, \infty)$, there exist $r_t^{(4)} \in [0, 1)$ and $\varepsilon_t^{(4)} > 0$ such that $\varphi(s) \leq r_t^{(4)}$ for all $s \in [t, t + \varepsilon_t^{(4)}]$.

(f) For any nonincreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1$.

(g) $\varphi$ is a function of contractive factor; that is, for any strictly decreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1$.

In 2014, Lin and Chang [14] first introduced the new concept of Karapinar’s type $\mathcal{MT}$-cyclic contraction as follows.
**Definition 1.7 [14].** Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$. A cyclic mapping $T: A \cup B \to A \cup B$ is called Karapinar’s type $MT$-cyclic contraction, if there exists an $MT$-function $\varphi : [0, \infty) \to [0,1)$, such that
\[
d(Tx,Ty) \leq \frac{1}{3} \varphi(d(x,y))[d(x,y) + d(Tx,x) + d(Ty,y)] + (1 - \varphi(d(x,y)))d(A,B)
\]
holds for all $x \in A$ and $y \in B$.

Lin and Chang [14] proved the following convergence theorems for Karapinar’s type $MT$-cyclic contraction.

**Theorem 1.8 [14].** Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$. Suppose $T: A \cup B \to A \cup B$ is a Karapinar’s type $MT$-cyclic contraction, then there exists a sequence $\{x_n\}$, such that
\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = d(A,B).
\]

In this paper, we establish some new convergence theorems for new nonlinear mappings satisfying the following condition:

(S) there exists an $MT$-function $\varphi : [0, \infty) \to [0,1)$ such that
\[
d(Tx,Ty) \leq \varphi(d(x,y)) \max \left\{d(x,y), \frac{1}{5}[d(Tx,x) + 2d(Ty,y) + d(y,Tx)]\right\}
\]
\[
+ (1 - \varphi(d(x,y)))\text{dist}(A,B)
\]
for all $x \in A$ and $y \in B$.

2. Some new convergence theorems

In this section, we establish some convergence theorems with regard to the best proximity points.

**Theorem 2.1.** Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$ and $T : A \cup B \to A \cup B$ be a cyclic mapping. Suppose that

(S) there exists an $MT$-function $\varphi : [0, \infty) \to [0,1)$ such that
\[
d(Tx,Ty) \leq \varphi(d(x,y)) \max \left\{d(x,y), \frac{1}{5}[d(Tx,x) + 2d(Ty,y) + d(y,Tx)]\right\}
\]
\[
+ (1 - \varphi(d(x,y)))\text{dist}(A,B)
\]
for all $x \in A$ and $y \in B$. 


Then there exists a sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \( A \cup B \) such that
\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) = \text{dist}(A, B).
\]

**Proof.** Because \( T \) is cyclic, we have \( T(A) \subset B \) and \( T(B) \subset A \). Let \( x_1 \in A \) be given. Define \( x_{n+1} = Tx_n \) for all \( n \in \mathbb{N} \). Then \( \{x_{2n-1}\}_{n \in \mathbb{N}} \subset A \) and \( \{x_{2n}\}_{n \in \mathbb{N}} \subset B \). We want to prove
\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) = \text{dist}(A, B).
\]
If \( \varphi(d(x_1, x_2)) = 0 \), then, by (S), we have
\[
d(x_2, x_3) \leq \text{dist}(A, B) \leq d(x_1, x_2)
\]
and
\[
d(x_2, x_3) \leq \varphi(d(x_1, x_2))d(x_1, x_2) + (1 - \varphi(d(x_1, x_2)))\text{dist}(A, B).
\]
If \( \varphi(d(x_1, x_2)) > 0 \), then, by (S) again, we have
\[
d(x_2, x_3) \leq \varphi(d(x_1, x_2)) \max \left\{ d(x_1, x_2), \frac{1}{5}[d(x_2, x_1) + 2d(x_3, x_2) + d(x_2, x_2)] \right\}
+ (1 - \varphi(d(x_1, x_2)))\text{dist}(A, B)
\leq \varphi(d(x_1, x_2)) \max \left\{ d(x_1, x_2), \frac{1}{5}[d(x_1, x_2) + 2d(x_2, x_3)] \right\}
+ (1 - \varphi(d(x_1, x_2)))\text{dist}(A, B).
\]
(2.1)
Assume that
\[
\frac{1}{5}[d(x_1, x_2) + 2d(x_2, x_3)] > d(x_1, x_2)
\]
which implies
\[
\frac{1}{2}d(x_2, x_3) > d(x_1, x_2).
\]
It follows from (2.1) and above inequalities that
\[
d(x_2, x_3) \leq \frac{1}{5}\varphi(d(x_1, x_2))[d(x_1, x_2) + 2d(x_2, x_3)] + (1 - \varphi(d(x_1, x_2)))\text{dist}(A, B)
< \frac{1}{2}\varphi(d(x_1, x_2))d(x_2, x_3) + (1 - \varphi(d(x_1, x_2)))\text{dist}(A, B)
< \varphi(d(x_1, x_2))d(x_2, x_3) + (1 - \varphi(d(x_1, x_2)))\text{dist}(A, B)
\leq \varphi(d(x_1, x_2))d(x_2, x_3) + (1 - \varphi(d(x_1, x_2)))d(x_2, x_3)
= d(x_2, x_3)
\]
which leads a contradiction. Hence it must be
\[ d(x_1, x_2) \geq \frac{1}{5}[d(x_1, x_2) + 2d(x_2, x_3)]. \]  
(2.2)

By (2.1) and (2.2), we get
\[ d(x_2, x_3) \leq \varphi(d(x_1, x_2))d(x_1, x_2) + (1 - \varphi(d(x_1, x_2)))\text{dist}(A, B) \]
\[ \leq \varphi(d(x_1, x_2))d(x_1, x_2) + (1 - \varphi(d(x_1, x_2)))d(x_1, x_2) \]
\[ = d(x_1, x_2). \]

Next, if \( \varphi(d(x_3, x_2)) = 0 \), then, by (S), we have
\[ d(x_4, x_3) \leq \text{dist}(A, B) \leq d(x_2, x_3) \]
and
\[ d(x_4, x_3) \leq \varphi(d(x_3, x_2))d(x_3, x_2) + (1 - \varphi(d(x_3, x_2)))\text{dist}(A, B). \]

If \( \varphi(d(x_3, x_2)) > 0 \), then, from (S) again, we have
\[ d(x_4, x_3) \leq \varphi(d(x_3, x_2)) \max \left\{ d(x_3, x_2), \frac{1}{5}[d(x_4, x_3) + 2d(x_3, x_2) + d(x_2, x_4)] \right\} \]
\[ + (1 - \varphi(d(x_3, x_2)))\text{dist}(A, B) \]
\[ \leq \varphi(d(x_2, x_3)) \max \left\{ d(x_2, x_3), \frac{1}{5}[2d(x_2, x_3) + d(x_3, x_4) + d(x_2, x_4)] \right\} \]
\[ + (1 - \varphi(d(x_2, x_3)))\text{dist}(A, B). \]
(2.3)

If \( \frac{1}{5}[2d(x_2, x_3) + d(x_3, x_4) + d(x_2, x_4)] > d(x_2, x_3) \), then we obtain
\[ d(x_3, x_4) + d(x_2, x_4) > 3d(x_2, x_3) \]
which deduces
\[ d(x_2, x_3) + 2d(x_3, x_4) \geq d(x_3, x_4) + d(x_2, x_4) > 3d(x_2, x_3) \]
and hence
\[ d(x_3, x_4) > d(x_2, x_3). \]

By (2.3) and above inequalities, we have
\[ d(x_3, x_4) \leq \frac{1}{5}\varphi(d(x_2, x_3))[2d(x_2, x_3) + d(x_3, x_4) + d(x_2, x_4)] \]
\[ + (1 - \varphi(d(x_2, x_3)))\text{dist}(A, B) \]
\[ \leq \frac{1}{5}\varphi(d(x_2, x_3))[3d(x_2, x_3) + 2d(x_3, x_4)] \]
\[ + (1 - \varphi(d(x_2, x_3)))\text{dist}(A, B) \]
\[ < \varphi(d(x_2, x_3))d(x_3, x_4) + (1 - \varphi(d(x_2, x_3)))\text{dist}(A, B) \]
\[ \leq \varphi(d(x_2, x_3))d(x_3, x_4) + (1 - \varphi(d(x_2, x_3)))d(x_3, x_4) \]
\[ < d(x_3, x_4). \]
which is a contradiction. So it must be

\[ d(x_2, x_3) \geq \frac{1}{5}[2d(x_2, x_3) + d(x_3, x_4) + d(x_2, x_4)]. \]  \hspace{1cm} (2.4)

Using (2.3) and (2.4), we have

\[ d(x_3, x_4) \leq \varphi(d(x_2, x_3))d(x_2, x_3) + (1 - \varphi(d(x_2, x_3))) \text{dist}(A, B) \leq d(x_2, x_3). \]

Similarly, if \( \varphi(d(x_3, x_4)) = 0 \), then, by (S), we have

\[ d(x_4, x_5) \leq \text{dist}(A, B) \leq d(x_3, x_4) \]

and hence

\[ d(x_4, x_5) \leq \varphi(d(x_3, x_4))d(x_3, x_4) + (1 - \varphi(d(x_3, x_4))) \text{dist}(A, B). \]

If \( \varphi(d(x_3, x_4)) > 0 \), then, by (S), we obtain

\[ d(x_4, x_5) \leq \varphi(d(x_3, x_4)) \max \left\{ d(x_3, x_4), \frac{1}{5}[d(x_3, x_4) + 2d(x_5, x_4) + d(x_4, x_4)] \right\} \]

\[ + (1 - \varphi(d(x_3, x_4))) \text{dist}(A, B) \leq \varphi(d(x_3, x_4)) \max \left\{ d(x_3, x_4), \frac{1}{5}[d(x_3, x_4) + 2d(x_4, x_5)] \right\} \]

\[ + (1 - \varphi(d(x_3, x_4))) \text{dist}(A, B). \] \hspace{1cm} (2.5)

If \( \frac{1}{5}[d(x_3, x_4) + 2d(x_4, x_5)] > d(x_3, x_4) \), then we obtain

\[ \frac{1}{2}d(x_4, x_5) > d(x_3, x_4). \]

By (2.5) and above inequalities, we get

\[ d(x_4, x_5) \leq \frac{1}{5}\varphi(d(x_3, x_4))[d(x_3, x_4) + 2d(x_4, x_5)] + (1 - \varphi(d(x_3, x_4))) \text{dist}(A, B) \]

\[ < \frac{1}{2}\varphi(d(x_3, x_4))d(x_4, x_5) + (1 - \varphi(d(x_3, x_4))) \text{dist}(A, B) \]

\[ < \varphi(d(x_3, x_4))d(x_4, x_5) + (1 - \varphi(d(x_3, x_4))) \text{dist}(A, B) \leq \varphi(d(x_3, x_4))d(x_4, x_5) + (1 - \varphi(d(x_3, x_4)))d(x_4, x_5) \]

\[ = d(x_4, x_5) \]

which leads a contradiction. Hence

\[ d(x_3, x_4) \geq \frac{1}{5}[d(x_3, x_4) + 2d(x_4, x_5)]. \] \hspace{1cm} (2.6)
From (2.5) and (2.6), we have

\[
d(x_4, x_5) \leq \varphi(d(x_3, x_4))d(x_3, x_4) + (1 - \varphi(d(x_3, x_4)))\text{dist}(A, B) \\
\leq d(x_3, x_4).
\]

If \(\varphi(d(x_5, x_4)) = 0\), then, by (S), we have

\[
d(x_5, x_4) \leq \text{dist}(A, B) \leq d(x_5, x_4)
\]

and

\[
d(x_6, x_5) \leq \varphi(d(x_5, x_4))d(x_5, x_4) + (1 - \varphi(d(x_5, x_4)))\text{dist}(A, B).
\]

If \(\varphi(d(x_5, x_4)) > 0\), thenFrom (S) again, we obtain

\[
d(x_6, x_5) \leq \varphi(d(x_5, x_4)) \max \left\{ d(x_5, x_4), \frac{1}{5}[d(x_6, x_5) + 2d(x_5, x_4) + d(x_4, x_6)] \right\} \\
+ (1 - \varphi(d(x_5, x_4)))\text{dist}(A, B) \\
\leq \varphi(d(x_4, x_5)) \max \left\{ d(x_4, x_5), \frac{1}{5}[2d(x_4, x_5) + d(x_5, x_6) + d(x_4, x_6)] \right\} \\
+ (1 - \varphi(d(x_4, x_5)))\text{dist}(A, B). \tag{2.7}
\]

If \(\frac{1}{5}[2d(x_4, x_5) + d(x_5, x_6) + d(x_4, x_6)] > d(x_4, x_5)\), then we obtain

\[
d(x_5, x_6) + d(x_4, x_6) > 3d(x_4, x_5)
\]

which deduces

\[
d(x_4, x_5) + 2d(x_5, x_6) \geq d(x_5, x_6) + d(x_4, x_6) > 3d(x_4, x_5)
\]

and hence

\[
d(x_5, x_6) > d(x_4, x_5).
\]

So we get

\[
d(x_5, x_6) \leq \frac{1}{5}\varphi(d(x_4, x_5))[2d(x_4, x_5) + d(x_5, x_6) + d(x_4, x_6)] \\
+ (1 - \varphi(d(x_4, x_5)))\text{dist}(A, B) \\
\leq \frac{1}{5}\varphi(d(x_4, x_5))[3d(x_4, x_5) + 2d(x_5, x_6)] \\
+ (1 - \varphi(d(x_4, x_5)))\text{dist}(A, B) \\
= \varphi(d(x_4, x_5))d(x_5, x_6) + (1 - \varphi(d(x_4, x_5)))d(x_5, x_6) \\
< d(x_5, x_6)
\]
which leads a contradiction. So we confirm that
\[ d(x_4, x_5) \geq \frac{1}{5}[2d(x_4, x_5) + d(x_5, x_6) + d(x_4, x_6)]. \]

From (2.7) and above inequalities, we have
\[ d(x_5, x_6) \leq \varphi(d(x_4, x_5))d(x_4, x_5) + (1 - \varphi(d(x_4, x_5)))\text{dist}(A, B) \]
\[ \leq d(x_4, x_5). \]

Hence, by induction, we obtain a sequence \( \{x_n\}_{n\in\mathbb{N}} \) satisfying
\[ d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}) \quad (2.8) \]
and
\[ d(x_{n+1}, x_{n+2}) \leq \varphi(d(x_n, x_{n+1}))d(x_n, x_{n+1}) + (1 - \varphi(d(x_n, x_{n+1})))\text{dist}(A, B) \quad \text{for all } n \in \mathbb{N}. \quad (2.9) \]

Clearly, (2.8) shows that the sequence \( \{d(x_n, x_{n+1})\} \) is nonincreasing in \([0, \infty)\), and hence we know
\[ \lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) \text{ exists.} \quad (2.10) \]

Since \( \varphi \) is an \( \mathcal{M}T \)-function, by Theorem 1.6, we obtain
\[ 0 \leq \sup_{n \in \mathbb{N}}(\varphi(d(x_n, x_{n+1}))) < 1. \]

Let
\[ \eta := \sup_{n \in \mathbb{N}}(\varphi(d(x_n, x_{n+1}))). \]

Then
\[ 0 \leq \varphi(d(x_n, x_{n+1})) \leq \eta < 1 \quad \text{for all } n \in \mathbb{N}. \quad (2.11) \]

Taking \( n = 1 \) in (2.9), we have from (2.11) that
\[ d(x_2, x_3) \leq \varphi(d(x_1, x_2))d(x_1, x_2) + (1 - \varphi(d(x_1, x_2)))\text{dist}(A, B) \]
\[ \leq \eta d(x_1, x_2) + \text{dist}(A, B). \quad (2.12) \]

Taking \( n = 2 \) in (2.9) and using (2.11) and (2.12), we obtain
\[ d(x_3, x_4) \leq \varphi(d(x_2, x_3))d(x_2, x_3) + (1 - \varphi(d(x_2, x_3)))\text{dist}(A, B) \]
\[ \leq \varphi(d(x_2, x_3)) [\eta d(x_1, x_2) + \text{dist}(A, B)] + (1 - \varphi(d(x_2, x_3)))\text{dist}(A, B) \]
\[ \leq \eta^2 d(x_1, x_2) + \text{dist}(A, B). \quad (2.13) \]
Taking $n = 3$ in (2.9) and using (2.11) and (2.13), we get

$$d(x_4, x_5) \leq \varphi(d(x_3, x_4))d(x_3, x_4) + (1 - \varphi(d(x_3, x_4)))\text{dist}(A, B)$$
$$\leq \varphi(d(x_3, x_4)) [\eta^2d(x_1, x_2) + \text{dist}(A, B)] + (1 - \varphi(d(x_3, x_4)))\text{dist}(A, B)$$
$$\leq \eta^2d(x_1, x_2) + \text{dist}(A, B). \quad (2.14)$$

Taking $n = 4$ in (2.9) and using (2.11) and (2.14), we have

$$d(x_5, x_6) \leq \varphi(d(x_4, x_5))d(x_4, x_5) + (1 - \varphi(d(x_4, x_5)))\text{dist}(A, B)$$
$$\leq \varphi(d(x_4, x_5)) [\eta^3d(x_1, x_2) + \text{dist}(A, B)] + (1 - \varphi(d(x_4, x_5)))\text{dist}(A, B)$$
$$\leq \eta^3d(x_1, x_2) + \text{dist}(A, B).$$

Continuing this process, we obtain

$$\text{dist}(A, B) \leq d(x_{n+1}, x_{n+2}) \leq \eta^n d(x_1, x_2) + \text{dist}(A, B). \quad (2.15)$$

Due to $\eta \in [0, 1)$, $\lim_{n \to \infty} \eta^n = 0$. By taking the limit as $n \to \infty$ in (2.15), we get

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = \text{dist}(A, B). \quad (2.16)$$

Finally, we finish the proof by combining (2.10) with (2.16). \qed

**Corollary 2.2.** Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$ and $T : A \cup B \to A \cup B$ be a cyclic mapping. Suppose that there exists a nondecreasing function $\mu : [0, \infty) \to [0, 1)$ such that

$$d(Tx, Ty) \leq \mu(d(x, y)) \max \left\{ d(x, y), \frac{1}{5}[d(Tx, x) + 2d(Ty, y) + d(y, Tx)] \right\}$$
$$+ (1 - \mu(d(x, y)))\text{dist}(A, B)$$

for all $x \in A$ and $y \in B$. Then there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $A \cup B$ such that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) = \text{dist}(A, B).$$

**Proof.** Since $\mu$ is a nondecreasing function, $\mu$ is an $\mathcal{MT}$-function. Therefore, the conclusion is immediate from Theorem 2.1. \qed

**Corollary 2.3.** Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$ and $T : A \cup B \to A \cup B$ be a cyclic mapping. Suppose that there exists a nonincreasing function $\tau : [0, \infty) \to [0, 1)$ such that

$$d(Tx, Ty) \leq \tau(d(x, y)) \max \left\{ d(x, y), \frac{1}{5}[d(Tx, x) + 2d(Ty, y) + d(y, Tx)] \right\}$$
$$+ (1 - \tau(d(x, y)))\text{dist}(A, B)$$
for all \(x \in A\) and \(y \in B\). Then there exists a sequence \(\{x_n\}_{n \in \mathbb{N}}\) in \(A \cup B\) such that
\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) = \text{dist}(A, B).
\]

**Proof.** Since \(\tau\) is a nonincreasing function, \(\tau\) is an \(\mathcal{MT}\)-function. Therefore, the conclusion is immediate from Theorem 2.1. \(\square\)

The following conclusions are immediate from Theorem 2.1.

**Theorem 2.4.** Let \(A\) and \(B\) be nonempty subsets of a metric space \((X, d)\) and \(T : A \cup B \to A \cup B\) be a cyclic mapping. Suppose that

(S) there exists an \(\mathcal{MT}\)-function \(\varphi : [0, \infty) \to [0, 1)\) such that
\[
d(Tx, Ty) \leq \frac{1}{5} \varphi(d(x, y))[d(Tx, x) + 2d(Ty, y) + d(y, Tx)]
+ (1 - \varphi(d(x, y)))\text{dist}(A, B)
\]
for all \(x \in A\) and \(y \in B\).

Then there exists a sequence \(\{x_n\}_{n \in \mathbb{N}}\) in \(A \cup B\) such that
\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) = \text{dist}(A, B).
\]

**Corollary 2.5.** Let \(A\) and \(B\) be nonempty subsets of a metric space \((X, d)\) and \(T : A \cup B \to A \cup B\) be a cyclic mapping. Suppose that there exists a nondecreasing function \(\mu : [0, \infty) \to [0, 1)\) such that
\[
d(Tx, Ty) \leq \frac{1}{5} \mu(d(x, y))[d(Tx, x) + 2d(Ty, y) + d(y, Tx)]
+ (1 - \mu(d(x, y)))\text{dist}(A, B)
\]
for all \(x \in A\) and \(y \in B\). Then there exists a sequence \(\{x_n\}_{n \in \mathbb{N}}\) in \(A \cup B\) such that
\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) = \text{dist}(A, B).
\]

**Corollary 2.6.** Let \(A\) and \(B\) be nonempty subsets of a metric space \((X, d)\) and \(T : A \cup B \to A \cup B\) be a cyclic mapping. Suppose that there exists a nonincreasing function \(\tau : [0, \infty) \to [0, 1)\) such that
\[
d(Tx, Ty) \leq \frac{1}{5} \tau(d(x, y))[d(Tx, x) + 2d(Ty, y) + d(y, Tx)]
+ (1 - \tau(d(x, y)))\text{dist}(A, B)
\]
for all $x \in A$ and $y \in B$. Then there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $A \cup B$ such that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) = \text{dist}(A, B).$$

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References


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