Nonlinear Analysis and Differential Equations, Vol. 6, 2018, no. 1, 1 - 14
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/nade.2018.831

The Study of Convergence Theorems for Nonlinear Cyclic Mappings

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Abstract

In this paper, we establish some new convergence theorems for new nonlinear mappings satisfying the following condition:

(S) there exists an $\mathcal{MT}$-function $\varphi : [0,\infty) \to [0,1)$ such that

$$\begin{aligned} d(Tx,Ty) &\le \varphi(d(x,y)) \max\left\{ d(x,y), \frac{1}{5}[d(Tx,x)+2d(Ty,y)+d(y,Tx)] \right\} \\ &\quad + (1-\varphi(d(x,y)))dist(A,B) \end{aligned}$$

for all $x \in A$ and $y \in B$.

Mathematics Subject Classification: 41A17, 41A65

Keywords: $\mathcal{MT}$-function ($\mathcal{R}$-function), cyclic mapping, best proximity point, convergence theorem

1. Introduction and preliminaries

Let M be a nonempty subset of a metric space (X,d). If the fixed point equation $Tx = x$ has at least one solution, then we state that the map $T : M \to X$ has a fixed point in M (that is, $d(x,Tx) = 0$). By our experience, the

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equation $Tx = x$ does not necessarily have a solution. In this case, we turn to find an element $x \in M$ such that $d(x, Tx)$ is minimum and call the x is the best approximation of the fixed point of T.

The cyclic mappings and the best proximity points were introduced by Kirk, Srinavasan and Veeramani [10] in 2003. Some new results on cyclic mappings have been established in the literature, see e.g. [1, 9, 11-12, 14].

Definition 1.1 [10]. Let A and B be nonempty subsets of a metric space (X, d). A mapping $S : A \cup B \to A \cup B$ is called a *cyclic* mapping if

$$S(A) \subset B \text{ and } S(B) \subset A. \tag{1.1}$$

Definition 1.2 [9]. Let A and B be nonempty closed subsets of a complete metric space (X, d). A cyclic mapping T: $A \cup B \to A \cup B$ is called a cyclic contraction if for some $\alpha \in (0,1)$, the condition

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha) dist(A, B)$$

holds for all $x \in A$, $y \in B$, where $dist(A, B) = \inf\{d(x, y) : x \in A \text{ and } y \in B\}$.

Theorem 1.3 [9]. Let A and B be nonempty closed subsets of a complete metric space X. T: $A \cup B \to A \cup B$ be a cyclic contraction mapping, $x_1 \in A$ and define $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. Suppose $\{x_{2n-1}\}$ has a convergent subsequence in A. Then there exists $x \in A$ such that $d(x, Tx) = dist(A, B)$.

Theorem 1.4 [12, Proposition 6]. Let A and B be nonempty subsets of a metric space (X, d). Suppose T: $A \cup B \to A \cup B$ is a generalized cyclic contraction mapping then starting with any $x_0 \in A \cup B$, we have $d(x_n, Tx_n) \to dist(A, B)$, where $Tx_n = x_{n+1}$, $n \in \mathbb{N} \cup \{0\}$,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = dist(A, B).$$

All over this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let f be a real-valued function defined on $\mathbb{R}$. For $c \in \mathbb{R}$, we recall that

$$\limsup_{x\to c} f(x) = \inf_{\varepsilon>0} \sup_{0<|x-c|<\epsilon} f(x)$$

and

$$\limsup_{x\to c^+} f(x) = \inf_{\varepsilon>0} \sup_{0<x-c<\varepsilon} f(x).$$

Definition 1.5 [3-8, 14]. A function $\varphi : [0, \infty) \to [0, 1)$ is said to be an $\mathcal{MT}$-*function* (or $\mathcal{R}$-*function*) if

$$\limsup_{s \to t^+} \varphi(s) < 1 \quad \text{for all } t \in [0, \infty)$$

Obviously, if $\varphi : [0, \infty) \to [0, 1)$ is a nondecreasing function or a nonincreasing function, then φ is an $\mathcal{MT}$-function. Hence the set of $\mathcal{MT}$-functions is an important class.

In 2012, Du [5] first proved the following characterizations of $\mathcal{MT}$-functions.

Theorem 1.6 [5]. Let $\varphi : [0, \infty) \to [0, 1)$ be a function. Then the following statements are equivalent.

(a) φ is an $\mathcal{MT}$-function.

(b) For each $t \in [0, \infty)$, there exist $r_t^{(1)} \in [0, 1)$ and $\varepsilon_t^{(1)} > 0$ such that $\varphi(s) \leq r_t^{(1)}$ for all $s \in (t, t + \varepsilon_t^{(1)})$.

(c) For each $t \in [0, \infty)$, there exist $r_t^{(2)} \in [0, 1)$ and $\varepsilon_t^{(2)} > 0$ such that $\varphi(s) \leq r_t^{(2)}$ for all $s \in [t, t + \varepsilon_t^{(2)}]$.

(d) For each $t \in [0, \infty)$, there exist $r_t^{(3)} \in [0, 1)$ and $\varepsilon_t^{(3)} > 0$ such that $\varphi(s) \leq r_t^{(3)}$ for all $s \in (t, t + \varepsilon_t^{(3)}]$.

(e) For each $t \in [0, \infty)$, there exist $r_t^{(4)} \in [0, 1)$ and $\varepsilon_t^{(4)} > 0$ such that $\varphi(s) \leq r_t^{(4)}$ for all $s \in [t, t + \varepsilon_t^{(4)})$.

(f) For any nonincreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1$.

(g) φ is a **function of contractive factor**; that is, for any strictly decreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1$.

In 2014, Lin and Chang [14] first introduced the new concept of Karapinar's type $\mathcal{MT}$-cyclic contraction as follows.

Definition 1.7 [14]. Let A and B be nonempty subsets of a metric space (X, d). A cyclic mapping T: $A \cup B \to A \cup B$ is called *Karapinar's type* $\mathcal{MT}$-*cyclic contraction*, if there exists an $\mathcal{MT}$-function $\varphi : [0, \infty) \to [0, 1)$, such that

$$d(Tx, Ty) \leq \frac{1}{3}\varphi(d(x, y))[d(x, y) + d(Tx, x) + d(Ty, y)] + (1 - \varphi(d(x, y))d(A, B)$$

holds for all $x \in A$ and $y \in B$.

Lin and Chang [14] proved the following convergence theorems for Karapinar's type $\mathcal{MT}$-cyclic contraction.

Theorem 1.8 [14]. Let A and B be nonempty subsets of a metric space (X, d). Suppose T: $A \cup B \to A \cup B$ is a Karapinar's type $\mathcal{MT}$-cyclic contraction, then there exists a sequence $\{x_n\}$, such that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = d(A, B).$$

In this paper, we establish some new convergence theorems for new nonlinear mappings satisfying the following condition:

(S) there exists an $\mathcal{MT}$-function $\varphi : [0, \infty) \to [0, 1)$ such that

$$\begin{aligned} d\,(Tx, Ty) \ &\leq \varphi(d(x, y)) \max\left\{d(x, y), \frac{1}{5}[d(Tx, x) + 2d(Ty, y) + d(y, Tx)]\right\} \\ &+ (1 - \varphi(d(x, y)))dist(A, B) \end{aligned}$$

for all $x \in A$ and $y \in B$.

2. Some new convergence theorems

In this section, we establish some convergence theorems with regard to the best proximity points.

Theorem 2.1. *Let A and B be nonempty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic mapping. Suppose that*

(S) *there exists an $\mathcal{MT}$-function $\varphi : [0, \infty) \to [0, 1)$ such that*

$$\begin{aligned} d\,(Tx, Ty) \ &\leq \varphi(d(x, y)) \max\left\{d(x, y), \frac{1}{5}[d(Tx, x) + 2d(Ty, y) + d(y, Tx)]\right\} \\ &+ (1 - \varphi(d(x, y)))dist(A, B) \end{aligned}$$

for all $x \in A$ and $y \in B$.

Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ *in* $A \cup B$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = dist(A, B).$$

Proof. Because T is cyclic, we have $T(A) \subset B$ and $T(B) \subset A$. Let $x_1 \in A$ be given. Define $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Then $\{x_{2n-1}\}_{n\in\mathbb{N}} \subset A$ and $\{x_{2n}\}_{n\in\mathbb{N}} \subset B$. We want to prove

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = dist(A, B).$$

If $\varphi(d(x_1, x_2)) = 0$, then, by (S), we have

$$d(x_2, x_3) \le dist(A, B) \le d(x_1, x_2)$$

and

$$d(x_2, x_3) \le \varphi(d(x_1, x_2))d(x_1, x_2) + (1 - \varphi(d(x_1, x_2)))dist(A, B).$$

If $\varphi(d(x_1, x_2)) > 0$, then, by (S) again, we have

$$\begin{aligned} d(x_2, x_3) &\le \varphi(d(x_1, x_2)) \max\left\{d(x_1, x_2), \frac{1}{5}[d(x_2, x_1) + 2d(x_3, x_2) + d(x_2, x_2)]\right\} \\ &\quad + (1 - \varphi(d(x_1, x_2)))dist(A, B) \\ &\le \varphi(d(x_1, x_2)) \max\left\{d(x_1, x_2), \frac{1}{5}[d(x_1, x_2) + 2d(x_2, x_3)]\right\} \\ &\quad + (1 - \varphi(d(x_1, x_2)))dist(A, B). \end{aligned} \tag{2.1}$$

Assume that

$$\frac{1}{5}[d(x_1, x_2) + 2d(x_2, x_3)] > d(x_1, x_2)$$

which implies

$$\frac{1}{2}d(x_2, x_3) > d(x_1, x_2).$$

It follows from (2.1) and above inequalities that

$$\begin{aligned} d(x_2, x_3) &\le \frac{1}{5}\varphi(d(x_1, x_2))[d(x_1, x_2) + 2d(x_2, x_3)] + (1 - \varphi(d(x_1, x_2)))dist(A, B) \\ &< \frac{1}{2}\varphi(d(x_1, x_2))d(x_2, x_3) + (1 - \varphi(d(x_1, x_2)))dist(A, B) \\ &< \varphi(d(x_1, x_2))d(x_2, x_3) + (1 - \varphi(d(x_1, x_2)))dist(A, B) \\ &\le \varphi(d(x_1, x_2))d(x_2, x_3) + (1 - \varphi(d(x_1, x_2)))d(x_2, x_3) \\ &= d(x_2, x_3) \end{aligned}$$

which leads a contradiction. Hence it must be

$$d(x_1,x_2) \geq \frac{1}{5}[d(x_1,x_2)+2d(x_2,x_3)]. \tag{2.2}$$

By (2.1) and (2.2), we get

$$\begin{aligned} d\,(x_2,x_3) &\leq \varphi(d(x_1,x_2))d(x_1,x_2)+(1-\varphi(d(x_1,x_2)))dist(A,B) \\ &\leq \varphi(d(x_1,x_2))d(x_1,x_2)+(1-\varphi(d(x_1,x_2)))d(x_1,x_2) \\ &= d(x_1,x_2). \end{aligned}$$

Next, if $\varphi(d(x_3,x_2))=0$, then, by (S), we have

$$d\,(x_4,x_3) \leq dist(A,B) \leq d(x_2,x_3)$$

and

$$d\,(x_4,x_3) \leq \varphi(d(x_3,x_2))d(x_3,x_2)+(1-\varphi(d(x_3,x_2)))dist(A,B).$$

If $\varphi(d(x_3,x_2))>0$, then, from (S) again, we have

$$\begin{aligned} d\,(x_4,x_3) \;&\leq \varphi(d(x_3,x_2))\max\left\{d(x_3,x_2),\frac{1}{5}[d(x_4,x_3)+2d(x_3,x_2)+d(x_2,x_4)]\right\} \\ &\quad+(1-\varphi(d(x_3,x_2)))dist(A,B) \\ &\leq \varphi(d(x_2,x_3))\max\left\{d(x_2,x_3),\frac{1}{5}[2d(x_2,x_3)+d(x_3,x_4)+d(x_2,x_4)]\right\} \\ &\quad+(1-\varphi(d(x_2,x_3)))dist(A,B). \end{aligned} \tag{2.3}$$

If $\frac{1}{5}[2d(x_2,x_3)+d(x_3,x_4)+d(x_2,x_4)]>d(x_2,x_3)$,then we obtain

$$d(x_3,x_4)+d(x_2,x_4)>3d(x_2,x_3)$$

which deduces

$$d(x_2,x_3)+2d(x_3,x_4) \geq d(x_3,x_4)+d(x_2,x_4)>3d(x_2,x_3)$$

and hence

$$d(x_3,x_4)>d(x_2,x_3).$$

By (2.3) and above inequalities, we have

$$\begin{aligned} d\,(x_3,x_4) &\leq \frac{1}{5}\varphi(d(x_2,x_3))[2d(x_2,x_3)+d(x_3,x_4)+d(x_2,x_4)] \\ &\quad+(1-\varphi(d(x_2,x_3)))dist(A,B) \\ &\leq \frac{1}{5}\varphi(d(x_2,x_3))[3d(x_2,x_3)+2d(x_3,x_4)] \\ &\quad+(1-\varphi(d(x_2,x_3)))dist(A,B) \\ &< \varphi(d(x_2,x_3))d(x_3,x_4)+(1-\varphi(d(x_2,x_3)))dist(A,B) \\ &\leq \varphi(d(x_2,x_3))d(x_3,x_4)+(1-\varphi(d(x_2,x_3)))d(x_3,x_4) \\ &< d(x_3,x_4) \end{aligned}$$

which is a contradiction. So it must be

$$d(x_2,x_3) \geq \frac{1}{5}[2d(x_2,x_3)+d(x_3,x_4)+d(x_2,x_4)]. \tag{2.4}$$

Using (2.3) and (2.4), we have

$$\begin{aligned} d\left(x_3,x_4\right) &\leq \varphi(d(x_2,x_3))d(x_2,x_3)+(1-\varphi(d(x_2,x_3)))dist(A,B) \\ &\leq d(x_2,x_3). \end{aligned}$$

Similarly, if $\varphi(d(x_3,x_4))=0$, then, by (S), we have

$$d\left(x_4,x_5\right) \leq dist(A,B) \leq d(x_3,x_4)$$

and hence

$$d\left(x_4,x_5\right) \leq \varphi(d(x_3,x_4))d(x_3,x_4)+(1-\varphi(d(x_3,x_4)))dist(A,B).$$

If $\varphi(d(x_3,x_4))>0$, then, by (S), we obtain

$$\begin{aligned} d\left(x_4,x_5\right) &\leq \varphi(d(x_3,x_4))\max\left\{d(x_3,x_4),\frac{1}{5}[d(x_4,x_3)+2d(x_5,x_4)+d(x_4,x_4)]\right\} \\ &\quad+(1-\varphi(d(x_3,x_4)))dist(A,B) \\ &\leq \varphi(d(x_3,x_4))\max\left\{d(x_3,x_4),\frac{1}{5}[d(x_3,x_4)+2d(x_4,x_5)]\right\} \\ &\quad+(1-\varphi(d(x_3,x_4)))dist(A,B). \end{aligned} \tag{2.5}$$

If $\frac{1}{5}[d(x_3,x_4)+2d(x_4,x_5)]>d(x_3,x_4)$, then we obtain

$$\frac{1}{2}d(x_4,x_5)>d(x_3,x_4).$$

By (2.5) and above inequalities, we get

$$\begin{aligned} d\left(x_4,x_5\right) &\leq \frac{1}{5}\varphi(d(x_3,x_4))[d(x_3,x_4)+2d(x_4,x_5)]+(1-\varphi(d(x_3,x_4)))dist(A,B) \\ &< \frac{1}{2}\varphi(d(x_3,x_4))d(x_4,x_5)+(1-\varphi(d(x_3,x_4)))dist(A,B) \\ &< \varphi(d(x_3,x_4))d(x_4,x_5)+(1-\varphi(d(x_3,x_4)))dist(A,B) \\ &\leq \varphi(d(x_3,x_4))d(x_4,x_5)+(1-\varphi(d(x_3,x_4)))d(x_4,x_5) \\ &= d(x_4,x_5) \end{aligned}$$

which leads a contradiction. Hence

$$d(x_3,x_4) \geq \frac{1}{5}[d(x_3,x_4)+2d(x_4,x_5)]. \tag{2.6}$$

From (2.5) and (2.6), we have

$$\begin{aligned} d\left(x_4, x_5\right) &\leq \varphi(d(x_3, x_4))d(x_3, x_4) + (1 - \varphi(d(x_3, x_4)))dist(A, B) \\ &\leq d(x_3, x_4). \end{aligned}$$

If $\varphi(d(x_5, x_4)) = 0$, then, by (S), we have

$$d(x_5, x_4) \leq dist(A, B) \leq d(x_5, x_4)$$

and

$$d\left(x_6, x_5\right) \leq \varphi(d(x_5, x_4))d(x_5, x_4) + (1 - \varphi(d(x_5, x_4)))dist(A, B).$$

If $\varphi(d(x_5, x_4)) > 0$, thenFrom (S) again, we obtain

$$\begin{aligned} d\left(x_6, x_5\right) &\leq \varphi(d(x_5, x_4)) \max\left\{d(x_5, x_4), \frac{1}{5}[d(x_6, x_5) + 2d(x_5, x_4) + d(x_4, x_6)]\right\} \\ &+ (1 - \varphi(d(x_5, x_4)))dist(A, B) \\ &\leq \varphi(d(x_4, x_5)) \max\left\{d(x_4, x_5), \frac{1}{5}[2d(x_4, x_5) + d(x_5, x_6) + d(x_4, x_6)]\right\} \\ &+ (1 - \varphi(d(x_4, x_5)))dist(A, B). \end{aligned} \tag{2.7}$$

If $\frac{1}{5}[2d(x_4, x_5) + d(x_5, x_6) + d(x_4, x_6)] > d(x_4, x_5)$, then we obtain

$$d(x_5, x_6) + d(x_4, x_6) > 3d(x_4, x_5)$$

which deduces

$$d(x_4, x_5) + 2d(x_5, x_6) \geq d(x_5, x_6) + d(x_4, x_6) > 3d(x_4, x_5)$$

and hence

$$d(x_5, x_6) > d(x_4, x_5).$$

So we get

$$\begin{aligned} d\left(x_5, x_6\right) &\leq \frac{1}{5}\varphi(d(x_4, x_5))[2d(x_4, x_5) + d(x_5, x_6) + d(x_4, x_6)] \\ &+ (1 - \varphi(d(x_4, x_5)))dist(A, B) \\ &\leq \frac{1}{5}\varphi(d(x_4, x_5))[3d(x_4, x_5) + 2d(x_5, x_6)] \\ &+ (1 - \varphi(d(x_4, x_5)))dist(A, B) \\ &< \varphi(d(x_4, x_5))d(x_5, x_6) + (1 - \varphi(d(x_4, x_5)))dist(A, B) \\ &\leq \varphi(d(x_4, x_5))d(x_5, x_6) + (1 - \varphi(d(x_4, x_5)))d(x_5, x_6) \\ &< d(x_5, x_6) \end{aligned}$$

which leads a contradiction. So we confirm that

$$d(x_4, x_5) \geq \frac{1}{5}[2d(x_4, x_5) + d(x_5, x_6) + d(x_4, x_6)].$$

From (2.7) and above inequalities, we have

$$\begin{aligned} d\left(x_5, x_6\right) &\leq \varphi(d(x_4, x_5))d(x_4, x_5) + (1 - \varphi(d(x_4, x_5)))dist(A, B) \\ &\leq d(x_4, x_5). \end{aligned}$$

Hence, by induction, we obtain a sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfying

$$d\left(x_{n+1}, x_{n+2}\right) \leq d(x_n, x_{n+1}) \tag{2.8}$$

and

$$d\left(x_{n+1}, x_{n+2}\right) \leq \varphi(d(x_n, x_{n+1}))d(x_n, x_{n+1}) + (1 - \varphi(d(x_n, x_{n+1})))dist(A, B) \quad \text{for all } n \in \mathbb{N}. \tag{2.9}$$

Clearly, (2.8) shows that the sequence $\{d(x_n, x_{n+1})\}$ is nonincreasing in $[0, \infty)$, and hence we know

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) \quad \text{exists.} \tag{2.10}$$

Since φ is an $\mathcal{MT}$-function, by Theorem 1.6, we obtain

$$0 \leq \sup_{n\in\mathbb{N}}(\varphi(d(x_n, x_{n+1}))) < 1.$$

Let

$$\eta := \sup_{n\in\mathbb{N}}(\varphi(d(x_n, x_{n+1}))).$$

Then

$$0 \leq \varphi(d(x_n, x_{n+1})) \leq \eta < 1 \quad \text{for all } n \in \mathbb{N}. \tag{2.11}$$

Taking $n = 1$ in (2.9), we have from (2.11) that

$$\begin{aligned} d\left(x_2, x_3\right) &\leq \varphi(d(x_1, x_2))d(x_1, x_2) + (1 - \varphi(d(x_1, x_2)))dist(A, B) \\ &\leq \eta d(x_1, x_2) + dist(A, B). \end{aligned} \tag{2.12}$$

Taking $n = 2$ in (2.9) and using (2.11) and (2.12), we obtain

$$\begin{aligned} d\left(x_3, x_4\right) &\leq \varphi(d(x_2, x_3))d(x_2, x_3) + (1 - \varphi(d(x_2, x_3)))dist(A, B) \\ &\leq \varphi(d(x_2, x_3))\left[\eta d(x_1, x_2) + dist(A, B)\right] + (1 - \varphi(d(x_2, x_3)))dist(A, B) \\ &\leq \eta^2 d(x_1, x_2) + dist(A, B). \end{aligned} \tag{2.13}$$

Taking $n=3$ in (2.9) and using (2.11) and (2.13), we get

$$\begin{aligned} d(x_4,x_5) &\le \varphi(d(x_3,x_4))d(x_3,x_4) + (1-\varphi(d(x_3,x_4)))dist(A,B)\\ &\le \varphi(d(x_3,x_4))\left[\eta^2 d(x_1,x_2)+dist(A,B)\right] + (1-\varphi(d(x_3,x_4)))dist(A,B)\\ &\le \eta^3 d(x_1,x_2)+dist(A,B). \end{aligned} \tag{2.14}$$

Taking $n=4$ in (2.9) and using (2.11) and (2.14), we have

$$\begin{aligned} d(x_5,x_6) &\le \varphi(d(x_4,x_5))d(x_4,x_5) + (1-\varphi(d(x_4,x_5)))dist(A,B)\\ &\le \varphi(d(x_4,x_5))\left[\eta^3 d(x_1,x_2)+dist(A,B)\right] + (1-\varphi(d(x_4,x_5)))dist(A,B)\\ &\le \eta^4 d(x_1,x_2)+dist(A,B). \end{aligned}$$

Continuing this process, we obtain

$$dist(A,B) \le d(x_{n+1},x_{n+2}) \le \eta^n d(x_1,x_2)+dist(A,B). \tag{2.15}$$

Due to $\eta\in[0,1)$, $\lim_{n\to\infty}\eta^n=0$. By taking the limit as $n\to\infty$ in (2.15), we get

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = dist(A,B). \tag{2.16}$$

Finally, we finish the proof by combining (2.10) with (2.16). □

Corollary 2.2. *Let A and B be nonempty subsets of a metric space (X,d) and $T: A\cup B\to A\cup B$ be a cyclic mapping. Suppose that there exists a nondecreasing function $\mu:[0,\infty)\to[0,1)$ such that*

$$\begin{aligned} d(Tx,Ty) &\le \mu(d(x,y))\max\left\{d(x,y),\frac{1}{5}[d(Tx,x)+2d(Ty,y)+d(y,Tx)]\right\}\\ &\quad + (1-\mu(d(x,y)))dist(A,B) \end{aligned}$$

for all $x\in A$ and $y\in B$. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $A\cup B$ such that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n,x_{n+1}) = dist(A,B).$$

Proof. Since μ is a nondecreasing function, μ is an $\mathcal{MT}$-function. Therefore, the conclusion is immediate from Theorem 2.1. □

Corollary 2.3. *Let A and B be nonempty subsets of a metric space (X,d) and $T: A\cup B\to A\cup B$ be a cyclic mapping. Suppose that there exists a nonincreasing function $\tau:[0,\infty)\to[0,1)$ such that*

$$\begin{aligned} d(Tx,Ty) &\le \tau(d(x,y))\max\left\{d(x,y),\frac{1}{5}[d(Tx,x)+2d(Ty,y)+d(y,Tx)]\right\}\\ &\quad + (1-\tau(d(x,y)))dist(A,B) \end{aligned}$$

for all $x \in A$ and $y \in B$. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $A \cup B$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = dist(A, B).$$

Proof. Since τ is a nonincreasing function, τ is an $\mathcal{MT}$-function. Therefore, the conclusion is immediate from Theorem 2.1. □

The following conclusions are immediate from Theorem 2.1.

Theorem 2.4. *Let A and B be nonempty subsets of a metric space (X, d) and $T : A\cup B \to A\cup B$ be a cyclic mapping. Suppose that*

(S) *there exists an $\mathcal{MT}$-function $\varphi : [0, \infty) \to [0, 1)$ such that*

$$\begin{aligned} d\,(Tx, Ty) \ &\leq \frac{1}{5}\varphi(d(x, y))[d(Tx, x) + 2d(Ty, y) + d(y, Tx)] \\ &+ (1 - \varphi(d(x, y)))dist(A, B) \end{aligned}$$

for all $x \in A$ and $y \in B$.

Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $A \cup B$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = dist(A, B).$$

Corollary 2.5. *Let A and B be nonempty subsets of a metric space (X, d) and $T : A\cup B \to A\cup B$ be a cyclic mapping. Suppose that there exists a nondecreasing function $\mu : [0, \infty) \to [0, 1)$ such that*

$$\begin{aligned} d\,(Tx, Ty) \ &\leq \frac{1}{5}\mu(d(x, y))[d(Tx, x) + 2d(Ty, y) + d(y, Tx)] \\ &+ (1 - \mu(d(x, y)))dist(A, B) \end{aligned}$$

for all $x \in A$ and $y \in B$. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $A \cup B$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = dist(A, B).$$

Corollary 2.6. *Let A and B be nonempty subsets of a metric space (X, d) and $T : A\cup B \to A\cup B$ be a cyclic mapping. Suppose that there exists a nonincreasing function $\tau : [0, \infty) \to [0, 1)$ such that*

$$\begin{aligned} d\,(Tx, Ty) \ &\leq \frac{1}{5}\tau(d(x, y))[d(Tx, x) + 2d(Ty, y) + d(y, Tx)] \\ &+ (1 - \tau(d(x, y)))dist(A, B) \end{aligned}$$

for all $x \in A$ *and* $y \in B$*. Then there exists a sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* $A \cup B$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = dist(A, B).$$

Acknowledgments. The authors wish to express their hearty thanks to Professor Wei-Shih Du for his valuable suggestions and comments.

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Received: March 25, 2018; Published: April 10, 2018