Application of the SBA Method for Solving Partial Differential Equations

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Abstract

In this paper, we use the SBA method to construct the exact solution of some linear and nonlinear partial differential equations. Four models from mathematical physics have been successfully investigated.

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1 Introduction

These last years, several types of linear and nonlinear partial differential equations have been solved, using different techniques like: Homotopy Perturbation [1], Adomian decomposition method [2]-[7], The Reduced Differential Transform Method [8]-[10] , and others. Here we use the SBA method to find the exact solution of: the nonlinear homogenous and non-homogenous gas dynamic equations [11]-[16] and the two-dimensional partial differential initial boundary value problem [1].
2 Application of the SBA method to solve the nonlinear homogenous and non-homogenous gas dynamic equations

The general theory of SBA method can be found in [4],[17],[18],[19].

2.1 The nonlinear non-homogenous gas dynamic equation [11],[12],[16].

Consider the following non-homogenous, nonlinear gas dynamic equation:

\[ \frac{\partial u(x,t)}{\partial t} + u(x,t) \frac{\partial u(x,t)}{\partial x} - u(x,t) + u^2(x,t) = -e^{t-x} \]  

(1)

with initial condition

\[ u(x,0) = 1 - e^{-x} \]  

(2)

Let’s note

\[ Nu(x,t) = u(x,t) - u(x,t) \frac{\partial u(x,t)}{\partial x} - u^2(x,t) \]  

(3)

From (1), we have:

\[ u(x,t) = u(x,0) + \int_0^t N(u(x,s))ds = 1 - e^{t-x} + \int_0^t N(u(x,s))ds \]  

(4)

According to the SBA method, we suppose that the solution of (1) has the following form:

\[ u(x,t) = \lim_{k \to +\infty} u^k(x,t) \]  

(5)

where

\[ u^k(x,t) = \sum_{n=0}^{+\infty} u^n(x,t) ; \quad k \geq 1 \]  

(6)

and, for every \( k \geq 1 \), we get \( u^k_n(x,t) \) for \( n \geq 0 \), through the following SBA algorithm:

\[
\begin{cases}
  u^0_0(x,t) = 1 - e^{-x} + \int_0^t N(u^{k-1}(x,s))ds ; \quad k \geq 1 \\
  u^k_{n+1}(x,t) = 0 ; \quad n \geq 0
\end{cases}
\]  

(7)
For $k = 1$, we have the following SBA algorithm:

\[
\begin{cases}
  u_0^1(x, t) = 1 - e^{-x} + \int_0^t N(u^0(x, s))ds \\
  u_{n+1}^1(x, t) = 0; \quad n \geq 0
\end{cases}
\]  

(8)

Let’s suppose that one can find $u^0$ as $N(u^0(x, t)) = 0$, we obtain the following SBA algorithm:

\[
\begin{cases}
  u_0^1(x, t) = 1 - e^{-x} \\
  u_{n+1}^1(x, t) = 0; \quad n \geq 0
\end{cases}
\]  

(9)

From (9), we get:

\[ u^1(x, t) = 1 - e^{-x} \]  

(10)

For $k = 2$, we have the following SBA algorithm:

\[
\begin{cases}
  u_0^2(x, t) = 1 - e^{-x} + \int_0^t N(u^1(x, s))ds; \quad k \geq 1 \\
  u_{n+1}^2(x, t) = 0; \quad n \geq 0
\end{cases}
\]  

(11)

We remark that

\[ Nu^1(x, t) = u^1(x, t) - u^1(x, t) \frac{\partial u^1(x, t)}{\partial x} - (u^1(x, t))^2 = 0 \]  

(12)

and (11) becomes:

\[
\begin{cases}
  u_0^2(x, t) = 1 - e^{-x}; \quad k \geq 1 \\
  u_{n+1}^2(x, t) = 0; \quad n \geq 0
\end{cases}
\]  

(13)

We remark that (13) is the same algorithm that (9), thus

\[ u^2(x, t) = 1 - e^{-x} \]  

(14)

Using the same the procedure, we get:

\[ u^1(x, t) = u^2(x, t) = \ldots = u^n(x, t) = 1 - e^{-x} \]  

(15)

and the exact solution of (1) is:

\[ u(x, t) = 1 - e^{-x} \]  

(16)

Proposition-1

Suppose that $x \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$. Then the exact solution of the following equation

\[
\frac{\partial u(x, t)}{\partial t} + \frac{1}{m} u(x, t) \sum_{j=1}^m \frac{\partial u(x, t)}{\partial x_j} - \alpha u(x, t) + u^2(x, t) = -e^{t - \sum_{j=1}^m x_j} \]  

(17)
with the initial condition

\[ u(x, 0) = \alpha - e^{-\sum_{j=1}^{m} x_j} \]  

(18)

is:

\[ u(x, t) = \alpha - \frac{e^{t}}{\prod_{j=1}^{m} e^{x_j}} \]  

(19)

Proof

If \( t = 0 \), we have \( u(x, t) = \alpha - \frac{1}{\prod_{j=1}^{m} e^{x_j}} = \alpha - e^{-\sum_{j=1}^{m} x_j} \).

If \( t \neq 0 \), we have:

\[ \frac{\partial u(x, t)}{\partial t} = u(x, t) - \alpha \]  

(20)

\[ \frac{\partial u(x, t)}{\partial x_j} = \alpha - u(x, t) \]  

(21)

\[ \frac{1}{m} u(x, t) \sum_{j=1}^{m} \frac{\partial u(x, t)}{\partial x_j} = \alpha u(x, t) - u^2(x, t) \]  

(22)

\[ \frac{\partial u(x, t)}{\partial t} + \frac{1}{m} u(x, t) \sum_{j=1}^{m} \frac{\partial u(x, t)}{\partial x_j} - \alpha u(x, t) + 
\]

\[ u^2(x, t) = u(x, t) - \alpha + \alpha u(x, t) - u^2(x, t) - \alpha u(x, t) + u^2(x, t) = 
\]

\[ u(x, t) - \alpha = -e^{-\sum_{j=1}^{m} x_j} \]  

(23)

2.2 The nonlinear homogenous gas dynamic equation [2], [12], [16].

Consider the following nonlinear gas dynamic equation:

\[ \frac{\partial u(x, t)}{\partial t} + \frac{1}{2} \frac{\partial u^2(x, t)}{\partial x} - u(x, t) + u^2(x, t) = 0 \]  

(24)

with initial condition

\[ u(x, 0) = e^{-x} \]  

(25)

Let’s note
\[ Nu(x, t) = \frac{1}{2} \frac{\partial u^2(x, t)}{\partial x} + u^2(x, t) \]  

From (24), we have:

\[ u(x, t) = u(x, 0) + \int_0^t u(x, s) ds - \int_0^t N(u(x, s)) ds = e^{-x} + \int_0^t u(x, s) ds - \int_0^t N(u(x, s)) ds \]  

According to the SBA Method, we suppose that the solution of (24) has the following form:

\[ u(x, t) = \lim_{k \to +\infty} u^k(x, t) \]  

where

\[ u^k(x, t) = \sum_{n=0}^{+\infty} u^k_n(x, t) ; \quad k \geq 1 \]  

and, for every \( k \geq 1 \), we get \( u^k_n(x, t) \) for \( n \geq 0 \), through the following SBA algorithm:

\[
\begin{cases}
  u^0_0(x, t) = e^{-x} - \int_0^t N(u^{k-1}(x, s)) ds ; \quad k \geq 1 \\
  u^k_{n+1}(x, t) = \int_0^t u^k_n(x, s) ds ; \quad n \geq 0
\end{cases}
\]  

For \( k = 1 \), we have the following SBA algorithm:

\[
\begin{cases}
  u^1_0(x, t) = e^{-x} + \int_0^t N(u^0(x, s)) ds \\
  u^1_{n+1}(x, t) = \int_0^t u^1_n(x, s) ds ; \quad n \geq 0
\end{cases}
\]  

let’s suppose that one can find \( u^0 \) as \( N(u^0(x, t)) = 0 \), we obtain the following SBA algorithm:

\[
\begin{cases}
  u^1_0(x, t) = e^{-x} \\
  u^1_{n+1}(x, t) = \int_0^t u^1_n(x, s) ds ; \quad n \geq 0
\end{cases}
\]  

From (32), we obtain:

\[
\begin{cases}
  u^1_1(x, t) = te^{-x} \\
  u^1_2(x, t) = \frac{t^2}{2!} e^{-x} \\
  \quad \vdots \\
  u^1_n(x, t) = \frac{t^n}{n!} e^{-x}
\end{cases}
\]
Thus the exact solution to the first step is

\[ u^1(x, t) = e^{-x} \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^{t-x} \]  

(34)

For \( k = 2 \), we have the following SBA algorithm:

\[
\begin{align*}
  u^2_0(x, y, z, t) &= e^{-x} - \int_0^t N(u^1(x, s)) ds \\
  u^2_{n+1}(x, t) &= \int_0^t u^2_n(x, s) ds ; \quad n \geq 0
\end{align*}
\]  

(35)

We remark that

\[ Nu^1(x, t) = \frac{1}{2} \frac{\partial(u^1(x, t))^2}{\partial x} + u^2(x, t) = 0 \]  

(36)

and (35) becomes:

\[
\begin{align*}
  u^2_0(x, t) &= e^{-x} \\
  u^2_{n+1}(x, t) &= \int_0^t u^2_n(x, s) ds ; \quad n \geq 0
\end{align*}
\]  

(37)

We remark that (37) is the same algorithm that (32), thus

\[ u^2(x, t) = e^{t-x} \]  

(38)

Using the same the same procedure, we get:

\[ u^1(x, t) = u^2(x, t) = \ldots = u^n(x, t) = e^{t-x} \]  

(39)

and the exact solution of (24) is:

\[ u(x, t) = e^{t-x} \]  

(40)

Proposition-2

Suppose that \( x \in \mathbb{R}^m, \alpha \in \mathbb{R} \). Then the exact solution of the following equation

\[
\frac{\partial u(x, t)}{\partial t} + \frac{1}{m\alpha} \sum_{j=1}^{m} \frac{\partial u^\alpha(x, t)}{\partial x_j} - u(x, t) + u^\alpha(x, t) = 0
\]  

(41)

with the initial condition

\[ u(x, 0) = e^{-\sum_{j=1}^{m} x_j} \]  

(42)

is:
Application of the SBA method for solving PDE

\[ u(x, t) = \frac{e^t}{\prod_{j=1}^{m} e^{x_j}} \]  

(43)

Proof

If \( t = 0 \), we have: \( u(x, t) = \frac{1}{\prod_{j=1}^{m} e^{x_j}} = e^{\sum_{j=1}^{m} x_j} \)

If \( t \neq 0 \), we have:

\[
\frac{\partial u(x, t)}{\partial t} = u(x, t) \quad (44)
\]

\[
\frac{\partial u^\alpha(x, t)}{\partial x_j} = -\alpha u^\alpha(x, t) \quad (45)
\]

\[
\frac{1}{m\alpha} \sum_{j=1}^{m} \frac{\partial u^\alpha(x, t)}{\partial x_j} = -u^\alpha(x, t)
\]

Thus

\[
\frac{\partial u(x, t)}{\partial t} + \frac{1}{m\alpha} \sum_{j=1}^{m} \frac{\partial u^\alpha(x, t)}{\partial x_j} - u(x, t) + u^\alpha(x, t) = u(x, t) - u^\alpha(x, t) - u(x, t) + u^\alpha(x, t) = 0
\]  

(46)

3 Application of the SBA method to solve an

two-dimensional partial differential initial boundary value problem [1]

Consider the two-dimensional nonlinear inhomogeneous initial boundary value problem

\[
\begin{aligned}
\frac{\partial^2 u(x, y, t)}{\partial x^2} - \frac{15}{2} x \left( \frac{\partial^2 u(x, y, t)}{\partial x^2} \right)^2 - \frac{15}{2} y \left( \frac{\partial^2 u(x, y, t)}{\partial y^2} \right)^2 &= 2x^2 + 2y^2 \\
\frac{\partial u(x, y, 0)}{\partial x} &= 0 \\
u(x, y, 0) &= 0
\end{aligned}
\]  

(47)

Let’s note

\[
Nu(x, y, t) = x \left( \frac{\partial^2 u(x, y, t)}{\partial x^2} \right)^2 + y \left( \frac{\partial^2 u(x, y, t)}{\partial y^2} \right)^2
\]  

(48)
From (47), we have:

\[ u(x, y, t) = (x^2 + y^2)t^2 + \frac{15}{2} \int_0^t \int_0^z N(u(x, y, s))dsdz \tag{49} \]

According to the SBA method, we suppose that the solution of (47) has the following form:

\[ u(x, y, t) = \lim_{k \to +\infty} u_k(x, y, t) \tag{50} \]

where

\[ u_k(x, y, t) = \sum_{n=0}^{+\infty} u_k^n(x, y, t) ; \quad k \geq 1 \tag{51} \]

and, for every \( k \geq 1 \), we get \( u_k^n(x, y, t) \) for \( n \geq 0 \), through the following SBA algorithm:

\[
\begin{cases}
  u_k^0(x, y, t) = (x^2 + y^2)t^2 + \frac{15}{2} \int_0^t \int_0^z N(u_{k-1}^0(x, y, s))dsdz; \quad k \geq 1 \\
  u_{n+1}^k(x, y, t) = 0 ; \quad n \geq 0
\end{cases}
\tag{52}
\]

For \( k = 1 \), we have the following SBA algorithm:

\[
\begin{cases}
  u_1^0(x, y, t) = (x^2 + y^2)t^2 + \frac{15}{2} \int_0^t \int_0^z N(u_0^0(x, y, s))dsdz \\
  u_{n+1}^1(x, y, t) = 0 ; \quad n \geq 0
\end{cases}
\tag{52}
\]

let’s suppose that one can find \( u_0^0(x, y, t) \) as \( N(u_0^0(x, y, t)) = 0 \), we obtain the following SBA algorithm:

\[
\begin{cases}
  u_0^0(x, y, t) = (x^2 + y^2)t^2 \\
  u_{n+1}^1(x, y, t) = 0 ; \quad n \geq 0
\end{cases}
\tag{53}
\]

From (53), we get:

\[ u^1(x, y, t) = (x^2 + y^2)t^2 \tag{54} \]

For \( k = 2 \), we have the following SBA algorithm:

\[
\begin{cases}
  u_0^2(x, y, t) = (x^2 + y^2)t^2 + \frac{15}{2} \int_0^t \int_0^z N(u_1^1(x, y, s))dsdz \\
  u_{n+1}^2(x, y, t) = 0 ; \quad n \geq 0
\end{cases}
\tag{55}
\]

We remark that

\[ Nu^1(x, y, t) = x \left( \frac{\partial^2 u^1(x, y, t)}{\partial x^2} \right)^2 + y \left( \frac{\partial^2 u^1(x, y, t)}{\partial y^2} \right)^2 = 4(x + y)t^2 \neq 0 \tag{56} \]
According to the SBA method, we should have

\[ \text{SBA}(x, y, t) = 0 \] (57)

We rewrite the problem (47) in the following form:

\[
\begin{aligned}
\frac{\partial^2 u(x,y,t)}{\partial t^2} - \frac{15}{2} x \left( \frac{\partial^2 u(x,y,t)}{\partial x^2} \right)^2 - \frac{15}{2} y \left( \frac{\partial^2 u(x,y,t)}{\partial y^2} \right)^2 + 30(x+y)t^2 - 30(x+y)t^2 & = 2x^2 + 2y^2 \\
u(x,y,0) & = 0 \\
\frac{\partial u(x,y,0)}{\partial x} & = 0 
\end{aligned}
\] (58)

(58) gives us:

\[
\begin{aligned}
\frac{\partial^2 u(x,y,t)}{\partial t^2} - \frac{15}{2} \tilde{\text{SBA}}(x, y, t) - 30(x+y)t^2 & = 2x^2 + 2y^2 \\
u(x,y,0) & = 0 \\
\frac{\partial u(x,y,0)}{\partial x} & = 0 
\end{aligned}
\] (59)

where

\[ \tilde{\text{SBA}}(x, y, t) = \text{SBA}(x, y, t) - 4(x+y)t^2 \] (60)

From (59), we have:

\[ u(x,y,t) = (x^2 + y^2)t^2 + (x+y)t^6 + \frac{15}{2} \int_0^t \int_0^s \tilde{\text{SBA}}(x, y, s) \, ds \, dz \] (61)

According to the SBA method, we suppose that the solution of (59) has the following form:

\[ u(x,y,t) = \lim_{k \to +\infty} u^k(x,y,t) \] (62)

where

\[ u^k(x,y,t) = \sum_{n=0}^{+\infty} u_n^k(x,y,t) ; \quad k \geq 1 \] (63)

and, for every \( k \geq 1 \), we get \( u_n^k(x,y,t) \) for \( n \geq 0 \), through the following SBA algorithm:

\[
\begin{aligned}
\begin{array}{l}
u_0^k(x,y,t) = (x^2 + y^2)t^2 + (x+y)t^6 + \frac{15}{2} \int_0^t \int_0^s \tilde{\text{SBA}}(u^{k-1}(x,y,s)) \, ds \, dz ; \quad k \geq 1 \\
u_{n+1}^k(x,y,t) = 0 ; \quad n \geq 0 
\end{array}
\end{aligned}
\] (64)
For $k = 1$, we have the following SBA algorithm:

\[
\begin{align*}
\{ & u^1_0(x, y, t) = (x^2 + y^2)t^2 + (x + y)t^6 + \frac{15}{2} \int_0^t \int_0^s \tilde{N}(u^0(x, y, s)) ds dz \\
& u^1_{n+1}(x, y, t) = 0 ; \ n \geq 0
\end{align*}
\]

let’s suppose that one can find $u^0$ as $N(u^0(x, y, t)) = 0$, we obtain the following SBA algorithm:

\[
\begin{align*}
\{ & u^1_0(x, y, t) = (x^2 + y^2)t^2 + (x + y)t^6 \\
& u^1_{n+1}(x, y, t) = 0 ; \ n \geq 0 \quad (65)
\end{align*}
\]

From (53), we get:

\[
\begin{align*}
& u^1(x, y, t) = (x^2 + y^2)t^2 + (x + y)t^6 \\
& (66)
\end{align*}
\]

For $k = 2$, we have the following SBA algorithm:

\[
\begin{align*}
\{ & u^2_0(x, y, t) = (x^2 + y^2)t^2 + (x + y)t^6 + \frac{15}{2} \int_0^t \int_0^s N(u^1(x, y, s)) ds dz \\
& u^2_{n+1}(x, y, t) = 0 ; \ n \geq 0 \\
& (67)
\end{align*}
\]

We remark that

\[
\begin{align*}
& Nu^1(x, y, t) = x \left( \frac{\partial^2 u^1(x, y, t)}{\partial x^2} \right)^2 + y \left( \frac{\partial^2 u^1(x, y, t)}{\partial y^2} \right)^2 = 4(x + y)t^2 \\
& (68)
\end{align*}
\]

so:

\[
\begin{align*}
& \tilde{N}u^1(x, y, t) = 0 \\
& (69)
\end{align*}
\]

and (67) becomes:

\[
\begin{align*}
\{ & u^2_0(x, y, t) = (x^2 + y^2)t^2 + (x + y)t^6 \\
& u^2_{n+1}(x, y, t) = 0 ; \ n \geq 0 \\
& (70)
\end{align*}
\]

We remark that the algorithms (65) and (70) are identical. Then we obtain $u^2(x, y, t) = (x^2 + y^2)t^2 + (x + y)t^6$.

Using the same procedure, we get:

\[
\begin{align*}
& u^1(x, y, t) = u^2(x, y, t) = \ldots = u^n(x, y, t) = (x^2 + y^2)t^2 + (x + y)t^6 \\
& (71)
\end{align*}
\]

and the exact solution of (47) is:

\[
\begin{align*}
& u(x, y, t) = (x^2 + y^2)t^2 + (x + y)t^6 \\
& (72)
\end{align*}
\]
4 Conclusion

Through these examples, we showed again the usefulness of the SBA method comparatively with the methods that have been used in [1] and [16], in the search of the exact solution.

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