Nonlinear Analysis and Differential Equations, Vol. 6, 2018, no. 1, 25 - 41
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/nade.2018.843

Solving a Class of Partial Differential Equations with Different Types of Boundary Conditions by Using a Generalized Inverse Operator: Decomposition Method

Nawal A. Al-Zaid [1], Huda O. Bakodah [1] and Abdelhalim Ebaid [2]

[1] Department of Mathematics, Faculty of Science-Al Faisaliah Campus
King Abdulaziz University, Jeddah, Saudi Arabia

[2] Department of Mathematics, Faculty of Science, University of Tabuk
P.O.Box 741, Tabuk 71491, Saudi Arabia

Abstract

In this paper, a class of partial differential equations has been studied by the Adomian decomposition method. A generalized inverse operator has been developed to solving various partial differential equations with different types of boundary conditions (Dirichlet, Neumann, and mixed). Several examples in physics and fluid mechanics have been analyzed by the present approach, where a remarkable accuracy has been achieved.

Keywords: Adomian decomposition method, initial boundary value problem, improved Adomian decomposition method, Dirichlet conditions, Neumann conditions, Mixed conditions

1 Introduction

Since the beginning of the 1980, Adomian has developed a so-called decomposition method [1, 2]. The main advantage of this method is that it can be used directly for solving all types of differential and integral equations, linear or nonlinear, with constant or variable coefficients. Over the last 30 years, the Adomian decomposition method (ADM) has been applied to obtain the solution of a wide class of initial or boundary value problems [3-6], but when initial and boundary conditions have to be imposed, there are still difficulties that can be encountered. Most researchers obtained the solutions of initial and boundary value problems by using either initial or boundary conditions. In 1987, Adomian [7] suggested the modified method (MADM) which has applied on the heat equation $u_t = u_{xx}$ with initial and boundary conditions, using two canonical forms for *u*, one inverting the L_t operator and the other inverting the L_x operator, adding and dividing by two. For the Dirichlet problem, for the heat equation, Adomian used the operator L_{xx}^{-1} defined by

$$L_{xx}^{-1}(.) = \int_{x_0}^{x} dx' \int_{x_0}^{x'} (.) dx''. \tag{1.1}$$

For Dirichlet problem, Lesnic [8] proposed the inverse operator defined by

$$L_{xx}^{-1}(.) = \int_{x_0}^{x} dx' \int_{x_0}^{x'} (.) dx'' - \frac{x - x_0}{1 - x_0} \int_{x_0}^{1} dx' \int_{x_0}^{x'} (.) dx'' \cdot \tag{1.2}$$

In [9], Aly et al., defined L_{xx}^{-1} as

$$L_{xx}^{-1}(.) = \int_{a}^{x} dx' \int_{c}^{x'} (.) dx'' - z(x) \int_{a}^{b} dx' \int_{e}^{x'} (\cdot) dx'', \tag{1.3}$$

$z(x)$ in Eq. (1.3) is to be determined such that L_{xx}^{-1} can be expressed only in terms of the boundary conditions. In this paper an improved formula shall be proposed for the generalized inverse operator based on the MADM [8]. We shall use the ADM with the improved inverse operator to deal with partial differential equations with Dirichlet, Neumann and mixed boundary conditions.

2 Derivation of the Inverse Operator

2.1 Dirichlet boundary conditions.

To study the generalized form for the inverse operator, we consider the L_{xx} operator in the form

$$L_{xx} = \frac{1}{p(x)} \frac{\partial}{\partial x} \left(q(x) \frac{\partial}{\partial x} \right), \; p(x) \neq 0 \cdot \tag{2.1}$$

We define the inverse operator L_{xx}^{-1} as

$$L_{xx}^{-1}(.) = \int_{a}^{x} \frac{1}{q(x')} dx' \int_{c}^{x'} p(x'')(.) dx'' - z(x) \int_{a}^{b} \frac{1}{q(x')} dx' \int_{c}^{x'} p(x'')(\cdot) dx'', \tag{2.2}$$

Where $z(x)$ is to be determined such that L_{xx}^{-1} can be expressed only in terms of the boundary conditions $u(t,a)$ and $u(t,b)$. With this definition, we can easily get

$$L_{xx}^{-1}L_{xx}u = u(t,x) - u(t,a) - q(c)\frac{\partial u(t,c)}{\partial x}\int_a^x \frac{1}{q(x')}dx' - z(x)\left[u(t,b) - u(t,a)\right]$$
$$+ z(x)q(c)\frac{\partial u(t,c)}{\partial x}\int_a^b \frac{1}{q(x')}dx'. \tag{2.3}$$

In order to express $L_{xx}^{-1}L_{xx}u$ in terms of the two boundary conditions only, we have to eliminate the coefficient multiplying $\frac{\partial u(t,c)}{\partial x}$ by setting

$$-q(c)\frac{\partial u(t,c)}{\partial x}\int_a^x \frac{1}{q(x')}dx' + z(x)q(c)\frac{\partial u(t,c)}{\partial x}\int_a^b \frac{1}{q(x')}dx' = 0.$$

Solving this equation for $z(x)$ assuming that $q(c)\frac{\partial u(t,c)}{\partial x} \neq 0$, gives

$$z(x) = \frac{\int_a^x \frac{1}{q(x')}dx'}{\int_a^b \frac{1}{q(x')}dx'}. \tag{2.4}$$

Substituting (2.4) into (2.3), yields

$$L_{xx}^{-1}L_{xx}u = u(t,x) - u(t,a) - \frac{\int_a^x \frac{1}{q(x')}dx'}{\int_a^b \frac{1}{q(x')}dx'}\left[u(t,b) - u(t,a)\right]. \tag{2.5}$$

Here, if $p(x) = q(x) = 1$, we get the case of Lesnic in [8].

2.2 Neumann boundary conditions.

To study the generalized form for the inverse operator, we consider the L_{xx} operator in the form (2.1) and the inverse operator L_{xx}^{-1} as

$$L_{xx}^{-1}(.) = \int_c^x \frac{1}{q(x')}dx' \int_a^{x'} p(x'')(.)dx'' - z(x)\int_0^c \frac{1}{q(x')}dx'\left(x' \int_b^{x'} p(x'')(\cdot)dx''\right)\cdot \tag{2.6}$$

In this case, $z(x)$ is to be determined such that L_{xx}^{-1} can be expressed only in terms of the boundary conditions $u_x(t,a)$ and $u_x(t,b)$, hence

$$L_{xx}^{-1}L_{xx}u = u(t,x) - u(t,c) - q(a)\frac{\partial u(t,a)}{\partial x}\int_c^x \frac{1}{q(x')}dx' - z(x)\left[cu(t,c) - \int_0^c u(t,x')dx'\right]$$
$$+z(x)q(b)\frac{\partial u(t,b)}{\partial x}\int_0^c \frac{x'}{q(x')}dx'. \tag{2.7}$$

In order to express L_{xx}^{-1} in terms of the two boundary conditions only, we have to eliminate the coefficient multiplying $u(t,c)$ by setting $-u(t,c) - z(x)cu(t,c) = 0$ Therefore, $z(x)$ is given as (assuming that $u(t,c) \neq 0$)

$$z(x) = \frac{-1}{c}. \tag{2.8}$$

Substituting (2.8) into (2.7), yields

$$L_{xx}^{-1}L_{xx}u = u(t,x) - q(a)\frac{\partial u(t,a)}{\partial x}\int_c^x \frac{1}{q(x')}dx' - \frac{1}{c}\int_0^c u(t,x')dx'$$
$$-\frac{1}{c}q(b)\frac{\partial u(t,b)}{\partial x}\int_0^c \frac{x'}{q(x')}dx'. \tag{2.9}$$

It should be also noted hare that when $p(x) = q(x) = 1$, the case of Aly et.al., [9] is recovered.

2.3 Mixed boundary conditions.

Here, we have two types of problems:

(i) In many practical cases, Cauchy data cannot be specified at the same location; instead only one boundary value can be prescribed, with another condition specified at an interior location inside the specimen under investigation. In such situations we have to solve a direct problem in the region which then provides the Cauchy data at $x = a$ for an inverse problem formulated in the region.

To study the generalized form for the inverse operator, we consider the L_{xx} operator in the form (2.1) and the inverse operator L_{xx}^{-1} as

$$L_{xx}^{-1}(.) = \int_a^x \frac{1}{q(x')}dx' \int_a^{x'} p(x'')(.)dx''. \tag{2.10}$$

Accordingly,

$$L_{xx}^{-1}L_{xx}u = u(t,x) - u(t,a) - q(a)\frac{\partial u(t,a)}{\partial x}\int_a^x \frac{1}{q(x')}dx', \tag{2.11}$$

which is expressed in terms of the two boundary conditions only, i.e., $u(t,a)$ and $u_x(t,a)$,

(ii) To study the generalized form for the inverse operator, we consider the L_{xx} operator in the form (2.1) and the inverse operator L_{xx}^{-1} as

$$L_{xx}^{-1}(.) = \int_a^x \frac{1}{q(x')} dx' \int_b^{x'} p(x'')(.)dx'', \tag{2.12}$$

and hence,

$$L_{xx}^{-1} L_{xx} u = u(t,x) - u(t,a) - q(b) \frac{\partial u(t,b)}{\partial x} \int_a^x \frac{1}{q(x')} dx', \tag{2.13}$$

which is expressed in terms of the two boundary conditions only, i.e., $u(t,a)$ and $u_x(t,b)$,

When $p(x) = q(x) = 1$, we recover the case of Lesnic [8].

3 Analysis of the Improved Adomian Decomposition Method

Consider the following general example of the single second-order nonlinear inhomogeneous temporal-spatial partial differential equation

$$L_{xx} u(t,x) + L_{tt} u(t,x) + Nu(t,x) = g(t,x), \tag{3.1}$$

where $L_{xx} = \frac{1}{p(x)} \frac{\partial}{\partial x}\left(q(x) \frac{\partial}{\partial x} \right),\ p(x) \neq 0,\ L_{tt} = \frac{\partial^2}{\partial t^2}$,

and $Nu(t,x) = f\ (t, x, u(t,x), u_t(t,x), u_x(t,x))$ is a nonlinear operator which is assumed to be analytic and $g(t,x)$ is an inhomogeneous term. Adomian [7] suggested a modified method (MADM) using two canonical equations for *u*, as follows

$$L_{tt} u(t,x) = g(t,x) - L_{xx} u(t,x) - Nu(t,x), \tag{3.2}$$

$$L_{xx} u(t,x) = g(t,x) - L_{tt} u(t,x) - Nu(t,x). \tag{3.3}$$

It is clear that L_{tt} and L_{xx} are invertible, so, Adomian [7] applied the inverse operator L_{tt}^{-1} on both sides of Eq (3.2) and the inverse operator L_{xx}^{-1} to both sides of Eq (3.3), then adding the two equations and dividing by two, we obtain a single equation for *u*.

The standard Adomian decomposition method defines the solution $u(t,x)$ by the decomposition series

$$u(t,x) = \sum_{n=0}^{\infty} u_n(t,x), \tag{3.4}$$

and the nonlinear term $Nu(t,x)$ is defined as

$$Nu(t,x) = \sum_{n=0}^{\infty} A_n(t,x), \tag{3.5}$$

where A_n denotes the Adomian polynomials which can be computed from the relation

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{\infty} \lambda_i u_i\right)\right]_{\lambda=0}, \quad n \geq 0 \cdot \tag{3.6}$$

We shall use this technique with the improved inverse operator to deal with partial differential equations with Dirichlet, Neumann and mixed boundary conditions.

3.1 Dirichlet boundary conditions.

Consider Eq (3.1) subject to the initial conditions

$$u(0,x) = p_1(x), \; u_t(0,x) = p_2(x), \tag{3.7}$$

and the Dirichlet boundary conditions

$$u(t,a) = h_1(t), \; u(t,b) = h_2(t). \tag{3.8}$$

Firstly, we consider the t -partial solution as

$$L_{tt}u(t,x) = g(t,x) - L_{xx}u(t,x) - Nu(t,x), \tag{3.9}$$

Applying the inverse operator L_{tt}^{-1} defined by $L_{tt}^{-1}(.) = \int_0^t \int_0^t (.)dt\, dt$ on both sides of Eq. (3.9) and using the initial conditions, gives

$$u(t,x) = u(0,x) + tu_t(0,x) + L_{tt}^{-1}g(t,x) - L_{tt}^{-1}L_{xx}u(t,x) - L_{tt}^{-1}Nu(t,x), \tag{3.10}$$

Secondly, we consider the x -partial solution as

$$L_{xx}u(t,x) = g(t,x) - L_{tt}u(t,x) - Nu(t,x). \tag{3.11}$$

Applying the inverse operator L_{xx}^{-1} defined as in Eq (2.2) on both sides of Eq. (3.11), and using the boundary conditions, yields

$$u(t,x) = u(t,a) - \frac{\int_a^x \frac{1}{q(x')}dx'}{\int_a^b \frac{1}{q(x')}dx'}\left[u(t,b) - u(t,a)\right] + L_{xx}^{-1}g(t,x) - L_{xx}^{-1}L_{tt}u(t,x)$$
$$-L_{xx}^{-1}Nu(t,x). \tag{3.12}$$

Next, we average the partial solutions, i.e., adding the two partial solutions in Eq. (3.10) and Eq. (3.12) and then divide by two to obtain

$$u(t,x)=\frac{1}{2}\left(u(0,x)+tu_t(0,x)+u(t,a)-\frac{\int_a^x \frac{1}{q(x')}dx'}{\int_a^b \frac{1}{q(x')}dx'}\left[u(t,b)-u(t,a)\right]\right.$$
$$L_{tt}^{-1}g(t,x)+L_{xx}^{-1}g(t,x)$$
$$\left.-L_{tt}^{-1}L_{xx}u(t,x)-L_{xx}^{-1}L_{tt}u(t,x)-L_{tt}^{-1}Nu(t,x)-L_{xx}^{-1}Nu(t,x)\right). \quad (3.13)$$

Substituting Eq. (3.4) and Eq. (3.5) into Eq. (3.13), gives

$$\sum_{n=0}^{\infty}u_n(t,x)=\frac{1}{2}\left(u(0,x)+tu_t(0,x)+u(t,a)-\frac{\int_a^x \frac{1}{q(x')}dx'}{\int_a^b \frac{1}{q(x')}dx'}\left[u(t,b)-u(t,a)\right]\right.$$
$$+L_{tt}^{-1}g(t,x)+L_{xx}^{-1}g(t,x)$$
$$\left.-L_{tt}^{-1}L_{xx}\sum_{n=0}^{\infty}u_n(t,x)-L_{xx}^{-1}L_{tt}\sum_{n=0}^{\infty}u_n(t,x)-L_{tt}^{-1}\sum_{n=0}^{\infty}A_n(t,x)-L_{xx}^{-1}\sum_{n=0}^{\infty}A_n(t,x)\right). \quad (3.14)$$

So that the recurrence relations are

$$u_0=\frac{1}{2}\left(u(0,x)+tu_t(0,x)+u(t,a)-\frac{\int_a^x \frac{1}{q(x')}dx'}{\int_a^b \frac{1}{q(x')}dx'}\left[u(t,b)-u(t,a)\right]\right.$$
$$\left.+L_{tt}^{-1}g(t,x)+L_{xx}^{-1}g(t,x)\right),$$
$$u_{n+1}=\frac{1}{2}\left[-L_{tt}^{-1}L_{xx}u_n-L_{xx}^{-1}L_{tt}u_n-L_{tt}^{-1}A_n-L_{xx}^{-1}A_n\right],n\geq 0. \quad (3.15)$$

3.2 Neumann boundary conditions.

Consider Eq. (3.1) subject to the initial conditions in Eq. (3.7) and the Neumann boundary conditions

$$u_x(t,a)=h_1(t)\ ,u_x(t,b)=h_2(t)\cdot \quad (3.16)$$

Firstly, we consider the t -partial solution as in Eq. (3.10). Secondly, applying the inverse operator L_{tt}^{-1} defined as in Eq. (2.6) on both sides of Eq. (3.11), and using the boundary conditions, gives

$$u(t,x)=q(a)\frac{\partial u(t,a)}{\partial x}\int_c^x \frac{1}{q(x')}dx'-\frac{1}{c}\int_0^c u(t,x')dx'$$

$$-\frac{1}{c}q(b)\frac{\partial u(t,b)}{\partial x}\int_0^c \frac{x'}{q(x')}dx' + L_{xx}^{-1}g(t,x) - L_{xx}^{-1}L_{tt}u(t,x) - L_{xx}^{-1}Nu(t,x). \quad (3.17)$$

On adding the two partial solutions in Eq. (3.10) and Eq. (3.17) and dividing by two, we obtain

$$\begin{aligned} u(t,x) = \frac{1}{2}\Bigg(& u(0,x) + tu_t(0,x) + q(a)\frac{\partial u(t,a)}{\partial x}\int_c^x \frac{1}{q(x')}dx' - \frac{1}{c}\int_0^c u(t,x')dx' \\ & -\frac{1}{c}q(b)\frac{\partial u(t,b)}{\partial x}\int_0^c \frac{x'}{q(x')}dx' + L_{tt}^{-1}g(t,x) + L_{xx}^{-1}g(t,x) \\ & -L_{tt}^{-1}L_{xx}u(t,x) - L_{tt}^{-1}Nu(t,x) - L_{xx}^{-1}L_{tt}u(t,x) - L_{xx}^{-1}Nu(t,x)\Bigg)\cdot \end{aligned} \quad (3.18)$$

Substituting Eq. (3.4) and Eq. (3.5) into Eq.(3.18), gives

$$\begin{aligned} \sum_{n=0}^{\infty}u_n(t,x) = \frac{1}{2}\Bigg(& u(0,x) + tu_t(0,x) + q(a)\frac{\partial u(t,a)}{\partial x}\int_c^x \frac{1}{q(x')}dx' - \frac{1}{c}\int_0^c u(t,x')dx' \\ & -\frac{1}{c}q(b)\frac{\partial u(t,b)}{\partial x}\int_0^c \frac{x'}{q(x')}dx' + L_{tt}^{-1}g(t,x) + L_{xx}^{-1}g(t,x) \\ & -L_{tt}^{-1}L_{xx}\sum_{n=0}^{\infty}u_n(t,x) - L_{tt}^{-1}\sum_{n=0}^{\infty}A_n(t,x) - L_{xx}^{-1}L_{tt}\sum_{n=0}^{\infty}u_n(t,x) - L_{xx}^{-1}\sum_{n=0}^{\infty}A_n(t,x)\Bigg)\cdot \end{aligned} \quad (3.19)$$

and hence, the recurrence relations become

$$\begin{aligned} u_0 = \frac{1}{2}\Bigg(& u(0,x) + tu_t(0,x) + q(a)\frac{\partial u(t,a)}{\partial x}\int_c^x \frac{1}{q(x')}dx' - \frac{1}{c}\int_0^c u(t,x')dx' \\ & -\frac{1}{c}q(b)\frac{\partial u(t,b)}{\partial x}\int_0^c \frac{x'}{q(x')}dx' + L_{tt}^{-1}g(t,x) + L_{xx}^{-1}g(t,x)\Bigg), \\ u_{n+1} = \frac{1}{2} & \left[-L_{tt}^{-1}L_{xx}u_n - L_{xx}^{-1}L_{tt}u_n - L_{tt}^{-1}A_n - L_{xx}^{-1}A_n\right], n \ge 0. \end{aligned} \quad (3.20)$$

3.3 Mixed boundary conditions.

(i) Consider Eq (3.1) subject to the initial conditions in Eq (3.7) and the mixed boundary conditions

$$u(t,a) = h_1(t)\ ,\ u_x(t,a) = h_2(t), \quad (3.21)$$

Firstly, we consider the t -partial solution as in Eq (3.10). Secondly, applying the inverse operator L_{xx}^{-1} defined as in Eq (2.10) on both sides of Eq. (3.11) and using the boundary conditions, we get

$$u(t,x) = u(t,a) + q(a)\frac{\partial u(t,a)}{\partial x}\int_a^x \frac{1}{q(x')}dx' + L_{xx}^{-1} g(t,x) - L_{xx}^{-1} L_{tt} u(t,x)$$
$$-L_{xx}^{-1} N u(t,x). \tag{3.22}$$

From Eq. (3.10) and Eq. (3.22) we obtain

$$u(t,x) = \frac{1}{2}\Bigg(u(0,x) + t u_t(0,x) + u(t,a) + q(a)\frac{\partial u(t,a)}{\partial x}\int_a^x \frac{1}{q(x')}dx' + L_{tt}^{-1} g(t,x) + L_{xx}^{-1} g(t,x)$$
$$-L_{tt}^{-1} L_{xx} u(t,x) - L_{tt}^{-1} N u(t,x) - L_{xx}^{-1} L_{tt} u(t,x) - L_{xx}^{-1} N u(t,x)\Bigg)\cdot \tag{3.23}$$

Substituting Eq. (3.4) and Eq. (3.5) into Eq. (3.23) gives

$$\sum_{n=0}^{\infty} u_n(t,x) = \frac{1}{2}\Bigg(u(0,x) + t u_t(0,x) + u(t,a) + q(a)\frac{\partial u(t,a)}{\partial x}\int_a^x \frac{1}{q(x')}dx'$$
$$+ L_{tt}^{-1} g(t,x) + L_{xx}^{-1} g(t,x) - L_{tt}^{-1} L_{xx} \sum_{n=0}^{\infty} u_n(t,x)$$
$$- L_{tt}^{-1} \sum_{n=0}^{\infty} A_n(t,x) - L_{xx}^{-1} L_{tt} \sum_{n=0}^{\infty} u_n(t,x) - L_{xx}^{-1} \sum_{n=0}^{\infty} A_n(t,x)\Bigg)\cdot \tag{3.24}$$

with the corresponding recurrence relations

$$u_0 = \frac{1}{2}\Bigg(u(0,x) + t u_t(0,x) + u(t,a) + q(a)\frac{\partial u(t,a)}{\partial x}\int_a^x \frac{1}{q(x')}dx'$$
$$+ L_{tt}^{-1} g(t,x) + L_{xx}^{-1} g(t,x)\Bigg),$$
$$u_{n+1} = \frac{1}{2}\left[-L_{tt}^{-1} L_{xx} u_n - L_{xx}^{-1} L_{tt} u_n - L_{tt}^{-1} A_n - L_{xx}^{-1} A_n \right], n \geq 0. \tag{3.25}$$

(ii) Consider Eq (3.1) subject to the initial conditions in Eq. (3.7) and the mixed boundary conditions

$$u(t,a) = h_1(t) \ , \ u_x(t,b) = h_2(t), \tag{3.26}$$

We first consider the t -partial solution as in Eq (3.10). Then, applying the inverse operator L_{xx}^{-1} defined as in Eq (2.12) on both sides of Eq.(3.11), and using the boundary conditions, we get

$$u(t,x) = u(t,a) + q(b)\frac{\partial u(t,b)}{\partial x}\int_a^x \frac{1}{q(x')}dx' + L_{xx}^{-1}g(t,x)$$
$$-L_{xx}^{-1}L_{tt}u(t,x) - L_{xx}^{-1}Nu(t,x). \tag{3.27}$$

Therefore, the two partial solutions in Eq. (3.10) and Eq. (3.27) lead to

$$u(t,x) = \frac{1}{2}\Bigg(u(0,x) + tu_t(0,x) + u(t,a) + q(b)\frac{\partial u(t,b)}{\partial x}\int_a^x \frac{1}{q(x')}dx' + L_{tt}^{-1}g(t,x) + L_{xx}^{-1}g(t,x)$$
$$-L_{tt}^{-1}L_{xx}u(t,x) - L_{tt}^{-1}Nu(t,x) - L_{xx}^{-1}L_{tt}u(t,x) - L_{xx}^{-1}Nu(t,x)\Bigg)\cdot \tag{3.28}$$

Substituting Eq. (3.4) and Eq. (3.5) into Eq. (3.28), we have

$$\sum_{n=0}^{\infty}u_n(t,x) = \frac{1}{2}\Bigg(u(0,x) + tu_t(0,x) + u(t,a) + q(b)\frac{\partial u(t,b)}{\partial x}\int_a^x \frac{1}{q(x')}dx' + L_{tt}^{-1}g(t,x) + L_{xx}^{-1}g(t,x)$$

$$-L_{tt}^{-1}L_{xx}\sum_{n=0}^{\infty}u_n(t,x) - L_{tt}^{-1}\sum_{n=0}^{\infty}A_n(t,x) - L_{xx}^{-1}L_{tt}\sum_{n=0}^{\infty}u_n(t,x) - L_{xx}^{-1}\sum_{n=0}^{\infty}A_n(t,x)\Bigg)\cdot \tag{3.29}$$

with the recurrence relations:

$$u_0 = \frac{1}{2}\Bigg(u(0,x) + tu_t(0,x) + u(t,a) + q(b)\frac{\partial u(t,b)}{\partial x}\int_a^x \frac{1}{q(x')}dx' + L_{tt}^{-1}g(t,x) + L_{xx}^{-1}g(t,x)\Bigg)$$

$$u_{n+1} = \frac{1}{2}\left[-L_{tt}^{-1}L_{xx}u_n - L_{xx}^{-1}L_{tt}u_n - L_{tt}^{-1}A_n - L_{xx}^{-1}A_n\right], n \geq 0. \tag{3.30}$$

4. Numerical illustrations

To demonstrate the applicability of the proposed method, it has been applied in this section on some classical problems with the corresponding numerical results.

Example 1
Consider the following linear homogeneous heat equation
$u_t = u_{xx}$, $0 \leq x \leq 1,\ t > 0,$
with the initial/boundary conditions:
$u(0,x) = x^2,\ 0 \leq x \leq 1,$
$u(t,0) = 2t, u(t,1) = 1 + 2t,\ t > 0.$
Rewrite the heat equation in the operator form as $L_t u(t,x) = L_{xx}u(t,x)$ where $L_t = \frac{\partial}{\partial t}$, and $L_{xx} = \frac{\partial^2}{\partial x^2}$. Here, $p(x) = q(x) = 1$, hence

$$L_{xx}=\frac{1}{p(x)}\frac{\partial}{\partial x}\left(q(x)\frac{\partial}{\partial x}\right)=\frac{\partial^2}{\partial x^2}.$$

We solve for L_{xx} equation by applying the inverse operator

$$L_{xx}^{-1}(.)=\int_0^x dx'\int_0^{x'}(.)dx''-x\int_0^1 dx'\int_0^x(.)dx'',$$

on both sides of the previous equation, then we get

$u(t,x)=2(1-x)t+x(1+2t)+L_{xx}^{-1}L_t u(t,x).$

This gives the recursive relation:

$u_0=2(1-x)t+x(1+2t),$

$u_{n+1}=L_{xx}^{-1}L_t u_n,\ n\geq 0.$

Hence,

$u_0=x+2t,$

$u_1=x^2-x,$

$u_{n+1}=0\ ,n\geq 1.$

Therefore, the exact solution $u(t,x)=x^2+2t$ is obtained from only two components.

Example 2

Consider the system

$$\omega_{tt}-\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial\omega}{\partial x}\right)+\alpha\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial\theta}{\partial x}\right)=f(t,x), \tag{4.1}$$

$$\theta_t-\gamma\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial\theta}{\partial x}\right)+\beta\omega_t=g(t,x), \tag{4.2}$$

with the initial conditions

$$\omega(0,x)=x^2,\ \omega_t(0,x)=0,\ \theta(0,x)=x^2,\qquad 0<x<1 \tag{4.3}$$

and the boundary conditions

$$\omega(t,0)=t^2,\ \omega(t,1)=1+t^2,\ 0<t<T$$

$$\theta(t,0)=-t^2,\ \theta(t,1)=1-t^2,\ 0<t<T \tag{4.4}$$

Rewrite the given system in the operator form as

$$L_{tt}\omega-L_{xx}\omega+\alpha L_{xx}\omega=f(t,x), \tag{4.5}$$

$$L_t\theta - \gamma L_{xx}\theta + \beta L_t\omega = g\left(t,x\right), \tag{4.6}$$

where $L_t = \dfrac{\partial}{\partial t}$, $L_{tt} = \dfrac{\partial^2}{\partial t^2}$, and $L_{xx} = \dfrac{1}{x^2}\dfrac{\partial}{\partial x}\left(x^2\dfrac{\partial}{\partial x}\right)$. Applying the inverse operators L_{tt}^{-1} and L_t^{-1} on both sides of Eq. (4.5) and Eq. (4.6), respectively, we obtain

$$\omega(t,x) = \omega(0,x) + t\,\omega_t\left(0,x\right) + L_{tt}^{-1}f\left(t,x\right) + L_{tt}^{-1}\left(L_{xx}\omega\right) - \alpha L_{tt}^{-1}\left(L_{xx}\theta\right), \tag{4.7}$$

$$\theta(t,x) = \theta(0,x) + L_t^{-1}g\left(t,x\right) + \gamma L_t^{-1}\left(L_{xx}\theta\right) - \beta L_t^{-1}\left(L_t\omega\right), \tag{4.8}$$

For $p\left(x\right) = q(x) = x^r$, L_{xx}^{-1} is defined as

$$L_{xx}^{-1}\left(.\right) = \int_a^x \frac{1}{x^r}dx'\int_0^{x'} x^r\left(.\right)dx'' - z\left(x\right)\int_a^b \frac{1}{x^r}dx'\int_0^x x^r\left(.\right)dx''\cdot \tag{4.9}$$

where $z\left(x\right)$ is defined as given in reference [10] by

$$z\left(x\right) = \begin{cases} \dfrac{x-a}{b-a}, & r = 0 \\ -1, & r = 1.2.3... \end{cases}$$

At $r = 2$, L_{xx}^{-1} takes the form

$$L_{xx}^{-1}\left(.\right) = \int_a^x \frac{1}{x^2}dx'\int_0^{x'} x^2\left(.\right)dx'' + \int_a^b \frac{1}{x^2}dx'\int_0^x x^2\left(.\right)dx''\cdot \tag{4.10}$$

Similarly, the inverse operator L_{xx}^{-1} as defined in (4.10) is applied on both sides of Eqs. (4.7) and (4.8) to give

$$\begin{aligned} L_{xx}^{-1}\omega(t,x) = {} & L_{xx}^{-1}\left[\omega(0,x) + t\,\omega_t\left(0,x\right)\right] + L_{xx}^{-1}L_{tt}^{-1}f\left(t,x\right) \\ & + L_{xx}^{-1}L_{tt}^{-1}\left(L_{xx}\omega\right) - \alpha L_{xx}^{-1}L_{tt}^{-1}\left(L_{xx}\theta\right), \end{aligned} \tag{4.11}$$

$$\begin{aligned} L_{xx}^{-1}\theta(t,x) = {} & L_{xx}^{-1}\left[\theta(0,x) + \beta\omega(0,x)\right] + L_{xx}^{-1}L_t^{-1}g\left(t,x\right) \\ & - \beta L_{xx}^{-1}\omega(t,x) + \gamma L_{xx}^{-1}L_t^{-1}\left(L_{xx}\theta\right). \end{aligned} \tag{4.12}$$

Hence

$$L_{xx}^{-1}L_{xx}\omega = \omega(t,x) - 2\omega(t,a) + \omega(t,b), \tag{4.13}$$

$$L_{xx}^{-1}L_{xx}\theta = \theta(t,x) - 2\theta(t,a) + \theta(t,b), \tag{4.14}$$

Substituting (4.13) and (3.14) into the system (4.11) - (4.12) gives

$$L_{xx}^{-1}\omega(t,x) = L_{xx}^{-1}\left[\omega(0,x)+t\,\omega_t(0,x)\right]+L_{xx}^{-1}L_{tt}^{-1}f\,(t,x)$$
$$+L_{tt}^{-1}\left[\omega(t,x)-2\omega(t,a)+\omega(t,b)\right]-\alpha L_{tt}^{-1}\left[\theta(t,x)-2\theta(t,a)+\theta(t,b)\right], \quad (4.15)$$
$$L_{xx}^{-1}\theta(t,x) = L_{xx}^{-1}\left[\theta(0,x)+\beta\omega(0,x)\right]+L_{xx}^{-1}L_{t}^{-1}g\,(t,x)$$
$$-\beta L_{xx}^{-1}\omega(t,x)+\gamma L_{t}^{-1}\left[\theta(t,x)-2\theta(t,a)+\theta(t,b)\right]. \quad (4.16)$$

We rearrange (4.15) and (4.16) as

$$L_{tt}^{-1}\omega(t,x) = L_{xx}^{-1}\omega(t,x)-L_{xx}^{-1}\left[\omega(0,x)+t\,\omega_t(0,x)\right]-L_{xx}^{-1}L_{tt}^{-1}f\,(t,x)$$
$$+L_{tt}^{-1}\left[2\omega(t,a)-\omega(t,b)\right]+\alpha L_{tt}^{-1}\left[\theta(t,x)-2\theta(t,a)+\theta(t,b)\right], \quad (4.17)$$

$$L_{t}^{-1}\theta(t,x) = \frac{1}{\gamma}\left[L_{xx}^{-1}\theta(t,x)-L_{xx}^{-1}\theta(0,x)\right]$$
$$-\frac{1}{\gamma}L_{xx}^{-1}L_{t}^{-1}g\,(t,x)+\frac{1}{\gamma}\beta L_{xx}^{-1}\left[\omega(t,x)-\omega(0,x)\right]+L_{t}^{-1}\left[2\theta(t,a)-\theta(t,b)\right]. \quad (4.18)$$

The new system (4.17)-(4.18) includes all conditions (initial and boundary), but the problem that appears now is the inapplicability of Adomian decomposition method, so we define two functions $u(t,x)$ with the conditions $u(0,x)=u_t(0,x)=0$ such that

$$\omega(t,x) = \frac{\partial^2 u}{\partial t^2}, \quad (4.19)$$

The other function $v(t,x)$ with the condition $v(0,x)=0$ such that

$$\theta(t,x) = \frac{\partial v}{\partial t}. \quad (4.20)$$

Substituting (4.19) and (4.20) into the system (4.17)-(4.18), we obtain

$$u(t,x) = L_{tt}^{-1}\left[2\omega(t,a)-\omega(t,b)\right]-\alpha L_{tt}^{-1}\left[2\theta(t,a)-\theta(t,b)\right]$$
$$-L_{xx}^{-1}\left[\omega(0,x)+t\,\omega_t(0,x)\right]-L_{xx}^{-1}L_{tt}^{-1}f\,(t,x)+L_{xx}^{-1}u_{tt}+\alpha L_{tt}^{-1}v_t, \quad (4.21)$$
$$v(t,x) = L_{t}^{-1}\left[2\theta(t,a)-\theta(t,b)\right]-\frac{1}{\gamma}L_{xx}^{-1}\left[\theta(0,x)+\beta\omega(0,x)\right]$$
$$-\frac{1}{\gamma}L_{xx}^{-1}L_{t}^{-1}g\,(t,x)+\frac{1}{\gamma}L_{xx}^{-1}\left[v_t+\beta u_{tt}\right]. \quad (4.22)$$

The standard Adomian decomposition method defines the solutions $u(t,x)$ and $v(t,x)$ by the decomposition series

$$u(t,x)=\sum_{n=0}^{\infty}u_n, \quad \text{and} \quad v(t,x)=\sum_{n=0}^{\infty}v_n. \quad (4.23)$$

Substituting (4.23) into (4.21) and (4.22), we obtain the recurrence relations as follows

$$u_0 = L_{tt}^{-1}\left[2\omega(t,a)-\omega(t,b)\right]-\alpha L_{tt}^{-1}\left[2\theta(t,a)-\theta(t,b)\right]$$
$$-L_{xx}^{-1}\left[\omega(0,x)+t\,\omega_t(0,x)\right]-L_{xx}^{-1}L_{tt}^{-1}f\,(t,x),$$
$$v_0 = L_t^{-1}\left[2\theta(t,a)-\theta(t,b)\right]-\frac{1}{\gamma}L_{xx}^{-1}\left[\theta(0,x)+\beta\omega(0,x)\right]\;-\frac{1}{\gamma}L_{xx}^{-1}L_t^{-1}g\,(t,x),$$
$$u_{n+1} = L_{xx}^{-1}u_{n_{tt}} + \alpha L_{tt}^{-1}v_{n_t},\; n \geq 0,$$
$$v_{n+1} = \frac{1}{\gamma}L_{xx}^{-1}\left[v_{n_t}+\beta u_{n_{tt}}\right],\; n \geq 0. \tag{4.24}$$

Suppose that $\alpha=\beta=\gamma=1, f\,(t,x)=2, g\,(t,x)=-6,$ hence after calculating the components $u_i, i=0,1,2,\ldots$ and $v_i, i=0,1,2,\ldots$ then substituting (4.24) into (4.23), the solution is obtained after solving the resulting equations. Applying the proposed method, we obtain the following terms

$$u_0 = \frac{1}{6}t^4 - \frac{1}{20}x^4 - \frac{1}{20} - \frac{1}{2}\left(\frac{1}{3}x^2+\frac{1}{3}\right)t^2,$$
$$v_0 = -\frac{1}{3}t^3 - \frac{1}{10}x^4 - \frac{1}{10} + x^2 t,$$
$$u_1 = \frac{-1}{60}x^4 + \frac{5}{6}t^2x^2 - \frac{1}{18}x^2 - \frac{13}{180} + \frac{1}{3}t^2 - \frac{1}{12}t^4,$$
$$v_1 = \frac{1}{30}x^4 + \frac{1}{6}t^2x^2 - \frac{1}{18}x^2 - \frac{1}{45} + \frac{1}{6}t^2,$$
$$u_2 = \frac{1}{12}x^4 - \frac{1}{6}t^2x^2 + \frac{1}{9}x^2 + \frac{7}{36} - \frac{1}{6}t^2 + \frac{1}{3}\left(\frac{1}{6}x^2+\frac{1}{6}\right)t^3,$$
$$v_2 = \frac{1}{60}x^4t + \frac{1}{12}x^4 - \frac{1}{6}x^2t^2 + \frac{1}{18}x^2t + \frac{1}{9}x^2 + \frac{13}{180}t + \frac{7}{36} - \frac{1}{6}t^2,$$

We note that using only 10 components then the absolute errors becomes zero. In Fig. 1, the solution by the proposed method is compared with the exact solution.

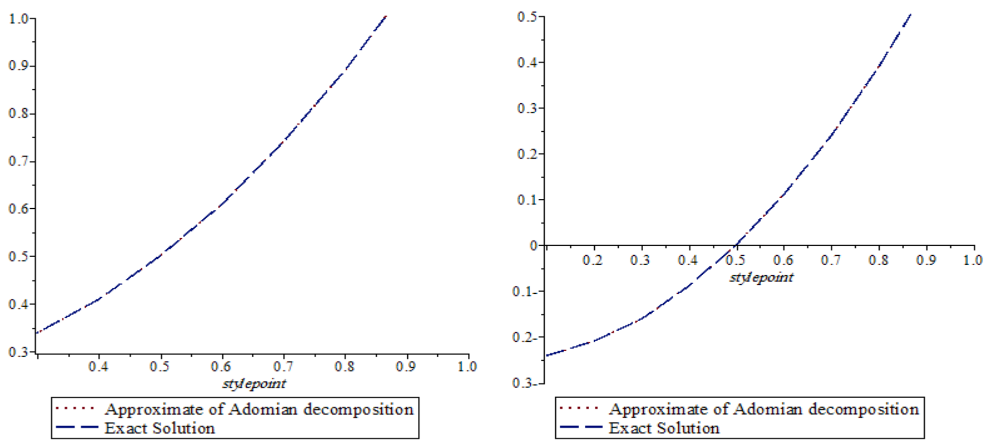

Fig. 1. The graphs for the exact and the approximate solutions example 2.

Example 3.

Consider the following non dimensional BVP [11]

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu(r)\frac{\partial\omega}{\partial r}\right)=\frac{dp}{dz}+M^{2}\omega,$$

with the mixed boundary conditions:

$$\frac{\partial\omega}{\partial r}=0 \text{ at } r=0,\ \omega=-1 \text{ at } r=h(z),$$

where $h(z)=1, \frac{dp}{dz}=-1, M=1$. Firstly, we rewrite the given equation in the operator form $L_r\omega=-1+\omega$, where the differential operator L_r is defined in the form $L_r=\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu(r)\frac{\partial}{\partial r}\right)$, and the inverse operator L_r^{-1} is defined by $L_r^{-1}(.)=\int_1^r\left[\frac{1}{r\mu(r)}\int_0^r r(.)dr''\right]dr'.$

Applying L_r^{-1}, we get $L_r^{-1}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu(r)\frac{\partial\omega}{\partial r}\right)\right]=\omega(r,z)-\omega(1,z).$

Operating with L_r^{-1}, it then follows

$$\omega(r,z)=\omega(1,z)+L_r^{-1}(-1+\omega).$$

Now we decompose $\omega(r,z)$ as $\sum_{n=0}^{\infty}\omega_n(t,x)$ and according to the modified decomposition method, the solution $\omega(r,z)$ can be elegantly computed by using the recurrence relation

$$\omega_0=\omega(1,z)+L_r^{-1}(-1),$$

$$\omega_{n+1}=L_r^{-1}(\omega_n),\ n\geq 0.$$

This gives

$$\omega_0=-1+\left(-\frac{r^2}{4}-\frac{1}{4}\right)=-\frac{r^2}{4}-\frac{3}{4},$$

$$\omega_1=L_r^{-1}(\omega_0)=-\frac{r^4}{64}-\frac{3r^2}{16}+\left(\frac{1}{64}+\frac{3}{16}\right),$$

$$\vdots$$

The other solution-components $\omega_2,\omega_3,...$ can be obtained. The absolutes errors are given at different values of r in Table 1.

Table 1. The absolutes errors at $0 \le r \le 1$

r	N=3	N=5	N=10
0.0	-6.81559245e-03	-2.04780013e-04	-1.83105434e-07
0.1	-6.71933832e-03	-2.01832306e-04	-1.80467659e-07
0.2	-6.43436500e-03	-1.93115973e-04	-1.72668237e-07
0.3	-5.97192208e-03	-1.79005419e-04	-1.60043422e-07
0.4	-5.35036578e-03	-1.60105033e-04	-1.43135696e-07
0.5	-4.59456444e-03	-1.37220728e-04	-1.22667814e-07
0.6	-3.73505667e-03	-1.11321946e-04	-9.95083075e-08
0.7	-2.80695258e-03	-8.34959543e-05	-7.46301849e-08
0.8	-1.84856578e-03	-5.48965935e-05	-4.90648978e-08
0.9	-8.99761046e-04	-2.66900003e-05	-2.38538696e-08
1.0	7.80000000e-34	-8.76543000e-34	-2.09014927e-37

5 Conclusion

In this paper, the generalized inverse operator based on the modified Adomian decomposition method was presented to solving partial differential equation with Dirichlet, Neumann, and mixed boundary conditions. The proposed method has been applied on several examples in physics and fluid mechanics and the obtained numerical results were highly accurate.

References

[1] G. Adomian, *Stochastic Systems*, Academic Press, New York, 1983.

[2] G. Adomian, Solving frontier problems modeled by nonlinear partial differential equations, *Comp. Math. Appl*., **22** (1991), no. 8, 91–94. https://doi.org/10.1016/0898-1221(91)90017-x

[3] A. Ebaid, A new numerical solution for the MHD peristaltic flow of a bio-fluid with variable viscosity in a circular cylindrical tube via Adomian decomposition method, *Phys. Lett. A*, **372** (2008), 5321–5328. https://doi.org/10.1016/j.physleta.2008.05.086

[4] H O. Bakodah, Modified Adomain Decomposition Method for the Generalized Fifth Order KdV Equations, *American Journal of Computational Mathematics*, **3** (2013), 53-58. https://doi.org/10.4236/ajcm.2013.31008

[5] H.O. Bakodah, New Approach for numerical solution of Kawahara and modified Kawahara equations by Adomian decomposition method, *International Journal of Conceptions on Computing and Information Technology*, **3** (2015), 40-43.

[6] H.O. Bakodah, F.A. Hendi and N.A. Al-zaid, Application of the New Modified Decomposition Method to the Regularized Long-Wave Equation, *Life Science Journal*, **9** (2012), no. 4, 5862-5866.

[7] G. Adomian, A new approach to the heat equation- An application of the decomposition method, *J. Math. Anal. Appl.*, **113**, (1986), 202-209. https://doi.org/10.1016/0022-247x(86)90344-6

[8] D. Lesnic, A computational algebraic investigation of the decomposition method for time-dependent problems, *Appl. Math. Comput*., **119** (2001), 197–206. https://doi.org/10.1016/s0096-3003(99)00257-x

[9] E. H. Aly, A Ebaid, R. Rach, Advances in the Adomian decomposition method for solving two-point nonlinear boundary value problems with Neumann boundary conditions, *Computers and Mathematics with Applications* **63** (2012), 1056–1065. https://doi.org/10.1016/j.camwa.2011.12.010

[10] A. Ebaid, A new analytical and numerical treatment for singular two-point boundary value problems via the Adomian decomposition method, *J. Computational and Applied Mathematics*, **235** (2011), 1914–1924. https://doi.org/10.1016/j.cam.2010.09.007

[11] N. El-Dabe, E. Elghazy, A. Ebaid, Closed form solution to a second order boundary value problem and its application in fluid mechanics, *Physics Letters A*, **363** (2007), 257–259. https://doi.org/10.1016/j.physleta.2006.11.010

Received: April 29, 2018; Published: May 27, 2018