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Positive Solution of a Nonlinear Four-Point Boundary-Value Problem

Weijiao Xu[1]

School of Science
Tianjin University of Commerce
Tianjin 300134, P.R. China

Huaying Wang

College of Mathematics and Statistics
Hebei University of Economics and Business
Hebei 050061, P.R. China

Abstract

This paper studies equation $\ddot{u}(t)+q(t)f(t)=0,\ t\in(0,1)$ with four-point boundary conditions $\dot{u}(0)=0, u(1)=a_1u(\xi)+a_2u(\eta)$, where $0<\xi,\eta<1, a_1+a_2<1$. The existence result of positive solution is obtained by applying the fixed point theorem in cones. The approaches developed here extend the ideas and techniques derived in recent literatures.

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1 Introduction

It is well known, boundary value problems have been becoming an important aspect in differential equations, one can read [1-8], etc. Many papers are

[1]Corresponding author

concerned about them, especially to the existence of solution (include multiple solution, positive solution, periodic solution, and extreme solution). A wide variety of approaches have been derived to this goal, such as nonlinear alternative of Leray-Schauder [1,9], Mawhin's continuation theorem [1,2], Krasnoselskii fixed point theorem [1], upper and lower solution [1,10], monotone iterative method [3], etc. Among them, multiple-point boundary value problems have attracted much attention for their widely background in theory and practical application. The study of multi-point boundary value problems for linear second order ordinary differential equations was initiated by II'in and Moiseev. Subsequently the more conclusions of nonlinear multi-point boundary value problems appeared. Ma [12] studied positive solutions of a nonlinear three-point boundary-value problem

$$\ddot{u}(t) + a(t)f(t) = 0, t \in (0,1)$$

with the boundary conditions

$$u(0) = 0, u(1) = \alpha u(\eta),$$

where $0 < \eta < 1$, and $0 < \alpha < \frac{1}{\eta}$.

Motivated by [12], in this paper, we consider the following problems

$$\ddot{u}(t) + q(t)f(t) = 0 \tag{1}$$

with the boundary conditions

$$\dot{u}(0) = 0, u(1) = a_1 u(\xi) + a_2 u(\eta), \tag{2}$$

where $0 < \xi, \eta < 1$.

For convenience, we denote that

$$\lambda = max\{\xi, \eta\}, \quad a = a_1 + a_2.$$

From now on, we assume the following:

$$(A_1) \quad f \in C([0,\infty),[0,\infty));$$

$$(A_2) \quad q \in C([0,\infty),[0,\infty)),$$

there exists $x_0 \in [\lambda, 1]$ such that $q(x_0) > 0$.

Set

$$f_0 = \lim_{u\to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u\to\infty} \frac{f(u)}{u}.$$

The paper is organized as follows: In section 2, we show some lemmas which are necessary to the proof of our main result. In section 3, by applying the fixed point theorem in cones, we obtain the main results for BVPs (1), (2).

2 Preliminary Notes

In this section, we establish some lemmas which are necessary to develop the main results in this paper.

Lemma 2.1 *Let $a_1 + a_2 \neq 1$, then for $y \in C[0,1]$, the problem*

$$\ddot{u} + y(t) = 0, \quad t \in (0,1) \tag{3}$$

$$\dot{u}(0) = 0, \quad u(1) = a_1 u(\xi) + a_2 u(\eta) \tag{4}$$

has a unique solution

$$\begin{aligned} u(t) \;\; = \;\; & -\int_0^t (t-s)y(s)ds + \frac{a_1}{a_1+a_2-1}\int_0^\xi (\xi - s)y(s)ds \\ & + \frac{a_2}{a_1+a_2-1}\int_0^\eta (\eta - s)y(s)ds - \frac{1}{a_1+a_2-1}\int_0^1 (1-s)y(s)ds. \end{aligned} \tag{5}$$

Lemma 2.2 [11] *Assume that $a_i, i = 1, 2, 3..., m-2$ have the same signal, if the function x satisfies:*

$$x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i),$$

then exists $\eta \in [\xi_1, \xi_{m-2}]$ such that $x(1) = \alpha x(\eta)$, where $\alpha = \sum_{i=1}^{m-2} a_i$.

Lemma 2.3 *Let $a_1 + a_2 < 1$. If for $y \in C[0,1]$ and $y \geq 0$, then the problem* (3) *and* (4) *has the unique solution u satisfies $u \geq 0$, $t \in C[0,1]$.*

Proof. From the fact that $\ddot{u} = -y(t) \leq 0$ we know that the graph of $u(t)$ is concave down on $(0,1)$ and by the unique solution we known the unique solution is positive when $t = 0$.

If $u(1) \geq 0$ then the concavity of u and the boundary condition $\dot{u}(0) = 0$ imply that $\dot{u}(t) = -\int_0^t y(t)ds < 0$ then $u(t)$ is monotonous decrease function, so it has $u \geq 0, t \in [0,1]$.

If $u(1) < 0$,then it has $u(\beta) < 0$ and $u(1) = au(\beta) > u(\beta), \beta \in (0,1)$. This contradicts the monotonous decrease function of u.

Lemma 2.4 *Let $a_1 + a_2 > 1$, if $y \in C[0,1]$ and $y(t) \geq 0, t \in (0,1)$. then* (3) *and* (4) *has no positive solution.*

Proof. Assume that (3) and (4) has a positive solution u.

If $u(1) > 0$ then $u(\beta) > 0$ and $u(1) = au(\beta) > u(\beta)$ this contradicts the monotonous decrease function of u.

If $u(1) = 0$, and $u(\tau) > 0$ for some $\tau \in (0, 1)$. Then $u(\beta) = u(1) = 0,\ \tau \neq \beta$.

If $\tau \in (0, \beta)$, then $u(\tau) > u(\beta) = u(1)$ this contradicts the definition of the concavity.

If $\tau \in (\beta, 1)$, then $u(\tau) > u(\beta)$ this contradicts the monotonous decrease function of u.

If $u(1) < 0, u(0) < 0$ it has no solution;

$u(1) < 0, u(0) \geq 0$, let $u(\bar{t}) = 0, \beta \in (0, \bar{t}\ ], u(t) = au(\beta)$ contradicts; $\beta \in (\bar{t}, 1), u(1) = au(\beta)$, it exists $t_1 \in (0, \bar{t}\ ]$ such that $u(t_1) > 0$. There is

$$\frac{u(t_1) - u(\beta)}{\beta - t_1} = \frac{u(t_1) - \dfrac{u(1)}{a}}{t_1 - \beta} < \frac{u(t_1) - u(1)}{t_1 - \beta} < \frac{u(t_1) - u(1)}{t_1 - 1}.$$

This contradicts the concavity of u, so $u(t_1) > 0$ is not true.

Lemma 2.5 *Let $a_1 + a_2 < 1$.If for $y \in C[0, 1]$ and $y \geq 0$,then the problem* (3) *and* (4) *has the unique solution u satisfies $\inf\limits_{t\in[\beta,1]} u(t) \geq \gamma\|u\|$ where $\gamma = min\{\dfrac{a(1-\beta)}{1-a\beta}, a\}$ and $a = a_1 + a_2$.*

Proof. When $0 < a < 1$ by the monotonous

$$u(\beta) \geq u(1).$$

Let

$$u(\bar{t}) = \|u\|.$$

If

$$\bar{t} \leq \beta < 1,$$

then

$$\min_{t\in[\beta,1]} u(t) = u(1),$$

and

$$u(\bar{t}) \leq u(1) + \frac{u(1) - u(\beta)}{1 - \beta}(0 - 1) \leq u(1)(1 - \frac{1 - \dfrac{1}{a}}{1 - \beta}) = u(t)\frac{1 - a\beta}{a(1 - \beta)},$$

$$\min_{t\in[\beta,1]} u(t) \geq \frac{a(1-\beta)}{1-a\beta}\|u\|.$$

If

$$\beta < \bar{t} < 1,$$

then

$$\min_{t\in[\beta,1]} u(t) = u(1),$$

by the monotonous

$$u(\beta) \geq u(\bar{t}),$$

$$\frac{u(1)}{a} \geq u(\bar{t}),$$

$$u(1) \geq au(\bar{t}),$$

$$\min_{t\in[\beta,1]} u(t) \geq au(\bar{t}) = a\|u\|.$$

3 Main Results

Theorem 3.1 *Assume* (A_1) *and* (A_2) *hold. Then the problem* (1) *and* (2) *has at least one positive solution in the case*

$$(i) f_0 = 0 \quad and \quad f_\infty = \infty \quad (superlinear)$$

or

$$(ii) f_0 = \infty \quad and \quad f_\infty = 0 \quad (sublinear)$$

where $0 < a_1 + a_2 < 1$.

Proof. (i) Superlinear case.

Suppose that $f_0 = 0$ *and* $f_\infty = \infty$. We wish to show the existence of a positive solution of (1) and (2).

Now (1) and (2) has a solution $y = y(t)$ if and only if y solves the operator equation

$$\begin{aligned} y(t) &= -\int_0^t (t-s)q(s)f(y(s))ds + \frac{a_1}{a_1+a_2-1}\int_0^\xi (\xi-s)q(s)f(y(s))ds + \\ &\quad \frac{a_2}{a_1+a_2-1}\int_0^\eta (\eta-s)q(s)f(y(s))ds - \frac{1}{a_1+a_2-1}\int_0^1 (1-s)q(s)f(y(s))ds \\ &= Ay(t). \end{aligned} \tag{6}$$

Denote

$$K = \{y \mid y \in C[0,1], y \geq 0, \min_{\lambda\leq t\leq 1} y(t) \geq \gamma\|y\|\},$$

$\lambda = max\{\xi, \eta\}$. It is obvious that K is a cone in $C[0,1]$. Moreover $AK \subset K$. It is also easy to check that $A : K \to K$ is completely continuous.

Since $f_0 = 0$, we may choose $H_1 > 0$ such that $f(y) \leq \varepsilon y$, for $0 < y < H_1$ when $\varepsilon > 0$ satisfies $\frac{\varepsilon}{1-a_1-a_2}\int_0^1 (1-s)q(s)ds \leq 1$. Thus, if $y \in K$ and $\|y\| = H_1$, it has

$$\begin{aligned}
Ay(t) &= -\int_0^t (t-s)q(s)f(y(s))ds + \frac{a_1}{a_1+a_2-1}\int_0^\xi (\xi-s)q(s)f(y(s))ds \\
&\quad + \frac{a_2}{a_1+a_2-1}\int_0^\eta (\eta-s)q(s)f(y(s))ds \\
&\quad - \frac{1}{a_1+a_2-1}\int_0^1 (1-s)q(s)f(y(s))ds \\
&\leq -\frac{1}{a_1+a_2-1}\int_0^1 (1-s)q(s)f(y(s))ds \\
&\leq -\frac{1}{a_1+a_2-1}\int_0^1 (1-s)q(s)\varepsilon y(s)ds \\
&\leq \frac{\varepsilon}{1-a_1-a_2}\int_0^1 (1-s)q(s)ds\|y\| \\
&\leq \frac{\varepsilon}{1-a_1-a_2}\int_0^1 (1-s)q(s)dsH_1,
\end{aligned} \tag{7}$$

and we let

$$\Omega_1 = \{y \in C[0,1] \mid \|y\| < H_1\}. \tag{8}$$

Then (7) shows that $\|Ay\| \leq y$, for $y \in K \bigcap \partial\Omega_1$.

Since $f_\infty = \infty$, there exists $\widehat{H_2} > 0$ such that $f(u) \geq \rho u$, for $u \geq \widehat{H_2}$ where $\rho > 0$ such that

$$\frac{\rho\gamma}{1-a_1-a_2}\int_\lambda^1 (1-s)q(s)ds \geq 1. \tag{9}$$

Let $H_2 = max\{2H_1, \frac{\widehat{H_2}}{r}\}$ and $\Omega_2 = \{y \in C[0,1] \mid \|y\| < H_2\}$, the $y \in K$ and $\|y\| = H_2$ implies $\min\limits_{\lambda<t<1} y(t) \geq \gamma\|y\| \geq H_2$.

$$\begin{aligned}
Ay(\lambda) &= -\int_0^\lambda (\lambda-s)q(s)f(y(s))ds + \frac{a_1}{a_1+a_2-1}\int_0^\xi (\xi-s)q(s)f(y(s))ds \\
&\quad + \frac{a_2}{a_1+a_2-1}\int_0^\eta (\eta-s)q(s)f(y(s))ds \\
&\quad - \frac{1}{a_1+a_2-1}\int_0^1 (1-s)q(s)f(y(s))ds
\end{aligned}$$

$$
\begin{aligned}
&\geq -\int_0^\lambda (\lambda - s)q(s)f(y(s))ds + \frac{a_1}{a_1+a_2-1}\int_0^\lambda (\lambda - s)q(s)f(y(s))ds \\
&\quad + \frac{a_2}{a_1+a_2-1}\int_0^\lambda (\lambda - s)q(s)f(y(s))ds \\
&\quad - \frac{1}{a_1+a_2-1}\int_0^1 (1 - s)q(s)f(y(s))ds \\
&= \frac{1}{a_1+a_2-1}\int_0^\lambda (\lambda - s)q(s)f(y(s))ds \\
&\quad - \frac{1}{a_1+a_2-1}\int_0^1 (1 - s)q(s)f(y(s))ds \\
&= \frac{1}{a_1+a_2-1}\Big(\int_0^\lambda (\lambda - s)q(s)f(y(s))ds - \int_0^1 (1 - s)q(s)f(y(s))ds\Big) \\
&= \frac{1}{a_1+a_2-1}\Big(\int_0^\lambda \lambda q(s)f(y(s))ds - \int_0^\lambda sq(s)f(y(s))ds \\
&\quad - \int_0^1 q(s)f(y(s))ds + \int_0^1 sq(s)f(y(s))ds\Big) \\
&\geq \frac{1}{a_1+a_2-1}\Big(\int_\lambda^1 sq(s)f(y(s))ds - \int_\lambda^1 q(s)f(y(s))ds\Big) \\
&= \frac{1}{a_1+a_2-1}\int_\lambda^1 (s-1)q(s)f(y(s))ds \\
&\geq \frac{\rho}{a_1+a_2-1}\int_\lambda^1 (s-1)q(s)y(s)ds.
\end{aligned} \tag{10}
$$

Hence, for $y \in K \bigcap \partial\Omega_2$,

$$
\|Ay\| \geq \frac{\rho\gamma}{1-a_1-a_2}\int_\lambda^1 (1-s)q(s)ds\|y\| \geq \|y\|.
$$

Therefore, by the Fixed Point Theorem, it follows that A has a fixed point in $K \bigcap (\overline{\Omega} \setminus \Omega_1)$ such that $H_1 \leq \|u\| \leq H_2$. This completes the superlinear part of the theorem.

(ii) Sublinear case.

Suppose next that $f_0 = \infty$ and $f_\infty = 0$ choose $H_3 > 0$ such that $f(y) \geq My$ for $0 < y < H_3$ where

$$
\frac{M\gamma}{1-a_1-a_2}\int_\lambda^1 (s-1)q(s)y(s)ds \geq 1. \tag{11}
$$

By using the method to get (10), we can get that

$$\begin{aligned}
Ay(\lambda) &= -\int_0^\lambda(\lambda-s)q(s)f(y(s))ds+\frac{a_1}{a_1+a_2-1}\int_0^\xi(\xi-s)q(s)f(y(s))ds \\
&\quad +\frac{a_2}{a_1+a_2-1}\int_0^\eta(\eta-s)q(s)f(y(s))ds \\
&\quad -\frac{1}{a_1+a_2-1}\int_0^1(1-s)q(s)f(y(s))ds \\
&\geq \frac{1}{a_1+a_2-1}\int_\lambda^1(s-1)q(s)f(y(s))ds \\
&\geq \frac{M}{a_1+a_2-1}\int_\lambda^1(s-1)q(s)y(s)ds \\
&\geq \frac{M\gamma}{a_1+a_2-1}\int_\lambda^1(s-1)q(s)y(s)ds\|y\| \\
&= H_3.
\end{aligned} \tag{12}$$

Thus, we may let $\Omega_3=\{y\in C[0,1] \mid \|y\|<H_3\}$ such that

$$\|Ay\|\geq\|y\|, y\in K\bigcap\partial\Omega_3.$$

Now, since $f_\infty=0$, exists $\overline{H_4}>0$ such that $f(y)\leq\varphi y$ for $y\geq\overline{H_4}$ where $\varphi>0$ satisfies $\dfrac{N}{1-a_1-a_2}\displaystyle\int_0^1(1-s)q(s)y(s)ds\leq 1$.

We consider two cases:

Case(i). Support f is bounded, say $f(y)\leq N$ for all $y\in[0,\infty)$ choose $H_4=max\{2H_3,\dfrac{N}{1-a_1-a_2}\displaystyle\int_0^1(1-s)q(s)ds\}$ such that, for $y\in K$ with $\|y\|=H_4$. We have

$$\begin{aligned}
Ay(t) &= -\int_0^t(t-s)q(s)f(y(s))ds+\frac{a_1}{a_1+a_2-1}\int_0^\xi(\xi-s)q(s)f(y(s))ds \\
&\quad +\frac{a_2}{a_1+a_2-1}\int_0^\eta(\eta-s)q(s)f(y(s))ds \\
&\quad -\frac{1}{a_1+a_2-1}\int_0^1(1-s)q(s)f(y(s))ds \\
&\leq \frac{1}{a_1+a_2-1}\int_0^1(s-1)q(s)f(y(s))ds \\
&\leq \frac{N}{a_1+a_2-1}\int_0^1(s-1)q(s)y(s)ds \\
&\leq \frac{N}{a_1+a_2-1}\int_0^1(s-1)q(s)ds\|y\| \\
&\leq H_4,
\end{aligned} \tag{13}$$

therefore $\|Ay\| \leq \|y\|$.

Case(ii). If f is unbounded, we know from (A_1) that there is H_4: $H_4 > max\{2H_3, \frac{1}{\gamma}H_4\}$ such that $f(y) \leq f(H_4)$ for $0 < y \leq H_4$(we are able to do this since f is unbounded). Then for $y \in K$ and $\|y\| = H_4$,we have

$$\begin{aligned}
Ay(t) &= -\int_0^t (t-s)q(s)f(y(s))ds + \frac{a_1}{a_1+a_2-1}\int_0^\xi (\xi-s)q(s)f(y(s))ds \\
&\quad + \frac{a_2}{a_1+a_2-1}\int_0^\eta (\eta-s)q(s)f(y(s))ds \\
&\quad - \frac{1}{a_1+a_2-1}\int_0^1 (1-s)q(s)f(y(s))ds \\
&\leq \frac{1}{a_1+a_2-1}\int_0^1 (s-1)q(s)f(y(s))ds \\
&\leq \frac{1}{a_1+a_2-1}\int_0^1 (s-1)q(s)f(H_4)ds \\
&\leq \frac{1}{a_1+a_2-1}\int_0^1 (s-1)q(s)\varphi H_4 ds \\
&\leq H_4.
\end{aligned} \tag{14}$$

Therefore, in either case we may put $\Omega_4 = \{y \in C[0,1] \mid \|y\| < H_4\}$ and for $y \in K \bigcap \partial\Omega_4$, we may have $\|Ay\| \leq \|y\|$. By the Fixed Point Theorem, it follows that the bounded valuable problem (1) and (2) has a positive solution. Therefore, we complete the proof of the theorem.

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