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On Impulsive Sequential Fractional Differential Equations with Separated Boundary Conditions

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Abstract

In this article, we investigate a new type of an Impulsive Sequential Fractional Differential Equations (ISFDEs) with Boundary Value Problem (BVP). Some new existence and uniqueness results are obtained by using the contraction mapping principle and the Schauder fixed point theorem.

Keywords: Impulsive sequential fractional differential equations, Boundary value problem, Banach fixed point theorem

1 Introduction

The study of Fractional Differential Equations appended with Boundary Conditions (BCs) have received considerable attention among the researchers. These attentions can be attributed to widely-used of fractional differential equations tools in many scientific fields such as chemistry, biology, physics, control theory, viscoelasticity, electrochemistry, signal processing, nuclear dynamics, etc; see[[1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13]]. In fact, both Sequential Fractional Differential Equations (SFDEs), and Impulsive Fractional Differential Equations (IFDEs), have received a great deal of attention from many authors see [[14], [15], [16], [17]]. However, a new topic has been investigated by combining both SFDEs and IFDEs, which in turn produced ISFDEs to be a much wider case. We develop the existence theory for our problem , which defined below by (1).

In 2013 [14], fixed point theory used to establish the existence results for a sequential integro-differential equation of fractional order with some BCs, which is given as:

$$\begin{gathered}({}^cD^\alpha + \lambda{}^cD^{\alpha-1}\ )u(t) = pf\left(t, u(t)\right) + qI^\beta g\left(t, u(t)\right),\ 0 < t < 1,\\ u(t) = 0,\ u(1) = 0, x(1) = 0,\end{gathered}$$

where ${}^cD^\alpha$ denotes the Caputo fractional derivative of order α.

In [15], the standard fixed point theorem has been used to obtain some existence results of the solutions for following problem:

$$\begin{gathered}({}^cD^\alpha + k\ {}^cD^{\alpha-1})x(t) = f\left(t, x(t)\right),\ t \in [0,1],\ 2 < \alpha \leq 3,\\ x(t) = 0,\ x('0) = 0,\ x(\zeta) = a\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}x(s)\,ds,\ \beta > 0,\end{gathered}$$

where ${}^cD^\alpha$denotes the is the Caputo fractional derivative of order $\alpha,\ 0 < \eta < \zeta < 1$, is given continuous function.

In 2012 [16], studied the following problem that consists of IFDEs with nonlinear BCs:

$$\begin{gathered}{}^cD^q u(t) = f(t, u(t), u\left(\omega\left(t\right)\right)),\ t \in J',\\ \Delta u(t)\left.\right|_{t=t_k} = I_k(u(t_k)),\ k = 1, ..., m,\\ \Delta u'(t)\left.\right|_{t=t_k} = \bar{I}_k(u(t_k)),\ k = 1, ..., m,\\ u(t) = (0),\ t \in [-r, 0],\end{gathered}$$

subject to the nonlinear boundary value condition as follows: $g_0(u(0), u\left(T\right)) = 0, g_0(u'(0), u'\left(T\right)) = 0$, with ${}^c\mathrm{D}^q$ is the Caputo derivative of order q.

In [17] Shuai and Shuqin, discussed a BVP for an IFDEs. They transferred the BVP into equivalent integral equation. Banach fixed point theorem and Schauder fixed point theorem. were

used to acquire the existence of the solutions of the following problem:

$$\begin{gathered}{}^cD^\alpha x(t) = f(t),\ < 1\alpha \leq 2,\ t \in J = [0,1],\ t \neq t_k,\\ \Delta x(t)\left.\right|_{t=t_k} = I_k(x(t_k)),\ k = 1, ..., m,\ t_k \in (0,1),\\ \Delta x'(t)\left.\right|_{t=t_k} = \bar{I}_k(x(t_k)),\ k = 1, ..., m, t_k \in (0,1),\\ x(0) = h\left(x\right),\ x(1) = g\left(x\right),\end{gathered}$$

$g(x) = \max\limits_{j} \frac{|x(\xi_j)|}{\lambda+|x(\xi_j)|}, h(x) = \min\limits_{j} \frac{|x(\zeta_j)|}{\kappa+|x(\zeta_j)|}$, where $0 < \xi_j ; \zeta_j < 1, \zeta_j \neq t_i, j = 1, 2, ..., n$ and λ, κ are given positive constants.

Based on previous studies, in this topic we concentrate on the existence results of solutions of ISFDEs with BCs which is given as follows:

$$\begin{gathered} {}^cD^{\beta-1}(D+\kappa\ )u(t) = f\left(t, u(t)\right), \quad 1<\beta\leq 2, \quad 0<t<T, \\ au(0) + \ {}^cD^{\beta-1}u(0) = w_0, \quad bu(T) + \ {}^cD^{\beta-1}(T) = w_1, \\ \Delta u(t)\left|_{t=t_k}\right. = u(t_k^+) - u(t_k^-) = \psi_k(u(t_k)), \quad k=1,...,q, \\ \Delta u'(t)\left|_{t=t_k}\right. = u'(t_k^+) - u'(t_k^-) = \psi_k^*(u(t_k)), \quad k=1,...,q. \end{gathered} \tag{1}$$

where ${}^c\mathrm{D}^\beta$ is the Caputo derivative of order $\beta \in (1,2]$, and D is the ordinary derivative and $f \in (J\times R, R),\ \psi_k, \psi_k^* \in C\left(R\times R\right),\ a, b \in R,\ \kappa \in R^+,\ \Delta u(t)\left|_{t=t_k}\right. = u\left(t_k^+\right) - u\left(t_k^-\right),\ \Delta u'(t)\left|_{t=t_k}\right. = u'\left(t_k^+\right) - u'\left(t_k^-\right)$. Here, respectively, the right and the left limits of $u(t)$ at $t = t_k^+$ are represented by $u\left(t_k^+\right)$ and $u\left(t_k^-\right)$.

2 Preliminary Notes

We introduce throughout this section some concepts of fractional calculus and fractional differential equations appended. Let $J = [0,T]\,, J' = [0,T]\setminus\{t_1, ..., t_q\}\,, 0 = t_0 < t_1 < \cdots < t_q < t_{q+1} = T$, and introduce the Banach space

$$PC(J) = \left\{u : J \to R \mid u \in C(J'), \text{and } u(t_k^+), u(t_k^-) \text{ exist, and } u(t_k^+) = u(t_k), 1 \leq k \leq q\right\},$$

with the norm $\|x\|_{PC} = \sup_{t\in j}|x(t)|$.

Definition 1 *The Riemann-Liouville fractional integral of order α is given as*

$$\mathbf{I}_{0+}^{\alpha} g(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-r)^{\alpha-1} g(r)dr,\ \ x>0\ ,\alpha>0,$$

provided the integral exists.

Definition 2 *The Caputo fractional derivative of order α for a continuous function $g(t)$ is given by*

$$\mathbf{D}_{0+}^{\alpha} g(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_0^x (x-r)^{n-\alpha-1} g(r)dr,$$

where $n = [\alpha]+1, [\alpha]$ and $[\alpha]$denotes the integer of α.

Lemma 3 *Let $\alpha > 0$. then the fractional differential equation$_1$*

$${}^c\mathbf{D}^{\alpha} g(t) = 0,$$

has the solution

$$g(x) = b_0 + b_1 x + b_2 x^2 + ... + b_{n-1} x^{n-1},$$

where $b_i \in R,\ i = 0, 1, 2, 3, ..., n-1;\ n = [\alpha] + 1.$

Lemma 4 *The set* $F \subset PC\left([0,T], R^n\right)$ *is relatively compact if and only if*

(i) F *is uniformly bounded , that is, there exists* $C > 0$ *such that* $\|u\| \leq C$ *for each* $u \in F$;

(ii) F *is quasi-equicontinuous in* $[0, T]$.

Lemma 5 *Let* M *be a closed convex and nonempty subset of a Banach space* X. *Let* $\Phi_1,\ \Phi_2$ *be the operators such that*

(i) $\Phi_1 u + \Phi_2 v \in M$ *whenever* $u, v \in M$;

(ii) Φ_1 *is compact and continuous;*

(iii) Φ_2 *is a contraction mapping.*
Then there exists $z \in M$ *such that* $z = \Phi_1 z + \Phi_2 z$.

Lemma 6 *For any* $z \in C\left(J\right)$, *the solution of the problem*

$$({}^c D^\beta + \kappa\ {}^c D^{\beta-1}) u(t) = z(t) \quad t \in J', \tag{2}$$

with BCs is given by

$$u\left(t\right) = \int_0^t M_1\left(t, m\right) z\left(m\right) dm + d_1\left(t\right) \int_0^T M_1\left(T, m\right) z\left(m\right) dm \tag{3}$$

$$+ d_2\left(t\right) \int_0^T M_2\left(T, m\right) z\left(m\right) dm + d_3\left(t\right) \int_0^T z\left(m\right) dm + d_4\left(t\right) \sum_{j=1}^{q} \psi_j(u(t_j))$$

$$+ d_5\left(t\right) \sum_{j=1j}^{q} \psi_j^*(u(t_j)) + \sum_{j=1}^{q} p_{j_1} \psi_j^*(u(t_j)) + \sum_{j=k+1}^{q} p_{j_2} \psi_j^*(u(t_j))$$

$$- \sum_{j=k+1}^{q} \psi_j(u(t_j)) + d_6\left(t\right),\ t \in \left[t_k, t_{k+1}\right), \quad k = 0, 1, ..., q,$$

where

$$d_1\left(t\right) = \frac{e\left(^{-\kappa t} - 1\right) b}{\left(b - \mu\right)}, \quad d_2\left(t\right) = \frac{\left(1 - e^{-\kappa t}\right) \kappa}{\left(b - \mu\right)}, \quad d_3\left(t\right) = \frac{e^{-\kappa t} - 1}{\left(b - \mu\right)},$$

$$d_4\left(t\right) = \frac{e^{-\kappa t} b - \mu}{\left(b - \mu\right)}, \quad d_5\left(t\right) = \frac{e^{-\kappa t} b - \mu}{\left(b - \mu\right) \kappa}, \quad p_{j_1} = \left(\frac{\left(\mu - e^{-\kappa t} b\right) e^{\kappa t_j}}{\left(b - \mu\right) \kappa}\right),$$

$$p_{j_2} = \left(\frac{e^{-\kappa t} e^{\kappa t_j}}{\kappa} - \frac{1}{\kappa}\right), \quad d_6\left(t\right) = \left(\frac{e^{-\kappa t} b - \mu}{\left(b - \mu\right) a}\right) w_0 + \left(\frac{1 - e^{-\kappa t}}{\left(b - \mu\right)}\right) w_1,$$

and

$$M_1(t,m) = \frac{1}{\Gamma(\beta-1)} \int_s^t e^{-\kappa(t-s)} (s-m)^{\beta-2} \, ds,$$

$$\int_0^t M_1(t,m)\, z(m)\, dm = \int_0^t e^{-\kappa(t-s)} \mathbf{I}^{\beta-1} z(s) ds,$$

$$M_2(t,s) = \frac{1}{\Gamma(2-\beta)} \int_s^t (t-m)^{1-\beta} M_1(m,s)\, dm$$

$$= \frac{1}{\Gamma(2-\beta)} \int_0^T (T-s)^{1-\beta} \left(\int_0^s e^{-\kappa(s-m)} \mathbf{I}^{\beta-1} z(m) dm \right) ds$$

$$= \frac{1}{\Gamma(2-\beta)} \int_0^T (T-s)^{1-\beta} \left(\int_0^s M_1(s,m)\, z(m)\, dm \right) ds$$

$$= \frac{1}{\Gamma(2-\beta)} \int_0^T \left(\int_m^T (T-m)^{1-\beta} M_1(s,m)\, ds \right) z(m)\, dm$$

$$= \int_0^T M_2(T,m)\, z(m)\, dm,$$

$$\mu = \left(b e^{-\kappa T} - \kappa \int_0^T \frac{(T-s)^{1-\beta}}{\Gamma(2-\beta)} e^{-\kappa s} ds \right),$$

and $\sum_{J=k+1}^{k} = 0$.

Proof. The general solution of (1) on $(t_k, t_{k+1}]$ $(k = 0, 1, 2, \ldots, q)$ can be expressed as

$$u(t) = e^{-\kappa t} a_k + b_k + \int_0^t e^{-\kappa(t-s)} \mathbf{I}^{\beta-1} z(s) ds, \tag{5}$$

where a_k and b_k are arbitrary constants. At this point, linear equation (5) is used in order to find the BCs. Considering (5) on $[t_0, t_1]$, leads to

$$u(t) = e^{-\kappa t} a_0 + b_0 + \int_0^t e^{-\kappa(t-s)} \mathbf{I}^{q-1} z(s) ds, \; t \in [t_0, t_1], \tag{6}$$

solving the Caputo derivative by using (5) on $[t_0, t_1]$, implies that

$$ {}^cD^{\beta-1} u(t) = -\kappa a_0 \int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)} e^{-\kappa s} ds - \int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)} \left(\kappa \int_0^s e^{-\kappa(s-m)} \mathbf{I}^{\beta-1} z(m) dm \right) ds \tag{7}$$

$$+ \int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)} \mathbf{I}^{\beta-1} z(s) ds, \quad t \in [t_0, t_1],$$

Now, applying the BC $au(0) + {}^cD^{\beta-1}u(0) = w_0$ by using obtained equations (6) and (7) on j_0, we find that

$$a(a_0 + b_0) = w_0. \tag{8}$$

Next, tor $t=T$ we write equation (5), as following

$$u(T)=e^{-\kappa T}a_q+b_q+\int_0^T e^{-\kappa(T-s)}\mathbf{I}^{\beta-1}z(s)ds. \tag{9}$$

Taking the Caputo derivative by using(9) on $t\in[t_q,t_{q+1})$, , we have

$$\begin{aligned}{}^cD^{\beta-1}u(T)=&-\kappa a_q\int_0^T\frac{(T-s)^{1-\beta}}{\Gamma(2-\beta)}e^{-\kappa s}-\kappa\int_0^T\frac{(T-s)^{1-\beta}}{\Gamma(2-\beta)}\left(\int_0^s e^{-\kappa(s-m)}\mathbf{I}^{\beta-1}z(m)dm\right)ds \\ &+\int_0^T\frac{(T-s)^{1-\beta}}{\Gamma(2-\beta)}\mathbf{I}^{\beta-1}z(s)ds.\end{aligned} \tag{10}$$

Now, applying the BC $bu(T)+^cD^{\beta-1}(T)=w_1$ and using obtained equations (9) and (10), we have

$$\mu a_q+bb_q=w_1-b\int_0^T M_1(T,m)\,z(m)\,dm+\kappa\int_0^T M_2(T,m)\,z(m)\,dm-\int_0^T z(m)\,dm. \tag{11}$$

Furthermore, using the impulsive condition $\Delta u'(t)\,|_{t=t_k}=\psi_k^*(u(t_k))=u'(t_k^+)-u'(t_k^-)$, we derive

$$\begin{aligned}a_k&=a_{k-1}-\frac{1}{\kappa}e_k^{\kappa t_k}\psi^*(u(t_k)),\\ a_k&=a_q+\frac{1}{\kappa}\sum_{j=k+1}^{q}e_j^{\kappa t_j}\psi_j^*(u(t_j)),k=1,...,q-1\end{aligned} \tag{12}$$

In the same way, using the impulsive condition $\Delta u(t)\,|_{t=t_k}=\psi_k(u(t_k))=u(t_k^+)-u(t_k^-)$, we derive

$$\begin{aligned}b_k&=b_{k-1}+\psi_k(u(t_k))+\frac{1}{\kappa}\psi_k^*(u(t_k)),\\ b_k&=b_q-\sum_{j=k+1}^{q}\psi_j(u(t_j))-\frac{1}{\kappa}\sum_{j=k+1}^{q}\psi_j^*(u(t_j)),\quad k=1,...,q-1.\end{aligned} \tag{13}$$

Also for $k=0$ from $a\,(a_0+b_0)=w_0$, we obtain that

$$a\,(a_q+b_q)=w_0+a\sum_{j=1}^{q}\psi_j(u(t_j))+\frac{a}{\kappa}\sum_{j=1}^{q}\psi_j^*(u(t_j))-\frac{a}{\kappa}\sum_{j=1}^{q}e^{\kappa t_j}\psi_j^*(u(t_j)).$$

Solving the last equation together with (11) for a_q and b_q , we get

$$\begin{aligned} a_q &= \frac{b}{(b-\mu)\,a}w_0 - \frac{1}{(b-\mu)}w_1 \\ &+ \frac{b}{(b-\mu)}\int_0^T M_1(T,m)\,z(m)\,dm - \frac{\kappa}{(b-\mu)}\int_0^T M_2(T,m)\,z(m)\,dm \\ &+ \frac{1}{(b-\mu)}\int_0^T z(m)\,dm + \frac{b}{(b-\mu)}\sum_{j=1}^{q}\psi_j(u(t_j)) + \frac{b}{(b-\mu)\,\kappa}\sum_{j=1}^{q}\psi_j^*(u(t_j)) \\ &- \frac{b}{(b-\mu)\,\kappa}\sum_{j=1}^{q}e^{\kappa t_j}\psi_j^*(u(t_j)), \end{aligned} \tag{14}$$

$$\begin{aligned} b_q &= -\frac{\mu}{(b-\mu)\,a}w_0 + \frac{1}{(b-\mu)}w_1 \\ &- \frac{b}{(b-\mu)}\int_0^T M_1(T,m)\,z(m)\,dm + \frac{\kappa}{(b-\mu)}\int_0^T M_2(T,m)\,z(m)\,dm \\ &- \frac{1}{(b-\mu)}\int_0^T z(m)\,dm - \frac{\mu}{(b-\mu)}\sum_{j=1}^{q}\psi_j(u(t_j)) - \frac{\mu}{(b-\mu)\,\kappa}\sum_{j=1}^{q}\psi_j^*(u(t_j)) \\ &+ \frac{\mu}{(b-\mu)\,\kappa}\sum_{j=1}^{q}e^{\kappa t_j}\psi_j^*(u(t_j)). \end{aligned} \tag{15}$$

Now, by (12), (13), (14) and (15), we have

$$\begin{aligned} a_k &= \frac{b}{(b-\mu)\,a}w_0 - \frac{1}{(b-\mu)}w_1 \\ &+ \frac{b}{(b-\mu)}\int_0^T M_1(T,m)\,z(m)\,dm - \frac{\kappa}{(b-\mu)}\int_0^T M_2(T,m)\,z(m)\,dm \\ &+ \frac{1}{(b-\mu)}\int_0^T z(m)\,dm + \frac{b}{(b-\mu)}\sum_{j=1}^{q}\psi_j(u(t_j)) + \frac{b}{(b-\mu)\,\kappa}\sum_{j=1}^{q}\psi_j^*(u(t_j)) \\ &- \frac{b}{(b-\mu)\,\kappa}\sum_{j=1}^{q}e^{\kappa t_j}\psi_j^*(u(t_j)) + \frac{1}{\kappa}\sum_{j=k+1}^{q}e^{\kappa t_j}\psi_j^*(u(t_j)), \end{aligned} \tag{16}$$

$$b_k = -\frac{\mu}{(b-\mu)\,a} w_0 + \frac{1}{(b-\mu)} w_1 \tag{17}$$
$$- \frac{b}{(b-\mu)} \int_0^T M_1(T,m)\, z(m)\, dm + \frac{\kappa}{(b-\mu)} \int_0^T M_2(T,m)\, z(m)\, dm$$
$$- \frac{1}{(b-\mu)} \int_0^T z(m)\, dm - \frac{\mu}{(b-\mu)} \sum_{j=1}^{q} \psi_j(u(t_j)) - \frac{\mu}{(b-\mu)\,\kappa} \sum_{j=1}^{q} \psi_j^*(u(t_j))$$
$$+ \frac{\mu}{(b-\mu)\,\kappa} \sum_{j=1}^{q} e^{\kappa t_j} \psi_j^*(u(t_j)) - \sum_{j=k+1}^{q} \psi_j(u(t_j)) - \frac{1}{\kappa} \sum_{j=k+1}^{q} \psi_j^*(u(t_j)).$$

Hence (16) and (17), imply

$$e^{-\kappa t} a_k + b_k = d_1(t) \int_0^T M_1(T,m)\, z(m)\, dm + d_2(t) \int_0^T M_2(T,m)\, z(m)\, dm \tag{18}$$
$$+ d_3(t) \int_0^T z(m)\, dm + +d_4(t) \sum_{j=1}^{q} \psi_j(u(t_j)) + d_5(t) \sum_{j=1}^{q} \psi_j^*(u(t_j))$$
$$+ \sum_{j=1}^{q} p_{1j} \psi_j^*(u(t_j)) + \sum_{j=k+1}^{q} p_{2j} \psi_j^*(u(t_j)) - \sum_{j=k+1}^{q} \psi_j(u(t_j)) + d_6(t),$$

Taking (18) into (5), we immediately derive the formula (3). Lemma is proved. ■

3 Main Results

Lemma 5 and 6, are exploited in this sectıon to prove existence and uniqueness results of solution of ISFDEs (1). Some assumptions have been formulated to prove our main results. As follow:

(H$_1$) $f : J \times R \to R$ is a jointly continuous function.

(H$_2$) There exists a constant $\mathcal{L}_f > 0$ such that

$$|f(t,u) - f(t,v)| \le \mathcal{L}_f\, |u - v|, \quad t \in J, \quad u, v \in R.$$

(H$_3$) $|\psi_k(u) - \psi_k(v)| \le \mathcal{L}_\psi\, |u - v|, |\psi_k^*(u) - \psi_k^*(v)| \le \mathcal{L}_{\psi^*}\, |u - v|; \quad |\psi_k(u)| \le M_{\psi}, |\psi_k^*(u)| \le M_{\psi^*}$.

(H$_4$) $|f(t,u)| \le \vartheta(t)$ for $(t,u) \in J \times R$ where $\vartheta \in L^{\frac{1}{\sigma}}(J \times R), \sigma \in (0, \beta - 1)$.

From (H_{21})-(H_3) it follows that

$$\begin{aligned}
|f(t,u)| &\leq \mathcal{L}_f |u| + M_f, \quad t \in J, \quad u \in R, \quad M_f := \sup\{|f(t,0)| : 0 < t \leq T\}, \\
|\psi_k(u)| &\leq \mathcal{L}_\psi |u| + M_\psi, \quad M_\psi := \sup\{|\psi_k(0)| : k = 1, ..., q\}, \\
|\psi_k^*(u)| &\leq \mathcal{L}_{\psi^*} |u| + M_{\psi^*}, \quad M_{\psi^*} := \sup\{|\psi_k^*(0)| : k = 1, ..., q\}.
\end{aligned}$$

(H_5) There exist a function $\varsigma \in PC(J, R)$,and nondecreasing function $\Delta : R^+ \to R^+$ (that is, $\Lambda(\gamma u) \leq \gamma\Lambda(u)$ for all $\gamma \geq 1$ and $u \in R^+$) such that

$$|f(t,u)| \leq \varsigma(t)\Lambda(\|u\|), for \text{ all } (t,u) \in J \times R,$$

(H_6) There exist a constant $Y > 0$ such that $\frac{Y}{\mathcal{L}_\Phi \|\varsigma\| \Delta(Y)} > 1$.

We presented some simple estimations that are used in the forthcoming theorems.

Lemma 7 *For any $z \in PC([0,T], R)$ we have*

$$\left|\int_0^t M_1(t,m)\, z(m)\, dm\right| \leq \left\{\frac{T^{\beta-1}}{\kappa\Gamma(\beta)}\left(1 - e^{-\kappa T}\right)\right\} \|z\|_{PC}, \tag{19}$$

$$\left|\int_0^T M_1(T,m)\, z(m)\, dm\right| \leq \left\{\frac{T^{\beta-1}}{\kappa\Gamma(\beta)}\left(1 - e^{-\kappa T}\right)\right\} \|z\|_{PC}, \tag{20}$$

$$\left|\int_0^T M_2(T,m)\, z(m)\, dm\right| \leq \left\{\frac{T}{\kappa}\left(T\kappa + e^{-\kappa T} - 1\right)\right\} \|z\|_{PC}. \tag{21}$$

Theorem 8 *Assume that (H_1)-(H_3) hold. If*

$$\begin{aligned}
\mathcal{L}_\Phi := &\left(\frac{T^{\beta-1}}{\kappa\Gamma(\beta)}\left(1 - e^{-\kappa T}\right)(1 + \|d_1\|) + \frac{T}{\kappa}\left(Tk + e^{-\kappa T} - 1\right)\|d_2\| + T\|d_3\|\right)\mathcal{L}_f \\
&+ (1 + \|d_4\|)\, r\mathcal{L}_\psi + (\|d_5\| + \|p_{1j}\| + \|p_{2j}\|)\, r\mathcal{L}_{\psi^*} + \|d_6\| < 1.
\end{aligned} \tag{22}$$

then the problem 1) has a unique solution on J.

Proof. Show that the operator Φ is contraction. To do so, let $u, v \in B_\varepsilon$. For $t \in [t_k, t_{k+1})$, we have

$$
\begin{aligned}
&|(\Phi u)(t) - (\Phi v)(t)| \\
&:= \left| \int_0^t M_1(t,m) f(m,u(m))\, dm + d_1(t) \int_0^T M_1(T,m) f(m,u(m))\, dm \right. \\
&+ d_2(t) \int_0^T M_2(T,m) f(m,u(m))\, dm + d_3(t) \int_0^T f(m,u(m))\, dm + d_4(t) \sum_{j=1}^q \psi_j(u(t_j)) \\
&\left. + d_5(t) \sum_{j=1}^q \psi_j^*(u(t_j)) + \sum_{j=1}^q p_{1j}\psi_j^*(u(t_j)) + \sum_{j=k+1}^q p_{2j}\psi_j^*(u(t_j)) + \sum_{j=k+1}^q \psi_j(u(t_j)) + p_3(t) \right| \\
&- \left| \int_0^t M_1(t,m) f(m,v(m))\, dm + d_1(t) \int_0^T M_1(T,m) f(m,v(m))\, dm \right. \\
&+ d_2(t) \int_0^T M_2(T,m) f(m,v(m))\, dm + d_3(t) \int_0^T f(m,v(m))\, dm \\
&+ d_4(t) \sum_{j=1}^q \psi_j(v(t_j)) + d_5(t) \sum_{j=1}^q \psi_j^*(v(t_j)) + \sum_{j=1}^q p_{1j}\psi_j^*(v(t_j)) \\
&\left. + \sum_{j=k+1}^q p_{2j}\psi_j^*(v(t_j)) + \sum_{j=k+1}^q \psi_j(v(t_j)) + d_6(t) \right|,
\end{aligned}
$$

consequently

$$
\begin{aligned}
&|(\Phi u)(t) - (\Phi v)(t)| \\
&\leq \left| \left(\frac{T^{\beta-1}}{\kappa\Gamma(\beta)} \left(1 - e^{-\kappa T}\right)(1 + \|d_1\|) + \frac{T}{\kappa}\left(Tk + e^{-\kappa T} - 1\right)\|d_2\| + T\|d_3\| \right) L_f \right. \\
&+ (1 + \|d_4\|)\, q\, (L_\psi \varepsilon + M_\psi) + (\|d_6\| + \|d_{1j}\| + \|d_{2j}\|)\, q\, (L_{\psi^*}\varepsilon + M_{\psi^*}))\, \|u - v\| \\
&= \mathcal{L}_\Phi \|u - v\|_{PC},
\end{aligned}
$$

therefore, $|(\Phi u)(t) - (\Phi v)(t)| \leq \mathcal{L}_\Phi \|u - v\|_{PC}$. Φ is a contraction mapping. So, the conclusion follows by the contraction mapping principle. ■

Theorem 9 *Assume that (H_3) and (H_4) are holds. If*

$$
(1 + \|d_4\|)\, q\mathcal{L}_\psi + (\|d_5\| + \|p_{1j}\| + \|p_{2j}\|)\, q\mathcal{L}_{\psi^*} < 1,
$$

then our BVP in (1) has at least one solution on J.

Proof. Choose

$$\varepsilon \geq \Omega := \|\vartheta\|_{L^{\frac{1}{\sigma}}} \left(\frac{1}{\Gamma(\beta)} \frac{T^{\beta-\sigma-1}\left(1-e^{-\kappa T}\right)}{\kappa\left(\frac{\beta-\sigma-1}{1-\sigma}\right)^{1-\sigma}} (1+\|d_1\|) + \frac{T^{\sigma-1}}{\kappa\left(\frac{\sigma-1}{1-\sigma}\right)^{1-\sigma}} \left(Tk+e^{-\kappa T}-1\right)\|d_2\| + \frac{T^{\sigma-1}}{\left(\frac{\sigma-1}{1-\sigma}\right)^{1-\sigma}} \|d_3\| \right),$$

and denote $B_\epsilon = \{u \in PC(J,R), \|u\|_{PC} \leq \varepsilon\}$. The operators Φ_1 and Φ_2 on B_ϵ are Defined as

$$(\Phi_1 u)(t) = \int_0^t M_1(t,m) f(m,u(m))\,dm + d_1(t) \int_0^T M_1(T,m) f(m,u(m))\,dm + d_2(t) \int_0^T M_2(T,m) f(m,u(m))\,dm + d_3(t) \int_0^T z(m)\,dm,$$

and

$$(\Phi_2 u)(t) := d_4(t) \sum_{j=1}^{q} \psi_j(u(t_j)) + d_5(t) \sum_{j=1}^{q} \psi_j^*(u(t_j)) + \sum_{i=1}^{q} p_{1j} \psi_j^*(u(t_j)) + \sum_{j=k+1}^{q} p_{2j} \psi_j^*(u(t_j)) - \sum_{j=k+1}^{q} \psi_j(u(t_j)), t \in J_k.$$

Now, for $u, v \in B_\varepsilon, t \in J$, we have

$$\|\Phi_1 u + \Phi_2 v\|_{PC} \leq \Omega + (1+\|d_4\|)\, q\mathcal{L}_\psi + (\|d_5\| + \|p_{1,j}\| + \|p_{2,j}\|)\, q\mathcal{L}_{\psi^*} = \mathcal{L}_\Phi \|u-v\|_{PC}.$$

The above depended on Hölder inequality. Thus $\Phi_1 u + \Phi_2 v \in B_\epsilon$. It is clear that the operator Φ_2 is a contraction mapping. Moreover, the continuity of f implies operator Φ_1 is continuous and also Φ_1 is uniformly bounded on B_ε since

$$\|\Phi u\|_{PC} \leq \Omega \leq \varepsilon.$$

Now, prove the Φ is equicontinuous. Let $\Upsilon = J \times R$, $f_{\max} = \sup_{(t,x)\in\Upsilon} |f(t,u)|$. For any $t_k < s_1 < s_2 < t_{k+1}$, we have

$$\begin{aligned}
&|(\Phi u)(s_2) - (\Phi u)(s_1)| \\
&\leq f_{\max} \left[\left| \int_0^{s_2} M_1(s_2,m) - M_1(s_1,m)\,dm + \int_{s_2}^{s1} M_1(s_1,m)\,dm \right| \right. \\
&+ |d_1(s_2) - d_1(s_1)| \int_0^T M_1(T,m) f(m,u(m))\,dm \\
&+ |d_2(s_2) - d_2(s_1)| \int_0^T M_2(T,m) f(m,u(m))\,dm \\
&\left. + |d_3(s_2) - d_3(s_1)| \int_0^T f(m,u(m))\,dm \right].
\end{aligned}$$

It tends to zero as $s_1 \to s_2$. This allude that Φ is equicontinuous on the $(t_k, t_{k+1}]$. Φ is compact on B_r by Lemma 4. ■

Theorem 10 *Assume that* (H_5) *and* (H_6) *are holds. Then our BVP in (1) has at least one solution on* J.

Proof. This is made of two main parts: First part, Φ maps bounded sets into bounded sets in $PC(J,R)$. ϵ is a positive number let $B_\epsilon = \{u \in PC(J,R) : \|u\| \leq \varepsilon\}$ in $PC(J,R)$ is a bounded set then

$$
\begin{aligned}
|(\Phi u)(t)| \leq & \left[\frac{1}{\Gamma(\beta)} \frac{T^{\beta-\sigma-1}\left(1-e^{-\kappa T}\right)}{\kappa\left(\frac{\beta-\sigma-1}{1-\sigma}\right)^{1-\sigma}}\left(1+\|d_1\|\right)\right. \\
& + \frac{T^{\sigma-1}}{\kappa\left(\frac{\sigma-1}{1-\sigma}\right)^{1-\sigma}}\left(Tk+e^{-\kappa T}-1\right)\|d_2\| + \frac{T^{\sigma-1}}{\left(\frac{\sigma-1}{1-\sigma}\right)^{1-\sigma}}\|d_3\| \\
& + \left(1+\|d_4\|\right) q\mathcal{L}_\psi + \left(\|d_5\|+\|p_{1j}\|+\|p_{2j}\|\right) q\mathcal{L}_{\psi^*}\big]\,\|\varsigma\|\,\Lambda(\|u\|),
\end{aligned}
$$

which, on taking the norm, for $t \in [0,1]$ yields

$$
\|\Phi u\| \leq \mathcal{L}_\Phi \Delta\left(\|u\|\right)\|\varsigma\|.
$$

Second one: Φ maps bounded sets into equicontinuous sets of $PC(J,R)$. Let $n_1, n_2 \in [0,T]$ with $n_1 < n_2$ and $u \in B_\epsilon$, where B_ϵ is bounded set of $PC(J,R)$. Then we have

$$
\begin{aligned}
& |(\Phi u)(n_2) - (\Phi u)(n_1)| \\
& \leq f_{\max}\left[\left|\int_0^{n_2} M_1(n_2,m) - M_1(n_1,m)\,dm + \int_{n_2}^{n_1} M_1(n_1,m)\,dm\right|\right. \\
& + \left|d_1(n_2) - d_1(n_1)\right| \int_0^T M_1(T,m) f(m,u(m))\,dm \\
& + \left|d_2(n_2) - d_2(n_1)\right| \int_0^T M_2(T,m) f(m,u(m))\,dm \\
& \left. + \left|d_3(n_2) - d_3(n_1)\right| \int_0^T f(m,u(m))\,dm\right].
\end{aligned}
$$

Clearly, the right-hand side of what we did above tend to zero independently of $u \in B_r$ as $n_2 \to n_1 \to 0$. Hence, Φ satisfies (H_5) and (H_6), it follows by Arzela - Ascoli theorem that Φ:$PC(J,R) \to PC(J,R)$ is completely continuous. Now, construct the set $\Lambda = \{u \in PC(J,R) : \|u\| < N\}$, the operator Φ: $\bar{\Lambda} \longrightarrow PC(J,R)$ is continuous and completely continuous , $\Lambda, \nexists u \in \partial\Lambda, u = \varsigma\, \Phi u, \varsigma \in [0,T]$. Therefore, by the nonlinear alternative of Leray-Schauder type. We conclude that Φ has fixed $u \in \bar{\Lambda}$ which is a solution of (BVP). ■

4 Example

Example. Consider the problem

$$
{}^cD^{\frac{2}{3}}(D+2\ )u(t)=\left(\sqrt{t+2}+\sin t+\tan^{-1}u(t)\right),1<\frac{5}{3}\leq 2,0<t<1, \quad (23)
$$
$$
u(0)+^c D^{\frac{3}{2}}u(0)=0,u(1)+^c D^{\frac{3}{2}}(1)=0,
$$
$$
\Delta u(t)\left|_{t=t_k}=\frac{\left|u\left(\frac{1}{2}\right)\right|}{5+\left|u\left(\frac{1}{2}\right)\right|},\Delta u'(t)\right|_{t=t_k}=\frac{\left|u\left(\frac{1}{2}\right)\right|}{10+\left|u\left(\frac{1}{2}\right)\right|},
$$

Here$t\in[0,1],\beta=\frac{5}{3},\lambda=2,T=1,a=1,b=1,w_0=0,w_1=0,\mathcal{L}_\psi=0.02,\mathcal{L}_{\psi}^{*}=0.02,\ f(t,u(t))=\left(\sqrt{t+2}+\sin t+\tan^{-1}u(t)\right)$ and

$$
|f(t,u)-f(t,v)|\leq\left|u-v+tan^{-1}u-tan^{-1}v\right|\leq\mathcal{L}_T|u-v|.
$$

With the given values, we find that

$$
\mathcal{L}_\Phi=\left(\frac{1}{2\,(0.902)}\,(1-0.135)\,(1+\|0.471\|)+\frac{1}{2}\,(1-0.135)\,\|1.277\|+\|0.471\|\right)0.02
$$
$$
+\,(1+\|1.652\|)\,0.02+(\|0.826\|+\|6.054\|+\|0.002\|)\,0.02+\|0\|
$$
$$
=0.224<1.
$$

Therefore, by Theorem 8, impulsive sequential fractional differential equation (23) has a unique solutionon $[0,1]$.

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