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Uniform Existence and Uniqueness for a Time-Dependent Ginzburg-Landau Model for Superconductivity

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Abstract

We study the initial boundary value problem for a time-dependent Ginzburg-Landau model of superconductivity. First, we prove the uniform boundedness of strong solutions with respect to diffusion parameter $\epsilon > 0$ in the case of Coulomb gauge for 2D case. Our second result is the uniqueness of axially symmetric weak solutions in 3D with L^2 initial data under Lorentz gauge.

Mathematics Subject Classifications: 35K55

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1 Introduction

This paper is mainly concerned with strong solutions to the following Ginzburg-Landau model for superconductivity:

$$\eta\psi_t + i\eta k\phi\psi + \left(\frac{i}{k}\nabla + A\right)^2 \psi + (|\psi|^2 - 1)\psi = 0, \tag{1.1}$$

$$A_t + \nabla\phi + \epsilon\operatorname{curl}^2 A + \operatorname{Re}\left\{\left(\frac{i}{k}\nabla\psi + \psi A\right)\overline{\psi}\right\} = \operatorname{curl} H \tag{1.2}$$

in $Q_T := (0,T)\times\Omega$, with boundary and initial conditions

$$\nabla\psi\cdot\nu = 0, \quad A\cdot\nu = 0, \quad \epsilon\operatorname{curl} A = H \quad \text{on} \quad (0,T)\times\partial\Omega, \tag{1.3}$$

$$(\psi, A)(x,0) = (\psi_0, A_0)(x) \quad \text{in} \quad \Omega. \tag{1.4}$$

Here $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the outward normal to $\partial\Omega$, and T is any given positive constant. The unknowns ψ, A, and ϕ are $\mathbb{C}$-valued, $\mathbb{R}^d$-valued, and $\mathbb{R}$-valued functions, respectively, and they stand for the order parameter, the magnetic potential, and the electric potential, respectively. $H := H(t,x)$ is the applied magnetic field, η and k are Ginzburg-Landau positive constants. $\overline{\psi}$ denotes the complex conjugate of $\psi, \operatorname{Re}\psi := (\psi+\overline{\psi})/2, |\psi|^2 := \psi\overline{\psi}$ is the density of superconducting carriers, $\epsilon > 0$ is a positive constant, and $i := \sqrt{-1}$.

It is well known that the Ginzburg-Landau equations are gauge invariant, namely if (ψ, A, ϕ) is a solution of (1.1)-(1.4), then for any real-valued smooth function $\chi, (\psi e^{ik\chi}, A+\nabla\chi, \phi-\chi_t)$ is also a solution of (1.1)-(1.4). So, in order to obtain the well-posedness of the problem, we need to impose suitable gauge condition. From the physical point of view, one usually has three types of the gauge conditions:

- Coulomb gauge: $\operatorname{div} A = 0$ in Ω and $\int_\Omega \phi dx = 0$.
- Lorentz gauge: $\phi = -\operatorname{div} A$ in Ω.
- Temporal gauge: $\phi = 0$ in Ω.

In the standard physical models, ϵ should be positive and there are many studies treating this case. For the initial data $\psi_0 \in H^1(\Omega), |\psi_0| \le 1, A_0 \in H^1(\Omega)$, Chen, Elliott and Tang [1], Chen, Hoffmann and Liang [3], Du [4] and Tang [12] proved the existence and uniqueness of global strong solutions to (1.1)-(1.4) in the case of the Coulomb and Lorentz as well as temporal gauges. For the initial data $\psi_0 \in H^1(\Omega), A_0 \in H^1(\Omega)$, Tang and Wang [13] obtained the existence and uniqueness of global strong solutions, while Fan and Jiang [7] showed the existence of global weak solutions when $\psi_0, A_0 \in L^2$. Fan and

Gao [5], Fan and Ozawa [8, 9, 10] established some uniqueness criteria on weak solutions. Zaouch [14] proved the existence of time-periodic solutions when the applied magnetic field H is time periodic. Phillips and Shin [11], Chen and Hoffmann [2] studied the well-posedness of classical solutions to the nonisothermal models for superconductivity.

When $\epsilon > 0$, the equation (1.2) has some parabolic nature and the dominant linear structure is given by the operator $\partial_t + \epsilon \mathrm{curl}^2$. By the standard energy estimates, it is easy to control $\|A\|_{H^1}$ by (1.2) only. In the physical models where the effect of the magnetic field itself curl A may be negligible as compared to other physical quantities, we may assume that $\epsilon = 0$. In this case, (1.2) loses its dissipative effect that guarantees the global existence of strong solutions.

The aim of this paper is to prove uniform boundedness of strong solutions in ϵ. For simplicity we take $H = 0$, and one of the main results in this paper reads as

Theorem 1.1. *Let $d = 2$. Let $\psi_0 \in H^2(\Omega), A_0 \in W^{1,q}(\Omega)$ for some $q > 2$ and $|\psi_0| \leq 1, \mathrm{div}\, A_0 = 0$ in Ω. Then for any $T > 0$, there exist unique strong solutions $(\psi_\epsilon, A_\epsilon, \phi_\epsilon)$ of (1.1)-(1.4) in the case of the Coulomb gauge, such that*

$$\begin{aligned}
&\psi_\epsilon \in L^\infty(0,T;H^2) \cap L^2(0,T;H^3), \quad \partial_t \psi_\epsilon \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1),\\
&A_\epsilon \in L^\infty(0,T;W^{1,q}(\Omega)), \quad \partial_t A_\epsilon \in L^2(0,T;L^2),\\
&\phi_\epsilon \in L^\infty(0,T;H^2) \cap L^2(0,T;H^3), \quad \partial_t \phi_\epsilon \in L^2(0,T;L^2),
\end{aligned} \tag{1.5}$$

with the corresponding norms that are uniformly bounded with respect to $\epsilon > 0$.

Recently, Fan and Gao [5], Fan and Ozawa [8, 9, 10] proved a conditional uniqueness result when Ω is a bounded domain or $\Omega = \mathbb{R}^d$, respectively.

Proposition 1.2. *([5, 8, 9, 10]). Let $\psi_0, A_0 \in L^2$. Assume that*

$$\psi, A \in L^r(0,T;L^p(\Omega)) \ \text{ with } \ \frac{2}{r} + \frac{d}{p} = 1, \ d < p \leq \infty. \tag{1.6}$$

Then there exists at most one weak solution (ψ, A) to the problem (1.1)-(1.4) in $\Omega \times (0,T) \subset \mathbb{R}^d \times (0,T)$ satisfying $\psi, A \in V_2(Q_T) := L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ with the Lorentz or Coulomb gauge.

In this paper, we will also be interested in axially symmetric weak solutions in 3D to (1.1)-(1.4) in the case of the Lorentz gauge. We study the uniqueness of solutions to (1.1)-(1.2) of the form

$$\psi(t,x) := \psi(t,r,z), \ A(t,x) := A_r e_r + A_\theta e_\theta + A_z e_z$$

with

$$e_r := \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right)^t, e_\theta := \left(\frac{x_2}{r}, -\frac{x_1}{r}, 0\right)^t, e_z := (0,0,1)^t$$

and $r := \sqrt{x_1^2 + x_2^2}$.

Let $\Omega := B_R(0) \times [0, h]$ and $B_R(0) := \{(x_1, x_2) : \ r^2 = x_1^2 + x_2^2 \le R^2\}$.

By straightforward calculations, we obtain

$$\begin{aligned}
\nabla \psi &= \left(\frac{\partial \psi}{\partial r}\frac{x_1}{r}, \frac{\partial \psi}{\partial r}\frac{x_2}{r}, \frac{\partial \psi}{\partial z} \right)^t, \\
\Delta \psi &= \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2}, \\
\operatorname{div} A &= \frac{1}{r}A_r + \frac{\partial A_r}{\partial r} + \frac{\partial A_z}{\partial z},
\end{aligned}$$

and we see that $\psi(t, r, z), A_r(t, r, z), A_\theta(t, r, z), A_z(t, r, z)$ satisfy the following system:

$$\eta \partial_t \psi + i \left(\frac{1}{k} - \eta k \right) \psi \left(\frac{\partial A_r}{\partial r} + \frac{1}{r}A_r + \frac{\partial A_z}{\partial z} \right) + \frac{2i}{k} \left(\frac{\partial \psi}{\partial r}A_r + \frac{\partial \psi}{\partial z}A_z \right)$$
$$- \frac{1}{k^2} \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} \right) + (A_r^2 + A_\theta^2 + A_z^2)\psi + (|\psi|^2 - 1)\psi = 0, \tag{1.7}$$
$$\partial_t A_r - \left(-\frac{1}{r^2}A_r + \frac{1}{r}\frac{\partial A_r}{\partial r} + \frac{\partial^2 A_r}{\partial r^2} + \frac{\partial^2 A_r}{\partial z^2} \right) + \mathrm{Re} \left(\frac{i}{k}\frac{\partial \psi}{\partial r}\overline{\psi} \right) + |\psi|^2 A_r = 0, \tag{1.8}$$
$$\partial_t A_\theta - \left(-\frac{1}{r^2}A_\theta + \frac{1}{r}\frac{\partial A_\theta}{\partial r} + \frac{\partial^2 A_\theta}{\partial r^2} + \frac{\partial^2 A_\theta}{\partial z^2} \right) + |\psi|^2 A_\theta = 0, \tag{1.9}$$
$$\partial_t A_z - \left(\frac{\partial^2 A_z}{\partial r^2} + \frac{1}{r}\frac{\partial A_z}{\partial r} + \frac{\partial^2 A_z}{\partial z^2} \right) + \mathrm{Re} \left(\frac{i}{k}\frac{\partial \psi}{\partial z}\overline{\psi} \right) + |\psi|^2 A_z = 0, \tag{1.10}$$
$$\left(\frac{\partial \psi}{\partial r}, \frac{\partial \psi}{\partial z} \right)\Big|_{\partial\Omega \times (0,T)} = 0, \tag{1.11}$$
$$\left(\frac{\partial A_r}{\partial z}, \frac{\partial A_\theta}{\partial z}, A_z \right)\Big|_{z=0,h} = 0, \quad \left(A_r, A_\theta + \frac{\partial A_\theta}{\partial r}, \frac{\partial A_z}{\partial r} \right)\Big|_{r=0,R} = 0, \tag{1.12}$$
$$(\psi, A_r, A_\theta, A_z)(\cdot, 0) = (\psi^0, A_r^0, A_\theta^0, A_z^0)(\cdot). \tag{1.13}$$

We will prove

Theorem 1.3. *Let $\psi^0, A_r^0, A_\theta^0, A_z^0 \in L^2$. Then there exists at most one weak solution $(\psi, A_r, A_\theta, A_z)$ to the problem (1.7)-(1.13) satisfying $\psi, A_r, A_\theta, A_z \in V_2(Q_T)$.*

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Since it has been proved that the problem (1.1)-(1.4) has a unique global-in-time smooth solution [1, 3, 4, 12], we only need to prove a priori estimates (1.5) uniformly in ϵ. From now on, we drop the subscript ϵ.

To begin with, it is easy to show that [1, 3, 4, 12]:

$$|\psi| \le 1 \quad \text{in} \quad Q_T. \tag{2.1}$$

Testing (1.1) by $\overline{\psi}$, taking the real part and using (2.1), we see that

$$\frac{1}{2}\eta\frac{d}{dt}\int|\psi|^2dx+\int\left|\frac{i}{k}\nabla\psi+\psi A\right|^2dx+\int|\psi|^4dx=\int|\psi|^2dx\le|\Omega|.$$

Integrating the above inequality in $(0,T)$, we get

$$\int_0^T\int\left|\frac{i}{k}\nabla\psi+\psi A\right|^2dxdt\le C. \tag{2.2}$$

Here and later on, C will denote a generic positive constant independent of $\epsilon>0$.

Testing (1.2) by A and using (2.1), (2.2) and $\operatorname{div}A=0$, we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int A^2dx+\epsilon\int|\operatorname{curl}A|^2dx &= -\operatorname{Re}\int\left(\frac{i}{k}\nabla\psi+\psi A\right)\overline{\psi}Adx\\ &\le \int\left|\frac{i}{k}\nabla\psi+\psi A\right||A|dx\\ &\le \left\|\frac{i}{k}\nabla\psi+\psi A\right\|_{L^2}\|A\|_{L^2},\end{aligned}$$

which yields

$$\|A\|_{L^\infty(0,T;L^2)}\le C. \tag{2.3}$$

Inequalities (2.1), (2.2) and (2.3) imply

$$\|\psi\|_{L^2(0,T;H^1)}\le C. \tag{2.4}$$

Taking div to (1.2), it is easy to infer that

$$-\Delta\phi=\operatorname{div}\operatorname{Re}\left\{\left(\frac{i}{k}\nabla\psi+\psi A\right)\overline{\psi}\right\} \quad \text{in} \quad Q_T, \tag{2.5}$$

$$\nabla\phi\cdot\nu=0 \quad \text{on} \quad (0,T)\times\partial\Omega. \tag{2.6}$$

Testing (2.5) by ϕ and using (2.1) and (2.2), we have

$$\|\nabla\phi\|_{L^2(0,T;L^2)}\le C. \tag{2.7}$$

Testing (1.1) by $-\Delta\overline{\psi}$, taking the real part and using (2.1) and (2.3), we deduce that

$$\begin{aligned}
&\frac{\eta}{2}\frac{d}{dt}\int|\nabla\psi|^2dx+\frac{1}{k^2}\int|\Delta\psi|^2dx\\
\leq\ &\eta k\int|\phi||\Delta\psi|dx+\frac{2}{k}\int|A||\nabla\psi||\Delta\psi|dx\\
&+\int|A|^2|\Delta\psi|dx+\int(|\psi|^2-1)|\psi||\Delta\psi|dx\\
\leq\ &C\|\phi\|_{L^2}\|\Delta\psi\|_{L^2}+C\|A\|_{L^4}\|\nabla\psi\|_{L^4}\|\Delta\psi\|_{L^2}\\
&+C\|A\|_{L^4}^2\|\Delta\psi\|_{L^2}+C\|\Delta\psi\|_{L^2}\\
\leq\ &\frac{1}{16}\frac{1}{k^2}\|\Delta\psi\|_{L^2}^2+C\|\phi\|_{L^2}^2+C\|A\|_{L^4}^4+C\\
\leq\ &\frac{1}{16}\frac{1}{k^2}\|\Delta\psi\|_{L^2}^2+C\|\nabla\phi\|_{L^2}^2+C\|\operatorname{curl}A\|_{L^2}^2+C. \qquad (2.8)
\end{aligned}$$

Here we have used the Gagliardo-Nirenberg inequalities

$$\|\nabla\psi\|_{L^4}^2\leq C\|\psi\|_{L^\infty}\|\Delta\psi\|_{L^2}, \qquad (2.9)$$

$$\|A\|_{L^4}^2\leq C\|A\|_{L^2}\|\operatorname{curl}A\|_{L^2}. \qquad (2.10)$$

Testing (1.2) by curl^2A and using (2.1), (2.3), (2.9) and (2.10), we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\operatorname{curl}A|^2dx+\epsilon\int|\operatorname{curl}^2A|^2dx\\
\leq\ &C(\|\nabla\psi\|_{L^4}^2+\|A\|_{L^4}\|\nabla\psi\|_{L^4})\|\operatorname{curl}A\|_{L^2}+C\|\operatorname{curl}A\|_{L^2}^2\\
\leq\ &\frac{1}{16}\frac{1}{k^2}\|\Delta\psi\|_{L^2}^2+C\|\operatorname{curl}A\|_{L^2}^2. \qquad (2.11)
\end{aligned}$$

Combining (2.8) and (2.11) and using the Gronwall inequality, we have

$$\|\psi\|_{L^\infty(0,T;H^1)}+\|\psi\|_{L^2(0,T;H^2)}\leq C, \qquad (2.12)$$

$$\|A\|_{L^\infty(0,T;H^1)}\leq C. \qquad (2.13)$$

It follows from (1.1), (2.1), (2.12) and (2.13) that

$$\|\psi_t\|_{L^2(0,T;L^2)}\leq C, \qquad (2.14)$$

$$\|\nabla\phi\|_{L^\infty(0,T;L^2)}\leq C, \qquad (2.15)$$

$$\|A_t\|_{L^2(0,T;L^2)}\leq C. \qquad (2.16)$$

Taking ∂_t to (1.1), testing then by $\overline{\psi}_t$, taking the real part, and using (2.1), (2.13), (2.15) and (2.16), we have

$$\begin{aligned}
&\frac{\eta}{2}\frac{d}{dt}\int|\psi_t|^2dx+\frac{1}{k^2}\int|\nabla\psi_t|^2dx+\int A^2|\psi_t|^2dx\\
\le\ &\eta k\int|\phi_t||\psi_t|dx+\frac{2}{k}\int|A_t||\nabla\psi||\psi_t|dx\\
&+\frac{2}{k}\int|A||\nabla\psi_t||\psi_t|dx+2\int|A||A_t||\psi_t|dx+C\int|\psi_t|^2dx\\
\le\ &C\|\phi_t\|_{L^2}\|\psi_t\|_{L^2}+C\|A_t\|_{L^2}\|\nabla\psi\|_{L^3}\|\psi_t\|_{L^6}\\
&+C\|A\|_{L^4}\|\nabla\psi_t\|_{L^2}\|\psi_t\|_{L^4}+2\|A\|_{L^4}\|A_t\|_{L^2}\|\psi_t\|_{L^4}+C\|\psi_t\|_{L^2}^2\\
\le\ &C(\|\nabla\psi_t\|_{L^2}+\|A_t\|_{L^2}+\|\nabla\psi\|_{L^3}\|\psi_t\|_{L^6}+\|A\|_{L^4}\|\psi_t\|_{L^4})\|\psi_t\|_{L^2}\\
&+C\|A_t\|_{L^2}\|\nabla\psi\|_{L^3}\|\psi_t\|_{L^6}+C\|\nabla\psi_t\|_{L^2}\|\psi_t\|_{L^4}+C\|A_t\|_{L^2}\|\psi_t\|_{L^4}+C\|\psi_t\|_{L^2}^2\\
\le\ &\frac{1}{16}\frac{1}{k^2}\|\nabla\psi_t\|_{L^2}^2+C\|\psi_t\|_{L^2}^2+C\|A_t\|_{L^2}^2+C\\
&+C\|\nabla\psi\|_{L^3}^2\|\psi_t\|_{L^2}^2+C\|\nabla\psi\|_{L^3}\|A_t\|_{L^2}^2. \qquad (2.17)
\end{aligned}$$

Taking ∂_t to (1.2), testing then by A_t, and using (2.1), (2.12), (2.13) and (2.10), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|A_t|^2dx+\epsilon\int|\mathrm{curl}\,A_t|^2dx+\int|\psi|^2|A_t|^2dx\\
\le\ &C\int(|\nabla\psi_t|+|\nabla\psi||\psi_t|+|A||\psi_t|)|A_t|dx\\
\le\ &C(\|\nabla\psi_t\|_{L^2}+\|\nabla\psi\|_{L^4}\|\psi_t\|_{L^4}+\|A\|_{L^4}\|\psi_t\|_{L^4})\|A_t\|_{L^2}\\
\le\ &\frac{1}{16}\frac{1}{k^2}\|\nabla\psi_t\|_{L^2}^2+C\|\nabla\psi\|_{L^4}^2\|A_t\|_{L^2}^2+C\|\psi_t\|_{L^2}^2. \qquad (2.18)
\end{aligned}$$

Combining (2.17) and (2.18) and using (2.14), (2.16) and the Gronwall inequality, we conclude

$$\|\psi_t\|_{L^\infty(0,T;L^2)}+\|\psi_t\|_{L^2(0,T;H^1)}\le C, \qquad (2.19)$$

$$\|A_t\|_{L^\infty(0,T;L^2)}\le C. \qquad (2.20)$$

Taking curl to (1.2), testing then by $|\mathrm{curl}\,A|^{q-2}\mathrm{curl}\,A$, and using (2.1), (2.12) and (2.13), we have

$$\frac{d}{dt}\int|\mathrm{curl}\,A|^qdx\le C(\|\nabla\psi\|_{L^{2q}}^2+\|\mathrm{curl}\,A\|_{L^q}+\|A\|_{L^{2q}}\|\nabla\psi\|_{L^{2q}})\|\mathrm{curl}\,A\|_{L^q}^{q-1},$$

and therefore

$$\frac{d}{dt}\|\mathrm{curl}\,A\|_{L^q}\le C(\|\nabla\psi\|_{L^{2q}}^2+\|\mathrm{curl}\,A\|_{L^q}+1),$$

which gives

$$\|A\|_{L^\infty(0,T;W^{1,q})} \le C. \tag{2.21}$$

Here we have used the well-known fact:

$$\|A\|_{W^{1,q}} \le C(\|\mathrm{div}\, A\|_{L^q} + \|\mathrm{curl}\, A\|_{L^q})$$

with $A \cdot \nu = 0$ on $\partial\Omega$.

On the other hand, it follows from (1.1), (2.1), (2.12), (2.15) and (2.19) that

$$\|\psi\|_{L^\infty(0,T;H^2)} + \|\psi\|_{L^2(0,T;H^3)} \le C, \tag{2.22}$$

while from (2.5), (2.6), (2.22) and (2.13) we get

$$\|\phi\|_{L^\infty(0,T;H^2)} + \|\phi\|_{L^2(0,T;H^3)} \le C,$$
$$\|\phi_t\|_{L^2(0,T;L^2)} \le C.$$

This completes the proof. □

3 Proof of Theorem 1.3

In this section, we will use Theorem 1.2 to prove Theorem 1.3. In fact, one only needs to prove that

$$\psi, A_r, A_\theta, A_z \in V_2(Q_T). \tag{3.1}$$

Note that

$$V_2(Q_T) \subset L^4(0,T;L^4(\Omega))$$

which satisfies the condition (1.6) for $r = p = 4$ and $d = 2$, where Ω denotes

$$\Omega := \{(r,z)|\ 0 \le r \le R,\ 0 \le z \le h\}.$$

Testing (1.7) by $\overline{\psi}$ and taking the real part, we see that

$$\frac{\eta}{2}\frac{d}{dt}\int_0^R\int_0^h |\psi|^2 drdz + \int_0^R\int_0^h \left(\left|\frac{i}{k}\frac{\partial\psi}{\partial r} + \psi A_r\right|^2 + \left|\frac{i}{k}\frac{\partial\psi}{\partial z} + \psi A_z\right|^2\right) drdz$$
$$+\frac{1}{2k^2}\int_0^R\int_0^h \frac{|\psi|^2}{r^2} drdz + \int_0^R\int_0^h A_\theta^2|\psi|^2 drdz + \int_0^R\int_0^h (|\psi|^2-1)^2 drdz$$
$$+\int_0^R\int_0^h |\psi|^2 drdz = \int_0^R\int_0^h drdz = Rh,$$

which gives

$$\int_0^R\int_0^h |\psi|^2 drdz + \int_0^T\int_0^R\int_0^h \left(\left|\frac{i}{k}\frac{\partial\psi}{\partial r}+\psi A_r\right|^2 + \left|\frac{i}{k}\frac{\partial\psi}{\partial z}+\psi A_z\right|^2\right) drdzdt$$
$$+\int_0^T\int_0^R\int_0^h |\psi|^4 drdzdt \leq C. \tag{3.2}$$

Testing (1.8) by A_r and using (3.2) and the Gagliardo-Nirenberg inequality, we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^R\int_0^h A_r^2 drdz + \int_0^R\int_0^h \left(\left|\frac{\partial A_r}{\partial r}\right|^2 + \left|\frac{\partial A_r}{\partial z}\right|^2\right) drdz + \int_0^R\int_0^h \frac{A_r^2}{2r^2} drdz\\
\leq\ & \int_0^R\int_0^h \left|\frac{i}{k}\frac{\partial\psi}{\partial r}+\psi A_r\right| |\psi||A_r| drdz\\
\leq\ & \int_0^R\int_0^h \left|\frac{i}{k}\frac{\partial\psi}{\partial r}+\psi A_r\right|^2 drdz + C\|\psi\|_{L^4}^2\|A_r\|_{L^4}^2\\
\leq\ & \int_0^R\int_0^h \left|\frac{i}{k}\frac{\partial\psi}{\partial r}+\psi A_r\right|^2 drdz + C\|\psi\|_{L^4}^2\|A_r\|_{L^2}\left\|\left|\frac{\partial A_r}{\partial r}\right| + \left|\frac{\partial A_r}{\partial z}\right|\right\|_{L^2}\\
\leq\ & \int_0^R\int_0^h \left|\frac{i}{k}\frac{\partial\psi}{\partial r}+\psi A_r\right|^2 drdz + \frac{1}{2}\int_0^R\int_0^h \left(\left|\frac{\partial A_r}{\partial r}\right|^2 + \left|\frac{\partial A_r}{\partial z}\right|^2\right) drdz + C\|\psi\|_{L^4}^4\|A_r\|_{L^2}^2,
\end{aligned}$$

which gives

$$A_r \in V_2(Q_T). \tag{3.3}$$

Similarly,

$$A_\theta, A_z \in V_2(Q_T). \tag{3.4}$$

Note that (3.3) and (3.4) yield

$$A_r, A_z \in L^4(Q_T), \tag{3.5}$$

while (3.2) and (3.5) lead to

$$\psi \in V_2(Q_T).$$

This completes the proof.

□

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