Uniform Existence and Uniqueness for a Time-Dependent Ginzburg-Landau Model for Superconductivity

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Abstract

We study the initial boundary value problem for a time-dependent Ginzburg-Landau model of superconductivity. First, we prove the uniform boundedness of strong solutions with respect to diffusion parameter \( \epsilon > 0 \) in the case of Coulomb gauge for 2D case. Our second result is the uniqueness of axially symmetric weak solutions in 3D with \( L^2 \) initial data under Lorentz gauge.

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1 Introduction

This paper is mainly concerned with strong solutions to the following Ginzburg-Landau model for superconductivity:

\[
\eta \psi_t + i \eta k \phi \psi + \left( \frac{i}{k} \nabla + A \right)^2 \psi + (|\psi|^2 - 1)\psi = 0, \tag{1.1}
\]

\[
A_t + \nabla \phi + \epsilon \text{curl}^2 A + \text{Re} \left\{ \left( \frac{i}{k} \nabla \psi + \psi A \right) \overline{\psi} \right\} = \text{curl} H \tag{1.2}
\]

in \(Q_T := (0, T) \times \Omega\), with boundary and initial conditions

\[
\nabla \psi \cdot \nu = 0, \quad A \cdot \nu = 0, \quad \epsilon \text{curl} A = H \quad \text{on} \quad (0, T) \times \partial \Omega, \tag{1.3}
\]

\[
(\psi, A)(x, 0) = (\psi_0, A_0)(x) \quad \text{in} \quad \Omega. \tag{1.4}
\]

Here \(\Omega \subset \mathbb{R}^d\) is a bounded domain with smooth boundary \(\partial \Omega\), \(\nu\) is the outward normal to \(\partial \Omega\), and \(T\) is any given positive constant. The unknowns \(\psi, A,\) and \(\phi\) are \(\mathbb{C}\)-valued, \(\mathbb{R}^d\)-valued, and \(\mathbb{R}\)-valued functions, respectively, and they stand for the order parameter, the magnetic potential, and the electric potential, respectively. \(H := H(t, x)\) is the applied magnetic field, \(\eta\) and \(k\) are Ginzburg-Landau positive constants. \(\overline{\psi}\) denotes the complex conjugate of \(\psi\), \(\text{Re} \psi := (\psi + \overline{\psi})/2\), \(|\psi|^2 := \psi \overline{\psi}\) is the density of superconducting carriers, \(\epsilon > 0\) is a positive constant, and \(i := \sqrt{-1}\).

It is well known that the Ginzburg-Landau equations are gauge invariant, namely if \((\psi, A, \phi)\) is a solution of (1.1)-(1.4), then for any real-valued smooth function \(\chi, (\psi e^{ik\chi}, A + \nabla \chi, \phi - \chi_t)\) is also a solution of (1.1)-(1.4). So, in order to obtain the well-posedness of the problem, we need to impose suitable gauge condition. From the physical point of view, one usually has three types of the gauge conditions:

- **Coulomb gauge**: \(\text{div} A = 0\) in \(\Omega\) and \(\int_\Omega \phi dx = 0\).

- **Lorentz gauge**: \(\phi = -\text{div} A\) in \(\Omega\).

- **Temporal gauge**: \(\phi = 0\) in \(\Omega\).

In the standard physical models, \(\epsilon\) should be positive and there are many studies treating this case. For the initial data \(\psi_0 \in H^1(\Omega), |\psi_0| \leq 1, A_0 \in H^1(\Omega),\) Chen, Elliott and Tang [1], Chen, Hoffmann and Liang [3], Du [4] and Tang [12] proved the existence and uniqueness of global strong solutions to (1.1)-(1.4) in the case of the Coulomb and Lorentz as well as temporal gauges. For the initial data \(\psi_0 \in H^1(\Omega), A_0 \in H^1(\Omega),\) Tang and Wang [13] obtained the existence and uniqueness of global strong solutions, while Fan and Jiang [7] showed the existence of global weak solutions when \(\psi_0, A_0 \in L^2\). Fan and

When $\epsilon > 0$, the equation (1.2) has some parabolic nature and the dominant linear structure is given by the operator $\partial_t + \epsilon \text{curl}^2$. By the standard energy estimates, it is easy to control $\|A\|_{H^1}$ by (1.2) only. In the physical models where the effect of the magnetic field itself curl $A$ may be negligible as compared to other physical quantities, we may assume that $\epsilon = 0$. In this case, (1.2) loses its dissipative effect that guarantees the global existence of strong solutions.

The aim of this paper is to prove uniform boundedness of strong solutions in $\epsilon$. For simplicity we take $H = 0$, and one of the main results in this paper reads as

**Theorem 1.1.** Let $d = 2$. Let $\psi_0 \in H^2(\Omega), A_0 \in W^{1,q}(\Omega)$ for some $q > 2$ and $|\psi_0| \leq 1, \text{div } A_0 = 0$ in $\Omega$. Then for any $T > 0$, there exist unique strong solutions $(\psi, A, \phi)$ of (1.1)-(1.4) in the case of the Coulomb gauge, such that

$$
\psi, A \in L^\infty(0, T; H^2) \cap L^2(0, T; H^3), \quad \psi, \partial_t \psi, A, \partial_t A \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1),
$$

$$
\phi \in L^\infty(0, T; H^2) \cap L^2(0, T; H^3), \quad \phi, \partial_t \phi \in L^2(0, T; L^2),
$$

(1.5)

with the corresponding norms that are uniformly bounded with respect to $\epsilon > 0$.

Recently, Fan and Gao [5], Fan and Ozawa [8, 9, 10] proved a conditional uniqueness result when $\Omega$ is a bounded domain or $\Omega = \mathbb{R}^d$, respectively.

**Proposition 1.2. ([5, 8, 9, 10]).** Let $\psi_0, A_0 \in L^2$. Assume that

$$
\psi, A \in L^r(0, T; L^p(\Omega)) \quad \text{with} \quad \frac{2}{r} + \frac{d}{p} = 1, \quad d < p \leq \infty.
$$

(1.6)

Then there exists at most one weak solution $(\psi, A)$ to the problem (1.1)-(1.4) in $\Omega \times (0, T) \subset \mathbb{R}^d \times (0, T)$ satisfying $\psi, A \in V_2(Q_T) := L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ with the Lorentz or Coulomb gauge.

In this paper, we will also be interested in axially symmetric weak solutions in 3D to (1.1)-(1.4) in the case of the Lorentz gauge. We study the uniqueness of solutions to (1.1)-(1.2) of the form

$$
\psi(t, x) := \psi(t, r, z), \quad A(t, x) := A_r e_r + A_\theta e_\theta + A_z e_z
$$

with

$$
e_r := \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right)^t, \quad e_\theta := \left(\frac{x_2}{r}, -\frac{x_1}{r}, 0\right)^t, \quad e_z := (0, 0, 1)^t
$$
and \( r := \sqrt{x_1^2 + x_2^2} \).

Let \( \Omega := B_R(0) \times [0, h] \) and \( B_R(0) := \{ (x_1, x_2) : \ r^2 = x_1^2 + x_2^2 \leq R^2 \} \).

By straightforward calculations, we obtain
\[
\nabla \psi = \left( \frac{\partial \psi}{\partial r}, \frac{x_1}{r} \frac{\partial \psi}{\partial r}, \frac{x_2}{r} \frac{\partial \psi}{\partial z} \right)^t,
\]
\[
\Delta \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2},
\]
\[
\text{div} A = \frac{1}{r} A_r + \frac{\partial A_r}{\partial r} + \frac{\partial A_z}{\partial z},
\]
and we see that \( \psi(t, r, z), A_r(t, r, z), A_\theta(t, r, z), A_z(t, r, z) \) satisfy the following system:

\[
\eta \partial_t \psi + i \left( \frac{1}{k} - \eta k \right) \psi \left( \frac{\partial A_r}{\partial r} + \frac{1}{r} A_r + \frac{\partial A_z}{\partial z} \right) + 2i \frac{k}{k} \left( \frac{\partial \psi}{\partial r} A_r + \frac{\partial \psi}{\partial z} A_z \right)
\]
\[
- \frac{1}{k^2} \left( \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} \right) + \left( A_r^2 + A_\theta^2 + A_z^2 \right) \psi + (|\psi|^2 - 1) \psi = 0, \quad (1.7)
\]
\[
\partial_t A_r = \left( -\frac{1}{r^2} A_r + \frac{1}{r} \frac{\partial A_r}{\partial r} + \frac{\partial^2 A_r}{\partial r^2} + \frac{\partial^2 A_r}{\partial z^2} \right) + \text{Re} \left( \frac{i}{k} \frac{\partial \psi}{\partial r} \psi \right) + |\psi|^2 A_r = 0, \quad (1.8)
\]
\[
\partial_t A_\theta = \left( -\frac{1}{r^2} A_\theta + \frac{1}{r} \frac{\partial A_\theta}{\partial r} + \frac{\partial^2 A_\theta}{\partial r^2} + \frac{\partial^2 A_\theta}{\partial z^2} \right) + |\psi|^2 A_\theta = 0, \quad (1.9)
\]
\[
\partial_t A_z = \left( \frac{\partial^2 A_z}{\partial r^2} + \frac{1}{r^2} \frac{\partial A_z}{\partial r} + \frac{\partial^2 A_z}{\partial z^2} \right) + \text{Re} \left( \frac{i}{k} \frac{\partial \psi}{\partial z} \psi \right) + |\psi|^2 A_z = 0, \quad (1.10)
\]
\[
\left. \left( \frac{\partial \psi}{\partial r}, \frac{\partial \psi}{\partial z} \right) \right|_{\partial \Omega \times (0, T)} = 0, \quad (1.11)
\]
\[
\left. \left( \frac{\partial A_r}{\partial z}, \frac{\partial A_\theta}{\partial z}, A_z \right) \right|_{z=0, h} = 0, \quad \left. \left( A_r, A_\theta + \frac{\partial A_\theta}{\partial r}, \frac{\partial A_z}{\partial r} \right) \right|_{r=0, R} = 0, \quad (1.12)
\]
\[
(\psi, A_r, A_\theta, A_z)(\cdot, 0) = (\psi^0, A_r^0, A_\theta^0, A_z^0)(\cdot). \quad (1.13)
\]

We will prove

**Theorem 1.3.** Let \( \psi^0, A_r^0, A_\theta^0, A_z^0 \in L^2 \). Then there exists at most one weak solution \( (\psi, A_r, A_\theta, A_z) \) to the problem (1.7)-(1.13) satisfying \( \psi, A_r, A_\theta, A_z \in V_2(Q_T) \).

## 2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Since it has been proved that the problem (1.1)-(1.4) has a unique global-in-time smooth solution \([1, 3,

To begin with, it is easy to show that $[1, 3, 4, 12]$:

$$|\psi| \leq 1 \text{ in } Q_T. \quad (2.1)$$

Testing (1.1) by $\overline{\psi}$, taking the real part and using (2.1), we see that

$$\frac{1}{2} \eta \frac{d}{dt} \int |\psi|^2 dx + \int \left| \frac{i}{k} \nabla \psi + \psi A \right|^2 dx + \int |\psi|^4 dx = \int |\psi|^2 dx \leq |\Omega|.$$ Integrating the above inequality in $(0, T)$, we get

$$\int_0^T \int \left| \frac{i}{k} \nabla \psi + \psi A \right|^2 dx dt \leq C. \quad (2.2)$$

Here and later on, $C$ will denote a generic positive constant independent of $\epsilon > 0$.

Testing (1.2) by $A$ and using (2.1), (2.2) and $\text{div} A = 0$, we find that

$$\frac{1}{2} \frac{d}{dt} \int A^2 dx + \epsilon \int |\text{curl} A|^2 dx = -\text{Re} \int \left( \frac{i}{k} \nabla \psi + \psi A \right) \overline{\psi} A dx$$

$$\leq \int \left| \frac{i}{k} \nabla \psi + \psi A \right| |A| dx$$

$$\leq \left\| \frac{i}{k} \nabla \psi + \psi A \right\|_{L^2} \|A\|_{L^2},$$

which yields

$$\|A\|_{L^\infty(0,T;L^2)} \leq C. \quad (2.3)$$

Inequalities (2.1), (2.2) and (2.3) imply

$$\|\psi\|_{L^2(0,T;H^1)} \leq C. \quad (2.4)$$

Taking $\text{div}$ to (1.2), it is easy to infer that

$$-\Delta \phi = \text{div} \text{Re} \left\{ \left( \frac{i}{k} \nabla \psi + \psi A \right) \overline{\psi} \right\} \text{ in } Q_T, \quad (2.5)$$

$$\nabla \phi \cdot \nu = 0 \text{ on } (0, T) \times \partial \Omega. \quad (2.6)$$

Testing (2.5) by $\phi$ and using (2.1) and (2.2), we have

$$\|\nabla \phi\|_{L^2(0,T;L^2)} \leq C. \quad (2.7)$$
Testing (1.1) by \(-\Delta \overline{\psi}\), taking the real part and using (2.1) and (2.3), we deduce that

\[
\frac{\eta}{2} \frac{d}{dt} \int |\nabla \psi|^2 dx + \frac{1}{k^2} \int |\Delta \psi|^2 dx \\
\leq \eta k \int |\phi||\Delta \psi| dx + \frac{2}{k} \int |A||\nabla \psi||\Delta \psi| dx \\
+ \int |A|^2 |\Delta \psi| dx + \int (|\psi|^2 - 1)|\psi||\Delta \psi| dx \\
\leq C\|\phi\|_{L^2} \|\Delta \psi\|_{L^2} + C\|A\|_{L^4} \|\nabla \psi\|_{L^4} \|\Delta \psi\|_{L^2} \\
+ C\|A\|_{L^4} \|\Delta \psi\|_{L^2} + C\|\Delta \psi\|_{L^2} \\
\leq \frac{1}{16} \frac{1}{k^2} \|\Delta \psi\|^2_{L^2} + C\|\phi\|^2_{L^2} + C\|A\|^2_{L^2} + C
\]

(2.8)

Here we have used the Gagliardo-Nirenberg inequalities

\[
\|\nabla \psi\|^2_{L^4} \leq C\|\psi\|_{L^\infty} \|\Delta \psi\|_{L^2}, \\
\|A\|^2_{L^4} \leq C\|A\|_{L^2} \|\text{curl} A\|_{L^2}
\]

(2.9) (2.10)

Testing (1.2) by \(\text{curl}^2 A\) and using (2.1), (2.3), (2.9) and (2.10), we find that

\[
\frac{1}{2} \frac{d}{dt} \int |\text{curl} A|^2 dx + \epsilon \int |\text{curl}^2 A|^2 dx \\
\leq C(\|\nabla \psi\|^2_{L^2} + \|A\|_{L^4} \|\nabla \psi\|_{L^4}) \|\text{curl} A\|_{L^2} + C\|\text{curl} A\|^2_{L^2} \\
\leq \frac{1}{16} \frac{1}{k^2} \|\Delta \psi\|^2_{L^2} + C\|\text{curl} A\|^2_{L^2}. \\
\]

(2.11)

Combining (2.8) and (2.11) and using the Gronwall inequality, we have

\[
\|\psi\|_{L^\infty(0,T;H^1)} + \|\psi\|_{L^2(0,T;H^2)} \leq C; \\
\|A\|_{L^\infty(0,T;H^1)} \leq C
\]

(2.12) (2.13)

It follows from (1.1), (2.1), (2.12) and (2.13) that

\[
\|\psi_t\|_{L^2(0,T;L^2)} \leq C, \\
\|\nabla \phi\|_{L^\infty(0,T;L^2)} \leq C, \\
\|A_t\|_{L^2(0,T;L^2)} \leq C
\]

(2.14) (2.15) (2.16)

Taking \(\partial_t\) to (1.1), testing then by \(\overline{\psi_t}\), taking the real part, and using (2.1),
Combining (2.17) and (2.18) and using (2.14), (2.16) and the Gronwall inequality, we conclude
\begin{align}
\|\psi_t\|_{L^\infty(0,T;L^2)} + \|\psi_t\|_{L^2(0,T;H^1)} & \leq C, \\
\|A_t\|_{L^\infty(0,T;L^2)} & \leq C.
\end{align}

Taking \(\partial_t\) to (1.2), testing then by \(A_t\), and using (2.1), (2.12), (2.13) and (2.10), we obtain
\begin{align}
\frac{1}{2} \frac{d}{dt} \int |A_t|^2 \, dx + \epsilon \int |\text{curl} A_t|^2 \, dx + \int |\psi|^2 |A_t|^2 \, dx \\
\leq C \int (|\nabla \psi_t| + |\nabla \psi| |\psi_t| + |A||\psi_t|) |A_t| \, dx \\
\leq C \|\nabla \psi_t\|_{L^2} + \|\nabla \psi\|_{L^4} \|\psi_t\|_{L^4} + \|A\|_{L^4} \|\psi_t\|_{L^4} \|A_t\|_{L^2} \\
\leq \frac{1}{16} \frac{1}{k^2} \|\nabla \psi_t\|_{L^2}^2 + C \|\nabla \psi\|_{L^4}^2 \|A_t\|_{L^2}^2 + C \|\psi_t\|_{L^2}^2.
\end{align}

Taking \(\partial_t\) to (1.2), testing then by \(|\text{curl} A|^q \text{curl} A_t\), and using (2.1), (2.12) and (2.13), we have
\begin{align}
\frac{d}{dt} \int |\text{curl} A|^q \, dx & \leq C(\|\nabla \psi\|_{L^2}^2 + \|\text{curl} A\|_{L^2}^2 + \|A\|_{L^2} \|\nabla \psi\|_{L^2} \|\text{curl} A\|_{L^{q-1}}),
\end{align}
and therefore
\begin{align}
\frac{d}{dt} \|\text{curl} A\|_{L^q} & \leq C(\|\nabla \psi\|_{L^2}^2 + \|\text{curl} A\|_{L^q} + 1),
\end{align}
which gives
\[ \|A\|_{L^\infty(0,T;W^{1,q})} \leq C. \] (2.21)
Here we have used the well-known fact:
\[ \|A\|_{W^{1,q}} \leq C(\|\text{div} A\|_{L^q} + \|\text{curl} A\|_{L^q}) \]
with \( A \cdot \nu = 0 \) on \( \partial \Omega \).

On the other hand, it follows from (1.1), (2.1), (2.12), (2.15) and (2.19) that
\[ \|\psi\|_{L^\infty(0,T;H^2)} + \|\psi\|_{L^2(0,T;H^3)} \leq C, \] (2.22)
while from (2.5), (2.6), (2.22) and (2.13) we get
\[ \|\phi\|_{L^\infty(0,T;H^2)} + \|\phi\|_{L^2(0,T;H^3)} \leq C, \]
\[ \|\phi_t\|_{L^2(0,T;L^2)} \leq C. \]

This completes the proof. \( \square \)

3 Proof of Theorem 1.3

In this section, we will use Theorem 1.2 to prove Theorem 1.3. In fact, one only needs to prove that
\[ \psi, A_r, A_\theta, A_z \in V_2(Q_T). \] (3.1)

Note that
\[ V_2(Q_T) \subset L^4(0,T;L^4(\Omega)) \]
which satisfies the condition (1.6) for \( r = p = 4 \) and \( d = 2 \), where \( \Omega \) denotes
\[ \Omega := \{(r,z)\mid 0 \leq r \leq R, 0 \leq z \leq h\}. \]

Testing (1.7) by \( \overline{\psi} \) and taking the real part, we see that
\[
\frac{\eta}{2} \int_0^T \int_0^h |\psi|^2 r dr dz + \int_0^T \int_0^h \left( \left| \frac{i}{k} \frac{\partial \psi}{\partial r} + \psi A_r \right|^2 + \left| \frac{i}{k} \frac{\partial \psi}{\partial z} + \psi A_z \right|^2 \right) r dr dz
\]
\[ + \frac{1}{2k^2} \int_0^T \int_0^h \frac{|\psi|^2}{r^2} r dr dz + \int_0^T \int_0^h A_\theta^2 |\psi|^2 r dr dz + \int_0^T \int_0^h (|\psi|^2 - 1)^2 r dr dz
\]
\[ + \int_0^T \int_0^h |\psi|^2 r dr dz = \int_0^T \int_0^h dr dz = R \eta, \]
which gives
\[ \int_0^R \int_0^h |\psi|^2 \, dr \, dz + \int_0^T \int_0^R \int_0^h \left( \frac{i}{k} \frac{\partial \psi}{\partial r} + \psi A_r \right)^2 + \left( \frac{i}{k} \frac{\partial \psi}{\partial z} + \psi A_z \right)^2 \, dr \, dz \, dt \]
\[ + \int_0^T \int_0^R \int_0^h |\psi|^4 \, dr \, dz \, dt \leq C. \]  
(3.2)

Testing (1.8) by $A_r$ and using (3.2) and the Gagliardo-Nirenberg inequality, we find that
\[
\frac{1}{2} \frac{d}{dt} \int_0^R \int_0^h A_r^2 \, dr \, dz + \int_0^R \int_0^h \left( \frac{\partial A_r}{\partial r} \right)^2 + \left( \frac{\partial A_r}{\partial z} \right)^2 \, dr \, dz + \int_0^R \int_0^h \frac{A_r^2}{2r^2} \, dr \, dz \]
\[ \leq \int_0^R \int_0^h \left| \frac{i}{k} \frac{\partial \psi}{\partial r} + \psi A_r \right| |\psi| |A_r| \, dr \, dz \]
\[ \leq \int_0^R \int_0^h \left| \frac{i}{k} \frac{\partial \psi}{\partial r} + \psi A_r \right|^2 \, dr \, dz + C \|\psi\|_{L^4}^2 \|A_r\|_{L^4}^2, \]
\[ \leq \int_0^R \int_0^h \left| \frac{i}{k} \frac{\partial \psi}{\partial r} + \psi A_r \right|^2 \, dr \, dz + \frac{1}{2} \int_0^R \int_0^h \left( \left| \frac{\partial A_r}{\partial r} \right|^2 + \left| \frac{\partial A_r}{\partial z} \right|^2 \right) \, dr \, dz + C \|\psi\|_{L^4}^4 \|A_r\|_{L^2}^2, \]
which gives
\[ A_r \in V_2(Q_T). \]  
(3.3)

Similarly,
\[ A_\theta, A_z \in V_2(Q_T). \]  
(3.4)

Note that (3.3) and (3.4) yield
\[ A_r, A_z \in L^4(Q_T), \]  
(3.5)

while (3.2) and (3.5) lead to
\[ \psi \in V_2(Q_T). \]

This completes the proof. \[ \square \]

References

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