\(\Delta\)-Convergence and Strong Convergence of Nonexpansive Multivalued Mappings in CAT(0) Spaces

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Abstract

By using the concept of \(\Delta\)-convergence introduced by Lim, we are able to give the CAT(0) space analogs of results on weak convergence of the proposed iterative processes proved in uniformly convex Banach spaces.

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1. Introduction and Preliminaries

The study of metric spaces without linear structure has played a vital role in various branches of pure and applied sciences. In particular, fixed point theorems in CAT(0) spaces for nonexpansive single valued, as well as for multivalued mappings have been studied extensively by many authors [3].

In 1976, Lim [18] introduced a concept of convergence in a general metric space setting which he called "\(\Delta\)-convergence". Kuczumow [15] introduced an identical notion of convergence in Banach spaces, which he called 'almost convergence'. Recently, Kirk and Panyanak [13] used the concept of \(\Delta\)-convergence introduced by Lim [18] to prove the CAT(0) space analogs of some Banach space results which involve weak convergence and Dhompongs and Panyanak [6] obtained \(\Delta\)-convergence theorems for the Picard, Mann and Ishikawa iterations in the CAT(0) space setting.
Inspired by Song and Wang [28], Laowang and Panyanak [17] extended results of [6] for multivalued nonexpansive mappings in CAT(0) spaces. On the other hand, in [9], Garcia-Falset et al. introduced two new conditions on single valued mappings, called conditions (E) and $(C_\lambda)$, which are weaker than nonexpansiveness and stronger than quasi- nonexpansiveness. Very recently, the current authors in [4] used modified conditions for multivalued mappings, and proved some fixed point theorems for multivalued mappings satisfying these condition in a CAT(0) space.

Let $(X,d)$ be a metric space and $x,y \in X$ with $l = d(x,y)$. A geodesic path from $x$ to $y$ is an isometry $c : [0,l] \to X$ such that $c(0) = x$ and $c(l) = y$. The image of a geodesic path is called a geodesic segment. A metric space $X$ is a (uniquely) geodesic space if every two points of $X$ are joined by only one geodesic segment. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space $X$ consists of three points $x_1, x_2, x_3$ of $X$ and three geodesic segments joining each pair of vertices. A comparison triangle of a geodesic triangle $\Delta(x_1, x_2, x_3)$ is the triangle $\Delta \setminus \Delta(x_1, x_2, x_3) = \Delta(x_1, x_2, x_3)$ in the Euclidean space $\mathbb{R}^2$ such that $d(X, y) = d_{\mathbb{R}^2}(x, y)$ for all $i, j = 1, 2, 3$.

A geodesic space $X$ is a CAT(0) space if for each geodesic triangle $\Delta := \Delta(x_1, x_2, x_3)$ in $X$ and its comparison triangle $\Delta := \Delta(x_1, x_2, x_3)$ in $\mathbb{R}^2$, the CAT(0) inequality

$$d(x, y) \leq d_{\mathbb{R}^2}(x, y)$$

is satisfied by all $x, y \in \Delta$ and $\overline{x}, \overline{y} \in \overline{\Delta}$. The meaning of the CAT(0) inequality is that a geodesic triangle in $X$ is at least thin as its comparison triangle in the Euclidean plane.

A geodesic metric space $X$ is called a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

Let $\Delta$ be a geodesic triangle in $X$ and let $\overline{\Delta}$ be its comparison triangle in $\mathbb{R}^2$. Then $\Delta$ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$, $d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y})$. The following properties of a CAT(0) space are useful:

(i) A CAT(0) space $X$ is uniquely geodesic;

(ii) For any $x \in X$ and any closed convex subset $D \subset X$, there is a unique closest point $y \in D$ to $x$.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space $X$. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x_n, x).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf \{ r(x, \{x_n\}) : x \in X \},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{ x \in X : r(x, \{x_n\}) = r(\{x_n\}) \}.$$
Definition 1. Let \( \{x_n\} \) be a bounded sequence in a complete \( CAT(0) \) space \( X \). Then \( \{x_n\} \) is said to \( \Delta \)-converge to \( x \) in \( X \) if \( x \) is the unique asymptotic center of \( \{x_m\} \) for every subsequence \( \{x_m\} \) of \( \{x_n\} \). In this case we write \( \Delta \lim_{n \to \infty} x_n = x \) and call \( x \) the \( \Delta \)-limit of \( \{x_n\} \).

It is known that in a \( CAT(0) \) space, \( A(\{x_n\}) \) consists of exactly one point[14]. A sequence \( \{x_n\} \) in a \( CAT(0) \) space \( X \) is said to be \( \Delta \)-convergent to \( x \in X \) if \( x \) is the unique asymptotic center of every subsequence of \( \{x_n\} \).

Recall that a bounded sequence \( \{x_n\} \) in a complete \( CAT(0) \) space \( X \) is said to be regular if \( r_a(X,\{x_n\}) = r_a(X,\{u_n\}) \) for every subsequence \( \{u_n\} \) of \( \{x_n\} \). It is known that every bounded sequence in a Banach space has a regular subsequence. Since every regular sequence \( \Delta \)-converges, we see immediately that every bounded sequence in a complete \( CAT(0) \) space \( X \) such that \( \{x_n\} \), \( \Delta \)-converges to \( x \) and given \( y \in X \) with \( y \neq x \),

\[
\limsup_{n \to \infty} d(x_n, x) < \limsup_{n \to \infty} d(x_n, y).
\]

Clearly, \( X \) satisfies a condition which is known in Banach space theory as the Opial property. We denote \( \omega_\omega(x_n) := \bigcup \{Z_a(\{u_n\})\} \), where the union is taken over all subsequences \( \{u_n\} \) of \( \{x_n\} \).

Lemma 1. [14] Every bounded sequence in a complete \( CAT(0) \) space has a \( \Delta \)-convergent subsequence.

Lemma 2. [14] If \( D \) is a closed convex subset of a complete \( CAT(0) \) space and if \( \{x_n\} \) is a bounded sequence in \( D \), then the asymptotic center of \( \{x_n\} \) is in \( D \).

Lemma 3. [6] If \( \{x_n\} \) is a bounded sequence in a complete \( CAT(0) \) space \( X \) with \( A(\{x_n\}) = \{x\} \) and \( \{u_n\} \) is a subsequence of \( \{x_n\} \) with \( A(\{u_n\}) = \{u\} \) and the sequence \( \{d(x_n, u)\} \) converges, then \( x = u \).

Fixed point theory in \( CAT(0) \) spaces was first studied by Kirk[8]. He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete \( CAT(0) \) space always has a fixed point. Since then fixed point theory for single-valued and multivalued mappings in \( CAT(0) \) spaces has been rapidly developed and many of papers have appeared.

Let \( K \) be a nonempty subset of a \( CAT(0) \) space \( X \). We denote by \( CB(D) \), \( K(D) \) and \( KC(D) \) the collection of all nonempty closed bounded subsets, nonempty compact subsets, and nonempty convex compact subset of \( D \), respectively. The Hausdorff metric \( H \) on \( CB(X) \) is defined by

\[
H(A, B) := \max \left\{ \sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(y, A) \right\}
\]

for all \( A, B \in CB(X) \), where \( \text{dist}(x, B) = \inf \{d(x, z) : z \in B\} \). Let \( T : X \to 2^X \) be a multivalued mapping. An element \( x \in X \) is said to be a fixed point of \( T \), if \( x \in Tx \). The set of its fixed points will be denoted by \( F(T) \).
Definition 2. A multivalued mapping $T : X \to CB(X)$ is called

(i) nonexpansive if $H(Tx, Ty) \leq d(x, y)$, $x, y \in X$,
(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and $H(TX, TP) \leq d(x, p)$ for all $x \in X$ and all $p \in F(T)$. A point $x \in X$ is called a fixed point of $T$ if $x = T(x).$ We shall denote by $\text{Fix}(T)$ the set of fixed points of $T$.

A multivalued mappings $T : K \to P(K)$ are said to satisfy Condition (I) if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $dist(x, Tx) \geq f(dist(x, F(T)))$ for all $x \in K$.

Lemma 4. [14] Let $C$ be a closed convex subset of a complete CAT(0) space $X$, and let $T : C \to X$ be a nonexpansive mapping. Then the conditions \( \{x_n\} \) $\Delta-$convergence to $x$ and $d(x_n, Tx_n) \to 0$, imply $x \in C$ and $Tx = x$.

Lemma 5. [6] Let $X$ be a CAT(0) space. Then for all $x, y, z \in X$ and all $t \in [0, 1]$ we have

(i) $d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z)$,
(ii) $d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x, z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2$.

Now the aim of this paper is to study the iterative scheme defined as follows: Let $K$ be a nonempty closed convex subset of a complete CAT(0) space $X$. Let $T : K \to P(K)$ be a multivalued mapping such that $F(T) \neq \emptyset$ and $P_T$ is a nonexpansive mapping. Suppose \( \{x_n\} \) is generated iteratively by

\[
\begin{align*}
x_1 & \in K, \; n \in \mathbb{N} \\
z_n & = (1 - \gamma_n)x_n + \gamma_nv_n \\
y_n & = (1 - \beta_n)v_n + \beta_nw_n \\
x_{n+1} & = (1 - \alpha_n)v_n + \alpha_nw_n
\end{align*}
\]

where $v_n \in P_T(x_n)$, $u_n \in P_T(y_n)$, $w_n \in P_T(z_n)$ and \{\( \alpha_n \)\}, \{\( \beta_n \)\}, \{\( \gamma_n \)\} $\in [a, b] \subset (0, 1)$. We show that the sequence \( \{x_n\} \) defined by (1.1) $\Delta-$convergence and strong convergence theorems to a fixed point of $T$ in a CAT(0) space.

we obtain more results, i.e., we show that many weak convergence theorems in a Banach space setting can be extended to a CAT(0) space setting.

Now we state some useful lemmas.

Lemma 6.

Lemma 7. [17] Let $C$ be a closed convex subset of a complete CAT(0) space $X$, and let $T : C \to X$ be a nonexpansive mapping. Then the conditions \( \{x_n\} \) $\Delta-$convergence to $x$ and $d(x_n, Tx_n) \to 0$, imply $x \in C$ and $Tx = x$. exists.

Lemma 8. [11] Let $T : K \to P(K)$ be a multivalued mapping and $P_T(x) = \{y \in Tx : ||x - y|| = d(x, Tx)\}$. Then the following are equivalent.

(1) $x \in F(T)$
(2) $P_T(x) = \{x\}$
(3) $x \in F(P_T)$. Moreover, $F(T) = F(P_T)$. 

Lemma 9. Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$. Let $T : K \to P(K)$ be a multivalued mapping such that $F(T) \neq \emptyset$ and $P_T$ is a nonexpansive mapping. Let $\{x_n\}$ be the sequence as defined in (1.1), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$. Then $\lim_{n \to \infty} d(x_n, p)$ exists for all $p \in F(T)$.

Proof. Let $p \in F(T)$. Then $p \in P_T(p) = \{p\}$ by Lemma 8. Then, using 1.1 and Lemma 5, we have
\[
d(z_n, p) = d((1 - \gamma_n) x_n \oplus \gamma_n v_n, p) \\
\leq (1 - \gamma_n) d(x_n, p) + \gamma_n d(v_n, p) \\
\leq (1 - \gamma_n) dist(x_n, P_T(p)) + \gamma_n dist(v_n, P_T(p)) \\
\leq (1 - \gamma_n) H(P_T(x_n), P_T(p)) + \gamma_n H(P_T(v_n), P_T(p)) \\
\leq (1 - \gamma_n) d(x_n, p) + \gamma_n d(x_n, p) \\
= d(x_n, p)
\]
and
\[
d(y_n, p) = d((1 - \beta_n) v_n \oplus \beta_n w_n, p) \\
\leq (1 - \beta_n) d(v_n, p) + \beta_n d(w_n, p) \\
\leq (1 - \beta_n) dist(v_n, P_T(p)) + \beta d(w_n, P_T(p)) \\
\leq (1 - \beta_n) H(P_T(x_n), P_T(p)) + \beta_n d(P_T(z_n), P_T(p)) \\
\leq (1 - \beta_n) d(x_n, p) + \beta_n d(z_n, p) \\
\leq d(x_n, p)
\]
and
\[
d(x_{n+1}, p) = d((1 - \alpha_n) v_n \oplus \alpha_n u_n, p) \\
\leq (1 - \alpha_n) d(v_n, p) + \alpha_n d(u_n, p) \\
\leq (1 - \alpha_n) dist(v_n, P_T(p)) + \alpha d(u_n, P_T(p)) \\
\leq (1 - \alpha_n) H(P_T(x_n), P_T(p)) + \alpha_n H(P_T(y_n), P_T(p)) \\
\leq (1 - \alpha_n) d(x_n, p) + \alpha_n d(y_n, p) \leq d(x_n, p)
\]

Consequently, we have $d(x_n, p) \leq d(x_1, p)$ for all $n \geq 1$. This implies that $\{d(x_n, p)\}_{n=1}^\infty$ is bounded and decreasing. Hence $\lim_{n \to \infty} d(x_n, p)$ exists. \(\square\)

Theorem 1. Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$. Let $T : K \to P(K)$ be a multivalued mapping such that $F(T) \neq \emptyset$ and $P_T$ is a nonexpansive mapping. Let $\{x_n\}$ be the sequence as defined in (1.1), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that $\lim_{n \to \infty} \alpha_n \beta_n \gamma_n (1 - \gamma_n) \neq 0$. Then $F(T)$ is nonempty if and only if $\{x_n\}$ bounded and $\lim_{n \to \infty} d(Tx_n, x_n) = 0$. 

2. Main results
Proof. Suppose that $F(T)$ is nonempty and let $p \in F(T)$. Then by Lemma 9, \( \lim_{n \to \infty} d(x_n, x^*) \) exists and \( \{x_n\} \) is bounded. From Lemma 5, we get

\[
d(z_n, p)^2 = d((1 - \gamma_n) x_n \oplus \gamma_n v_n, p)^2 \\
\leq (1 - \gamma_n) d(x_n, p)^2 + \gamma_n d(v_n, p)^2 - \gamma_n (1 - \gamma_n) d(x_n, v_n)^2 \\
\leq (1 - \gamma_n) \text{dist}(x_n, P_T(p))^2 + \gamma_n \text{dist}(v_n, P_T(p))^2 - \gamma_n (1 - \gamma_n) d(x_n, v_n)^2 \\
\leq (1 - \gamma_n) H(P_T(x_n), P_T(p))^2 + \gamma_n H(P_T(x_n), P_T(p))^2 - \gamma_n (1 - \gamma_n) d(x_n, v_n)^2 \\
\leq (1 - \gamma_n) d(x_n, p)^2 + \gamma_n d(x_n, p)^2 - \gamma_n (1 - \gamma_n) d(x_n, v_n)^2 \\
= d(x_n, p)^2 - \gamma_n (1 - \gamma_n) d(x_n, v_n)^2.
\]

It follows from lemma 5 that

\[
d(y_n, p)^2 = d((1 - \beta_n) v_n \oplus \beta_n w_n, p)^2 \\
\leq (1 - \beta_n) d(v_n, p)^2 + \alpha_n d(w_n, p)^2 - \beta_n (1 - \beta_n) d(v_n, w_n)^2 \\
\leq (1 - \beta_n) \text{dist}(v_n, P_T(p))^2 + \beta_n \text{dist}(w_n, P_T(p))^2 - \beta_n (1 - \beta_n) d(v_n, w_n)^2 \\
\leq (1 - \beta_n) H(P_T(x_n), P_T(p))^2 + \beta_n H(P_T(v_n), P_T(p))^2 - \beta_n (1 - \beta_n) d(v_n, w_n)^2 \\
\leq (1 - \beta_n) d(x_n, p)^2 + \beta_n d(z_n, p)^2 - \beta_n (1 - \beta_n) d(v_n, w_n)^2.
\]

Again, we apply lemma 5 to conclude that

\[
d(x_{n+1}, p)^2 = d((1 - \alpha_n) u_n \oplus \alpha_n u_n, p)^2 \\
\leq (1 - \alpha_n) d(u_n, p)^2 + \alpha_n d(u_n, p)^2 - \alpha_n (1 - \alpha_n) d(u_n, u_n)^2 \\
\leq (1 - \alpha_n) \text{dist}(u_n, P_T(p))^2 + \alpha_n \text{dist}(u_n, P_T(p))^2 - \alpha_n (1 - \alpha_n) d(u_n, u_n)^2 \\
\leq (1 - \alpha_n) H(P_T(x_n), P_T(p))^2 + \alpha_n H(P_T(y_n), P_T(p))^2 - \alpha_n (1 - \alpha_n) d(u_n, u_n)^2 \\
\leq (1 - \alpha_n) d(x_n, p)^2 + \alpha_n d(y_n, p)^2 - \alpha_n (1 - \alpha_n) d(u_n, u_n)^2.
\]

So, we have

\[
\alpha_n \beta_n \gamma_n (1 - \gamma_n) d(x_n, v_n)^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2
\]

for all $n \in \mathbb{N}$. This implies that

\[
\sum_{n=1}^{m} \alpha_n \beta_n \gamma_n (1 - \gamma_n) d(x_n, v_n)^2 \leq \sum_{n=1}^{m} [d(x_n, p)^2 - d(x_{n+1}, p)^2] \leq d(x_1, p)^2 < \infty.
\]

Since \( \sum_{n=1}^{m} \alpha_n \beta_n \gamma_n (1 - \gamma_n) < \infty \), hence \( \lim_{n \to \infty} d(x_n, v_n)^2 = 0 \). Thus \( \lim_{n \to \infty} d(x_n, v_n) = 0 \). From \( d(x_n, Tx_n) \leq d(x_n, P_T(x_n)) \leq d(x_n, v_n) = 0 \), we get \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \).

Conversely, suppose that \( \{x_n\} \) is bounded and \( \lim d(x_n, Tx_n) = 0 \). Let \( A(\{x_n\}) = \{x\} \). Then \( x \in C \) by Lemma 2. Since \( d(x_n, Tx) \leq d(x_n, Tx_n) + d(Tx_n, Tx) \) for all \( n \geq 1 \), then

\[
\lim \sup_{n \to \infty} d(x_n, Tx) \leq \lim \sup_{n \to \infty} d(x_n, x)
\]

By the unique of asymptotic centers, we have \( Tx = x \). Therefore, \( x \) is a fixed point of \( T \). \( \square \)
First, we prove the $\Delta$–convergence theorem.

**Theorem 2.** Let $K$ be a nonempty closed convex subset of a complete CAT(0) space $X$. Let $T : K \to P(K)$ be a multivalued mapping such that $F(T) \neq \emptyset$ and $P_T$ is a nonexpansive mapping. Let $\{x_n\}$ be the sequence as defined in (1.1), where \{\alpha_n\}, \{\beta_n\} and \{\gamma_n\} are sequences in $(0, 1)$ such that $\lim_{n \to \infty} \alpha_n \beta_n \gamma_n (1 - \gamma_n) \neq 0$. Then $\{x_n\}$ is $\Delta$–convergent to an element of $F(T)$.

**Proof.** $\{x_n\}$ is bounded by lemma9 and $\lim_{n \to \infty} d(x_n, Tx_n) = 0$ by Theorem1. We now let $\omega_\omega (x_n) := \bigcup A(\{ u_n \})$ where the union is taken over all subsequences $\{ u_n \}$ of $\{ x_n \}$. We claim that $\omega_\omega (x_n) \subset F(T)$. Let $u \epsilon \omega_\omega (x_n)$, then there exists a subsequence $\{ u_n \}$ of $\{ x_n \}$ such that $A(\{ u_n \}) = \{ u \}$. By lemma1 and lemma2 there exists a subsequence $\{ v_n \}$ of $\{ u_n \}$ such that $\Delta - \lim v_n = v \in C$. Since $\lim_{n \to \infty} d(v_n, Tv_n) = 0$, then $v \in F(T)$ by lemma4. We claim that $u = v$. Suppose not, since $T$ is nonexpansive and $v \in F(T)$, $\lim_{n \to \infty} d(x_n, v)$ exists by lemma9. Then by the uniqueness of asymptotic centers,

$$
\limsup_{n \to \infty} d(v_n, v) < \limsup_{n \to \infty} d(v_n, u)
\leq \limsup_{n \to \infty} d(u_n, u)
\leq \limsup_{n \to \infty} d(u_n, v)
= \limsup_{n \to \infty} d(x_n, v)
= \limsup_{n \to \infty} d(v_n, v)
$$

a contradiction, and hence $u = v \in F(T)$. To show that $\{ x_n \}$ $\Delta$–converges to a fixed point of $T$, it suffices to show that $\omega_\omega (x_n)$ consists of exactly one point. Let $\{ u_n \}$ be a subsequence of $\{ x_n \}$. By lemma1 and lemma2 there exists a subsequence $\{ v_n \}$ of $\{ u_n \}$ such that $\Delta - \lim v_n = v \in C$. Let $A(\{ u_n \}) = \{ u \}$ and $A(\{ x_n \}) = \{ x \}$. We have seen that $u = v$ and $v \in F(T)$. We can complete the proof by showing that $x = v$. Suppose not, since $\{ d(x_n, v) \}$ is convergent, then by the uniqueness of asymptotic centers,

$$
\limsup_{n \to \infty} d(v_n, v) < \limsup_{n \to \infty} d(v_n, x)
\leq \limsup_{n \to \infty} d(x_n, x)
\leq \limsup_{n \to \infty} d(x_n, v)
= \limsup_{n \to \infty} d(v_n, v)
$$

a contradiction, and hence the conclusion follows. □

**Theorem 3.** Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$. Let $T : K \to P(K)$ be a multivalued mapping such that $F(T) \neq \emptyset$ and $P_T$ is a nonexpansive mapping. Let $\{ x_n \}$ be the sequence as defined in (1.1), where \{\alpha_n\}, \{\beta_n\} and \{\gamma_n\} are sequences in $(0, 1)$ such that $\lim_{n \to \infty} \alpha_n \beta_n \gamma_n (1 - \gamma_n) \neq 0$. Then...
Then \( \{ x_n \} \) converges strongly to fixed point of \( F(T) \) if and only \( \liminf_{n \to \infty} d(x_n, F(T)) = 0. \)

**Proof.** The necessity is obvious. Conversely, suppose that \( \liminf_{n \to \infty} d(x_n, F(T)) = 0. \) By lemma 2, we have
\[
d(x_{n+1}, p) \leq d(x_n, p)
\]
This gives
\[
d(x_{n+1}, F(T)) \leq d(x_n, F(T)).
\]
Hence \( \lim_{n \to \infty} d(x_n, F(T)) \) exists. By hypothesis, \( \liminf_{n \to \infty} d(x_n, F(T)) = 0 \) so we must have \( \lim_{n \to \infty} d(x_n, F(T)) = 0. \) Next, we show that \( \{ x_n \} \) is a Cauchy sequence in \( K. \) Let \( \varepsilon > 0 \) be arbitrarily chosen. Since \( \lim_{n \to \infty} d(x_n, F(T)) = 0, \) there exists \( n_0 \) such that for all \( n \geq n_0. \) We have
\[
d(x_n, F(T)) < \frac{\varepsilon}{4}.
\]
In particular, \( \inf \{ d(x_{n_0}, p) : p \in F(T) \} < \frac{\varepsilon}{4} \); so there must exists a \( p^* \in F(T) \) such that
\[
d(x_{n_0}, p^*) < \frac{\varepsilon}{2}.
\]
Now for \( m, n \geq n_0, \) we have
\[
d(x_{n+m}, x_n) \leq d(x_{n+m}, p^*) + d(x_n, p^*)
\]
\[
\leq 2d(x_{n_0}, p^*)
\]
\[
< 2 \left( \frac{\varepsilon}{2} \right) = \varepsilon.
\]
Hence \( \{ x_n \} \) is a Cauchy sequence in closed subset \( K \) of a Complete CAT(0) space \( X, \) and therefore it must converge strongly to a point \( q \) in \( K. \) Let \( \lim_{n \to \infty} x_n = q. \) Now
\[
d(q, P_T(q)) \leq d(x_n, q) + d(x_n, P_T(x_n)) + H(P_T(x_n), P_T(q))
\]
\[
\leq d(x_n, q) + d(x_n, v_n) + d(x_n, q)
\]
\[
\to 0 \text{ as } n \to \infty
\]
which gives that implies \( d(q, P_T(q)) = 0. \) But \( P_T \) is a nonexpansive mapping so \( F(P_T) \) is closed. Therefore, \( q \in F(P_T) = F(T). \) \( \square \)

**Theorem 4.** Let \( C \) be a nonempty closed convex subset of a complete CAT(0) space \( X. \) Let \( T : K \to P(K) \) be a multivalued mapping satisfying Condition (I) such that \( F(T) \neq \emptyset \) and \( P_T \) is a nonexpansive mapping. Let \( \{ x_n \} \) be the sequence as defined in (1.1), where \( \{ \alpha_n \}, \{ \beta_n \} \) and \( \{ \gamma_n \} \) are sequences in \( (0, 1) \) such that \( \lim_{n \to \infty} \alpha_n \beta_n \gamma_n (1 - \gamma_n) = 0. \) Then \( \{ x_n \} \) converges strongly to fixed point of \( T. \)

**Proof.** As in proof of Theorem 1, we have \( \lim_{n \to \infty} d(x_n, Tx_n) = 0. \) Further, by Condition (I),
\[
\lim_{n \to \infty} d(x_n, Tx_n) \geq \lim_{n \to \infty} f(d(x_n, F(T))).
\]
It follows that \( \lim_{n \to \infty} d(x_n, F(T)) = 0. \) Therefore, the result follows from Theorem 2. \( \square \)
Δ—convergence in CAT(0) spaces

References


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