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New Best Proximity Point Theorems for a Pair of Nonlinear Non-self Mappings in Metric Spaces

Wei-Shih Du[1] **and Ming-Chih Yen**

Department of Mathematics
National Kaohsiung Normal University
Kaohsiung 82444, Taiwan

Abstract

In this paper, we establish new convergence theorems and best proximity point theorems for a pair of nonlinear non-self mappings in metric spaces.

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1. Introduction and preliminaries

From a mathematical point of view, it is well known that several problems arising form diverse areas of natural science involve the existence of solutions of nonlinear equations with the form $Tx = x$, where T is a selfmapping or non-self mapping defined on metric spaces or topological vector spaces. Fixed point theory plays an important role in solving such nonlinear equations. However, in our experience, if T is not a selfmapping, there is a high possibility that the equations of the form $Tx = x$ does not necessarily admit a solution. In this status, one would try to find an approximate solution x in the domain of T such that the error $d(x, Tx)$ is minimum, where d is the distance function.

[1]Corresponding author

Let A and B be nonempty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a self-mapping. A point $x \in A \cup B$ is called a *best proximity point* for T if

$$d(x, Tx) = dist(A, B) := \inf\{d(x, y) : x \in A, y \in B\}.$$

In this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$, the sets of positive integers and real numbers, respectively.

In 2003, Kirk, Srinavasan and Veeramani [18] introduced cyclic mappings and best proximity points for generalizing the Banach contraction principle. Let A and B be nonempty subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is called *cyclic* if

$$T(A) \subset B \text{ and } T(B) \subset A.$$

In the last decades, a number of existence and uniqueness theorems for best proximity point were investigated by several authors; see, e.g., [1-5, 8, 15-20] and references therein.

Let f be a real-valued function defined on $\mathbb{R}$. For $c \in \mathbb{R}$, we recall that

$$\limsup_{x \to c^+} f(x) = \inf_{\varepsilon > 0} \sup_{c < x < c+\varepsilon} f(x).$$

The concept of $\mathcal{MT}(\lambda)$-function [8] was introduced by Du in 2016.

Definition 1.1. Let $\lambda > 0$. A function $\mu : [0, \infty) \to [0, \lambda)$ is said to be an $\mathcal{MT}(\lambda)$-*function* [8, 10-15] if $\limsup_{s \to t^+} \mu(s) < \lambda$ for all $t \in [0, \infty)$. In particular, if $\lambda = 1$, then $\mu : [0, \infty) \to [0, 1)$ is called an $\mathcal{MT}$-*function* (or $\mathcal{R}$-*function*) [6-16].

Du established the following important and useful characterizations of $\mathcal{MT}(\lambda)$-functions; see [8, 10-15].

Theorem 1.1 (see [8, Theorem 2.4]). *Let $\lambda > 0$ and let $\mu : [0, \infty) \to [0, \lambda)$ be a function. Then the following statements are equivalent.*

(1) *μ is an $\mathcal{MT}(\lambda)$-function.*

(2) *$\lambda^{-1}\mu$ is an $\mathcal{MT}$-function.*

(3) *For each $t \in [0, \infty)$, there exist $\xi_t^{(1)} \in [0, \lambda)$ and $\epsilon_t^{(1)} > 0$ such that $\mu(s) \leq \xi_t^{(1)}$ for all $s \in (t, t + \epsilon_t^{(1)})$.*

(4) *For each $t \in [0, \infty)$, there exist $\xi_t^{(2)} \in [0, \lambda)$ and $\epsilon_t^{(2)} > 0$ such that $\mu(s) \leq \xi_t^{(2)}$ for all $s \in [t, t + \epsilon_t^{(2)}]$.*

(5) *For each* $t \in [0, \infty)$*, there exist* $\xi_t^{(3)} \in [0, \lambda)$ *and* $\epsilon_t^{(3)} > 0$ *such that* $\mu(s) \leq \xi_t^{(3)}$ *for all* $s \in (t, t + \epsilon_t^{(3)}]$.

(6) *For each* $t \in [0, \infty)$*, there exist* $\xi_t^{(4)} \in [0, \lambda)$ *and* $\epsilon_t^{(4)} > 0$ *such that* $\mu(s) \leq \xi_t^{(4)}$ *for all* $s \in [t, t + \epsilon_t^{(4)})$.

(7) *For any nonincreasing sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* $[0, \infty)$*, we have* $0 \leq \sup_{n\in\mathbb{N}} \mu(x_n) < \lambda$.

(8) *For any strictly decreasing sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* $[0, \infty)$*, we have* $0 \leq \sup_{n\in\mathbb{N}} \mu(x_n) < \lambda$.

(9) *For any eventually nonincreasing sequence* $\{x_n\}_{n\in\mathbb{N}}$ *(i.e. there exists* $\ell \in \mathbb{N}$ *such that* $x_{n+1} \leq x_n$ *for all* $n \in \mathbb{N}$ *with* $n \geq \ell$*) in* $[0, \infty)$*, we have* $0 \leq \sup_{n\in\mathbb{N}} \mu(x_n) < \lambda$.

(10) *For any eventually strictly decreasing sequence* $\{x_n\}_{n\in\mathbb{N}}$ *(i.e. there exists* $\ell \in \mathbb{N}$ *such that* $x_{n+1} < x_n$ *for all* $n \in \mathbb{N}$ *with* $n \geq \ell$*) in* $[0, \infty)$*, we have* $0 \leq \sup_{n\in\mathbb{N}} \mu(x_n) < \lambda$.

In this paper, we establish new convergence theorems and best proximity point theorems for a pair of nonlinear non-self mappings in metric spaces.

2. Main results

Recently, Du introduced the concept of approximate sequence [15] as follows.

Definition 2.1 [15]. Let A and B be nonempty subsets of a metric space (X, d) and $\tau : [0, \infty) \to [0, 1)$ be a function. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in $A \cup B$ is said to be *approximate with respect to* τ, if the following conditions are satisfied:

(a) one of the following conditions holds:

  (i) $\{x_{2n-1}\}_{n\in\mathbb{N}} \subset A$ and $\{x_{2n}\}_{n\in\mathbb{N}} \subset B$;

  (ii) $\{x_{2n}\}_{n\in\mathbb{N}} \subset A$ and $\{x_{2n-1}\}_{n\in\mathbb{N}} \subset B$,

(b) $d(x_{n+1}, x_{n+2}) \leq \tau(d(x_n, x_{n+1}))d(x_n, x_{n+1}) + (1 - \tau(d(x_n, x_{n+1})))dist(A, B)$ for all $n \in \mathbb{N}$.

By applying Theorem 1.1, Du established the following convergence theorem for approximate sequences in [15].

Theorem 2.1 [15]. *Let A and B be nonempty subsets of a metric space (X,d) and $\varphi : [0,\infty) \to [0,1)$ be an $\mathcal{MT}$-function. If $\{x_n\}_{n\in\mathbb{N}} \subset A \cup B$ is an approximate sequence with respect to φ, then $\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = dist(A,B)$.*

In this section, we first establish an existence theorems of approximate sequences for a pair of nonlinear non-self mappings.

Theorem 2.2. *Let A and B be nonempty subsets of a metric space (X,d) and $F : A \to B$ and $G : B \to A$ be two mappings. Suppose that*

(DY) *there exists a funtion $\varphi : [0,\infty] \to [0,1)$ such that*

$$d(Fx,Gy) \le \frac{1}{3}\varphi\left(d\left(x,y\right)\right)\max\{d\left(x,y\right)+d\left(Fx,y\right), d(x,Fx)+d\left(y,Gy\right) + d\left(Fx,Gy\right)\} + \left(1-\varphi\left(d\left(x,y\right)\right)\right) dist\left(A,B\right)$$

for all $x \in A$ and $y \in B$.

Then there exists an approximate sequence in $A \cup B$ with respect to φ.

Proof. Let $x_1 \in A$ be given. Define $x_{2n} = Fx_{2n-1}$ and $x_{2n+1} = Gx_{2n}$ for each $n \in \mathbb{N}$. Then $\{x_{2n-1}\}_{n\in\mathbb{N}} \subset A$ and $\{x_{2n}\}_{n\in\mathbb{N}} \subset B$. We want to show that the sequence $\{x_n\}_{n\in\mathbb{N}}$ in $A \cup B$ satisfies the following:

$$d(x_{n+1},x_{n+2}) \le \varphi(d(x_n,x_{n+1}))d(x_n,x_{n+1}) + (1-\varphi(d(x_n,x_{n+1})))dist(A,B) \tag{2.1}$$

for all $n \in \mathbb{N}$. If $\varphi\left(d\left(x_1,x_2\right)\right) = 0$, then, by (DY), we have

$$d(x_2,x_3) \le dist(A,B) \ \le \ d\left(x_1,x_2\right),$$

and hence

$$\begin{aligned} d(x_2,x_3) &\le dist\left(A,B\right) \\ &= \varphi\left(d\left(x_1,x_2\right)\right) d\left(x_1,x_2\right) + \left(1-\varphi\left(d\left(x_1,x_2\right)\right)\right)dist(A,B). \end{aligned}$$

Suppose that $\varphi\left(d\left(x_1,x_2\right)\right) > 0$. From (DY), we get

$$\begin{aligned} d(x_2,x_3) &\le \frac{1}{3}\varphi\left(d\left(x_1,x_2\right)\right)\max\left\{d\left(x_1,x_2\right), d\left(x_1,x_2\right)+2d\left(x_2,x_3\right)\right\} \\ &\quad + (1-\varphi(d(x_1,x_2)))dist(A,B) \qquad (2.2) \\ &= \frac{1}{3}\varphi\left(d\left(x_1,x_2\right)\right)\left[d\left(x_1,x_2\right)+2d\left(x_2,x_3\right)\right] + (1-\varphi(d\left(x_1,x_2\right)))dist(A,B). \end{aligned}$$

We assume that

$$d\left(x_2, x_3\right) > d(x_1, x_2). \tag{2.3}$$

Then inequalities (2.2) and (2.3) deduce

$$d(x_2, x_3) < \varphi\left(d\left(x_1, x_2\right)\right) d\left(x_2, x_3\right) + (1 - \varphi(d(x_1, x_2)))d(x_2, x_3) = d(x_2, x_3)$$

which leads to a contradiction. So it must be

$$d\left(x_2, x_3\right) \leq d(x_1, x_2) \tag{2.4}$$

So, by (2.2) and (2.4), we obtain

$$\begin{aligned} d(x_2, x_3) &\leq \frac{1}{3}\varphi\left(d\left(x_1, x_2\right)\right)\left\{d\left(x_1, x_2\right) + 2d\left(x_2, x_3\right)\right\} + (1 - \varphi(d(x_1, x_2)))dist(A, B) \\ &\leq \varphi\left(d\left(x_1, x_2\right)\right) d\left(x_1, x_2\right) + (1 - \varphi\left(d\left(x_1, x_2\right)\right))dist(A, B). \end{aligned}$$

Hence (2.1) holds for $n = 1$. Next, we claim that (2.1) holds for $n = 2$. If $\varphi\left(d\left(x_2, x_3\right)\right) = 0$, then, by (2.1) again, we obtain

$$d(x_4, x_3) \leq dist(A, B) \leq \ d\left(x_3, x_2\right),$$

and

$$d(x_4, x_3) \leq \varphi\left(d\left(x_3, x_2\right)\right) d\left(x_3, x_2\right) + (1 - \varphi\left(d\left(x_3, x_2\right)\right))dist(A, B).$$

Assume that $\varphi\left(d\left(x_2, x_3\right)\right) > 0$. By (DY), we get

$$\begin{aligned} d(x_4, x_3) \leq \frac{1}{3}\varphi\left(d\left(x_3, x_2\right)\right) \max\left\{d\left(x_4, x_3\right) + 2d\left(x_3, x_2\right), d\left(x_2, x_3\right) + 2d\left(x_4, x_3\right)\right\} \\ + (1 - \varphi(d(x_3, x_2)))dist(A, B). \end{aligned} \tag{2.5}$$

Suppose that

$$d\left(x_2, x_3\right) + 2d\left(x_4, x_3\right) > d\left(x_4, x_3\right) + 2d\left(x_3, x_2\right).$$

Then we obtain

$$d\left(x_4, x_3\right) > d\left(x_3, x_2\right). \tag{2.6}$$

Thus inequalities (2.5) and (2.6) imply

$$d(x_3, x_2) < \varphi\left(d\left(x_2, x_1\right)\right) d\left(x_3, x_2\right) + (1 - \varphi(d(x_2, x_1)))d\left(x_3, x_2\right) = d\left(x_3, x_2\right)$$

which is a contradiction. Hence it must be

$$d\left(x_4, x_3\right) \leq d\left(x_3, x_2\right). \tag{2.7}$$

Using (2.5) and (2.7), we obtain

$$\begin{aligned} d(x_4, x_3) \le & \frac{1}{3}\varphi\left(d\left(x_3, x_2\right)\right) \max\left\{d\left(x_4, x_3\right) + 2d\left(x_3, x_2\right), d\left(x_2, x_3\right) + 2d\left(x_4, x_3\right)\right\} \\ & + (1-\varphi(d(x_3, x_2)))dist(A, B). \\ \le & \varphi\left(d\left(x_3, x_2\right)\right) d(x_3, x_2) + (1-\varphi\left(d\left(x_3, x_2\right)\right))dist(A, B). \end{aligned}$$

So (2.1) is true for $n = 2$. By induction, we prove that our claim (2.1) holds for $n \in \mathbb{N}$. Hence, we show that $\{x_n\}_{n\in\mathbb{N}}$ is an approximate sequence in $A \cup B$ with respect to φ. □

Theorem 2.3. *Let A and B be nonempty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic mapping. Suppose that*

(H) *there exists a funtion $\varphi : [0, \infty] \to [0, 1)$ such that*

$$\begin{aligned} d(Tx, Ty) \le & \frac{1}{3}\varphi\left(d\left(x, y\right)\right) \max\{d\left(x, y\right) + d\left(Tx, y\right), d(x, Tx) + d\left(y, Ty\right) \\ & + d\left(Tx, Ty\right)\} + \left(1-\varphi\left(d\left(x, y\right)\right)\right) dist\left(A, B\right) \end{aligned}$$

for all $x \in A$ and $y \in B$.

Then there exists an approximate sequence in $A \cup B$ with respect to φ.

Proof. Since T is cyclic, we have $T(A) \subset B$ and $T(B) \subset A$. So we can define two mapping $F : A \to B$ *and* $G : B \to A$ by $F := T\mid_A$ and $G := T\mid_B$. It is easy to see that the condition (H) implies the condition (DY). Hence, the conclusion is immediate from Theorem 2.2. □

Applying Theorems 2.1 and 2.2, we establish immediately the following new convergence theorem.

Theorem 2.4. *Let A and B be nonempty subsets of a metric space (X, d) and $F : A \to B$ and $G : B \to A$ be two mappings. Suppose that there exists an $\mathcal{MT}$-funtion $\varphi : [0, \infty] \to [0, 1)$ such that*

$$\begin{aligned} d(Fx, Gy) \le & \frac{1}{3}\varphi\left(d\left(x, y\right)\right) \max\{d\left(x, y\right) + d\left(Fx, y\right), d(x, Fx) + d\left(y, Gy\right) \\ & + d\left(Fx, Gy\right)\} + \left(1-\varphi\left(d\left(x, y\right)\right)\right) dist\left(A, B\right) \end{aligned}$$

for all $x \in A$ and $y \in B$. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset A \cup B$ such that

(*a*) $d(x_{n+1}, x_{n+2}) \le \varphi(d(x_n, x_{n+1}))d(x_n, x_{n+1}) + [1-\varphi(d(x_n, x_{n+1}))]dist(A, B)$ for all $n \in \mathbb{N}$;

(b) $\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = dist(A, B)$.

Corollary 2.1. *Let A and B be nonempty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic mapping. Suppose that there exists an $\mathcal{MT}$-funtion $\varphi : [0, \infty] \to [0, 1)$ such that*

$$d(Fx, Gy) \leq \frac{1}{3}\varphi\left(d\left(x, y\right)\right)\left[d\left(x, y\right) + d\left(Fx, y\right)\right] + \left(1 - \varphi\left(d\left(x, y\right)\right)\right) dist\left(A, B\right)$$

for all $x \in A$ and $y \in B$. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset A \cup B$ such that

(a) $d(x_{n+1}, x_{n+2}) \leq \varphi(d(x_n, x_{n+1}))d(x_n, x_{n+1}) + [1-\varphi(d(x_n, x_{n+1}))]dist(A, B)$ for all $n \in \mathbb{N}$;

(b) $\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = dist(A, B)$.

Corollary 2.2. *Let A and B be nonempty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic mapping. Suppose that there exists an $\mathcal{MT}$-funtion $\varphi : [0, \infty] \to [0, 1)$ such that*

$$\begin{aligned} d(Fx, Gy) \leq{} & \frac{1}{3}\varphi\left(d\left(x, y\right)\right)\left[d(x, Fx) + d\left(y, Gy\right) + d\left(Fx, Gy\right)\right] \\ & + \left(1 - \varphi\left(d\left(x, y\right)\right)\right) dist\left(A, B\right) \end{aligned}$$

for all $x \in A$ and $y \in B$. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset A \cup B$ such that

(a) $d(x_{n+1}, x_{n+2}) \leq \varphi(d(x_n, x_{n+1}))d(x_n, x_{n+1}) + [1-\varphi(d(x_n, x_{n+1}))]dist(A, B)$ for all $n \in \mathbb{N}$;

(b) $\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = dist(A, B)$.

Some new existence theorems for best proximity points can be established by applying Theorem 2.4.

Theorem 2.5. *In Theorem 2.4, if we further assume*

(C) $d(Fx, Gy) \leq d(x, y)$ *for any $x \in A$ and $y \in B$.*

Then the following statements hold.

(a) *If* $\{x_{2n-1}\}_{n\in\mathbb{N}}$ *has a convergent subsequence in* A*, then there exists* $u \in A$ *such that* $d(u, Fu) = dist(A, B)$*.*

(b) *If* $\{x_{2n}\}_{n\in\mathbb{N}}$ *has a convergent subsequence in* B*, then there exists* $v \in B$ *such that* $d(v, Gv) = dist(A, B)$*.*

Proof. Applying Theorem 2.4 and following a similar argument as the proof of [15, Theorem 2.5] , we can prove the conclusion. □

Remark 2.1. As a direct application of Theorem 2.5, we can obtain easily some new (common) fixed point theorems.

Corollary 2.3. *Let* A *and* B *be nonempty subsets of a metric space* (X, d) *and* $T : A \cup B \to A \cup B$ *be a cyclic mapping. Suppose that*

(i) $d(Tx, Ty) \leq d(x, y)$ *for any* $x \in A$ *and* $y \in B$*.*

(ii) *there exists an* $\mathcal{MT}$*-funtion* $\varphi : [0, \infty] \to [0, 1)$ *such that*

$$d(Tx, Ty) \leq \frac{1}{3}\varphi\left(d\left(x, y\right)\right) \max\{d\left(x, y\right) + d\left(Tx, y\right), d(x, Tx) + d\left(y, Ty\right) + d\left(Tx, Ty\right)\} + \left(1 - \varphi\left(d\left(x, y\right)\right)\right) dist\left(A, B\right)$$

for all $x \in A$ *and* $y \in B$*.*

Then the following statements hold.

(a) *If* $\{x_{2n-1}\}_{n\in\mathbb{N}}$ *has a convergent subsequence in* A*, then there exists* $u \in A$ *such that* $d(u, Tu) = dist(A, B)$*.*

(b) *If* $\{x_{2n}\}_{n\in\mathbb{N}}$ *has a convergent subsequence in* B*, then there exists* $v \in B$ *such that* $d(v, Tv) = dist(A, B)$*.*

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