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Hadamard and Caputo-Hadamard FDE's with Three Point Integral Boundary Conditions

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Abstract

In this article, based on Banach and Schauder's fixed point theorems, the existence and uniqueness conditions of nonlinear (Hadamard (H) / Caputo-Hadamard (CH)) factional differential equations of order $1 < \alpha \leq 2$ with three point integral boundary conditions are obtained. Some examples are introduced to illustrate the applicability of the obtained results.

Mathematics Subject Classification: 34A08, 34A60, 34B10, 34B15

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1 Introduction

One of the important characteristics of fractional operators is their nonlocal nature. accounting for the hereditary properties of many phenomena and processes involved.

During the last decades, fractional calculus has gained remarkable importance due to the applications in almost all applied sciences. It should be pointed that fractional operators are more convenient for describing some physical problems. On the other hand most of the works on fractional differential equations are based on Riemann-Liouville and Caputo type fractional operators. Another kind of fractional operators that appears in the literature is the fractional derivative due to Hadamard. Hadamard derivative differs from the preceding ones in the sense that the kernel of the integral contains a logarithmic function of arbitrary exponent. Details and properties of the Hadamard fractional derivative and integral can be found in [11], [8], [9], [10], [12], [13]. However, differential equations with Hadamard derivatives is still studied less than that of Riemann-Liouville and Caputo fracional differential equations, see [7]-[18], and references therein.

In this paper, we propose the existence and uniqueness of solutions for nonlinear (Hadamard (H) / Caputo-Hadamard (CH)) factional differential equations of order $1 < q \leq 2$ with three point integral boundary conditions of the form.

- Nonlinear Hadamard fractional differential equations

$$^{H}D^{q}u\left(t\right)=g\left(t,u\left(t\right)\right),\quad 1<q\leq 2,\quad 0<a\leq t\leq T, \tag{1}$$

  associated with three point integral boundary conditions

$$u\left(a\right)=0,\quad u\left(T\right)=\beta\int_{a}^{\eta}u\left(s\right)ds,\quad a<\eta<T,\ \beta\in R, \tag{2}$$

and

- Nonlinear Caputo-Hadamard fractional differential equations

$$^{CH}D^{q}x\left(t\right)=g\left(t,x\left(t\right)\right),\quad 1<q\leq 2,\quad 0<a\leq t\leq T, \tag{3}$$

  associated with three point integral boundary conditions

$$x\left(a\right)=0,\quad x\left(T\right)=\beta\int_{a}^{\eta}x\left(s\right)ds,\quad a<\eta<T,\ \beta\in R. \tag{4}$$

Here $^{H}D^{q}$, $^{CH}D^{q}$ denote the Hadamard and the Caputo-Hadamard fractional derivatives of order $1 < q \leq 2$, respectively, $g : [a, T] \times R \to R$ is a continuous function, β is a given constant.

2 Preliminaries

In this section we introduce some definitions, lemmas and notations of fractional calculus

Definition 1 *The Hadamard fractional integral of order q for a continuous function $g : [a, \infty) \to R$ is defined as*

$$^{H}I^{q}g\left(t\right)=\frac{1}{\Gamma\left(q\right)}\int_{a}^{t}\left(\ln\frac{t}{s}\right)^{q-1}\frac{g\left(s\right)}{s}ds,\quad q>0,$$

provided that the integral exist.

Definition 2 *The Hadamard fractional derivative of order q for a continuous function $g:[a,\infty)\rightarrow R$ is defined as*

$$^{H}D^{q}g\left(t\right)=\delta^{n}\left(^{H}I^{q}g\right)\left(t\right)=\left(t\frac{d}{dt}\right)^{n}\frac{1}{\Gamma\left(n-q\right)}\int_{a}^{t}\left(\ln\frac{t}{s}\right)^{n-q-1}\frac{g\left(s\right)}{s}ds,$$

$n-1<q<n$, $n=[q]+1$, $\delta=t\frac{d}{dt}$, $[q]$ *denotes the integer part of* q.

Definition 3 *For at least n-times differentiable function $g:[a,\infty)\rightarrow R$, the Caputo-Hadamard fractional derivative of order q is defined as*

$$^{CH}D^{q}g\left(t\right)=\frac{1}{\Gamma\left(n-q\right)}\int_{a}^{t}\left(\ln\frac{t}{s}\right)^{n-q-1}\delta^{n}\frac{g\left(s\right)}{s}ds.$$

Lemma 4 *Let $u,x\in C_{\delta}^{n}\left(\left[a,T\right],R\right)$. Then*

$$^{H}I^{q}\left(^{H}D^{q}u\right)\left(t\right)=u\left(t\right)-\sum_{j=1}^{n}c_{j}\left(\ln\frac{t}{a}\right)^{n-j},$$

$$^{H}I^{q}\left(^{CH}D^{q}u\right)\left(t\right)=u\left(t\right)-\sum_{j=0}^{n-1}c_{j}\left(\ln\frac{t}{a}\right)^{j}.$$

Here $C_{\delta}^{n}\left(\left[a,T\right],R\right)=\left\{u:\left[a,T\right]\rightarrow R:\delta^{n-1}u\in C\left(\left[a,T\right],R\right)\right\}.$

Lemma 5 *Let $h\in C\left(\left[a,T\right],R\right)$ and $u\in C_{\delta}^{2}\left(\left[a,T\right],R\right)$. The linear Hadamard F.D.E given by*

$$\begin{cases} {}^{H}D^{q}u\left(t\right)=h\left(t\right), \quad 1<q\leq 2, \quad 0<a\leq t\leq T, \\ u\left(a\right)=0, \quad u\left(T\right)=\beta\int_{a}^{\eta}u\left(s\right)ds, \quad a<\eta<T, \ \beta\in R, \end{cases} \tag{5}$$

is equivalent to the integral equation given by:

$$u\left(t\right)=\frac{1}{\Gamma\left(q\right)}\int_{a}^{t}\left(\ln\frac{t}{s}\right)^{q-1}\frac{h\left(s\right)}{s}ds+\frac{\left(\ln\frac{t}{a}\right)^{q-1}}{\Gamma\left(q\right)\left(\left(\ln\frac{T}{a}\right)^{q-1}-\beta A_{1}\right)}$$
$$\times\left(\beta\int_{a}^{\eta}\int_{a}^{s}\left(\ln\frac{s}{r}\right)^{q-1}\frac{h\left(r\right)}{r}drds-\int_{a}^{T}\left(\ln\frac{T}{s}\right)^{q-1}\frac{h\left(s\right)}{s}ds\right), \tag{6}$$

where $A_{1}=\int_{a}^{\eta}\left(\ln\frac{s}{a}\right)^{q-1}ds$.

Proof. We consider the Hadamard case, applying $^{H}I^{q}$ to both sides of (5)

$$u\left(t\right)={}^{H}I^{q}h\left(t\right)+c_{1}\left(\ln\frac{t}{a}\right)^{q-1}+c_{2}\left(\ln\frac{t}{a}\right)^{q-2}.$$

The first boundary condition $u(0) = 0$, implies that $c_2 = 0$. Then

$$u(t) = {}^{H}I^{q}h(t) + c_1 \left(\ln \frac{t}{a}\right)^{q-1}. \tag{7}$$

The second boundary condition $u(T) = \beta \int_a^\eta u(s)\,ds$, implies

$${}^{H}I^{q}h(T) + c_1 \left(\ln \frac{T}{a}\right)^{q-1} = \beta \int_a^\eta {}^{H}I^{q}h(s)\,ds + \beta c_1 \int_a^\eta \left(\ln \frac{s}{a}\right)^{q-1} d,$$

$$c_1 = \frac{1}{\left(\ln \frac{T}{a}\right)^{q-1} - \beta A_1} \left(\beta \int_a^\eta {}^{H}I^{q}h(s)\,ds - {}^{H}I^{q}h(T)\right). \tag{8}$$

Substitute (8) in (7) and expand the Hadamard fractional integrals we get equation (6). By direct computation the converse statement can be easily obtained. This completes the proof. ■

Lemma 6 *Let $h \in C([a,T], R)$ and $u \in C_\delta^2([a,T], R)$. The linear Caputo-Hadamard F.D.E given by*

$$\begin{cases} {}^{CH}D^{q}u(t) = h(t), \quad 1 < q \le 2, \quad 0 < a \le t \le T, \\ u(a) = 0, \quad u(T) = \beta \int_a^\eta u(s)\,ds, \quad a < \eta < T, \ \beta \in R, \end{cases} \tag{9}$$

is equivalent to the integral equation given by:

$$u(t) = \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s}\right)^{q-1} \frac{h(s)}{s} ds + \frac{\ln \frac{t}{a}}{\Gamma(q)\left(\ln \frac{T}{a} - \beta A_2\right)}$$
$$\times \left(\beta \int_a^\eta \int_a^s \left(\ln \frac{s}{r}\right)^{q-1} \frac{h(r)}{r} dr ds - \int_a^T \left(\ln \frac{T}{s}\right)^{q-1} \frac{h(s)}{s} ds\right), \tag{10}$$

where $A_2 = \eta\left(\ln \frac{\eta}{a} - 1\right) + a$.

3 Existence results for the problem (1)-(2)

We define an operators $G_H : C([a,T], R) \to C([a,T], R)$ as follows

$$(G_H u)(t) = \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s}\right)^{q-1} \frac{g(s,u(s))}{s} ds + \frac{\left(\ln \frac{t}{a}\right)^{q-1}}{\Gamma(q)\left(\left(\ln \frac{T}{a}\right)^{q-1} - \beta A_1\right)}$$
$$\times \left(\beta \int_a^\eta \int_a^s \left(\ln \frac{s}{r}\right)^{q-1} \frac{g(r,u(r))}{r} dr ds - \int_a^T \left(\ln \frac{T}{s}\right)^{q-1} \frac{g(s,u(s))}{s} ds\right).$$

It should be noticed that problems (1)-(2) has solution if and only if the operators G_H has fixed point.

For the sake of convenience, we set

$$Q = \frac{1}{\Gamma(q+1)}\left(\ln\frac{T}{a}\right)^{q} + \frac{\left(\ln\frac{T}{a}\right)^{2q-1}\left(|\beta|(\eta-a)-1\right)}{\Gamma(q+1)\left|\left(\ln\frac{T}{a}\right)^{q-1}-\beta A_1\right|},$$

$$Q^{*} = \frac{\left(\ln\frac{T}{a}\right)^{2q-1}\left(|\beta|(\eta-a)-1\right)}{\Gamma(q+1)\left|\left(\ln\frac{T}{a}\right)^{q-1}-\beta A_1\right|}.$$

Theorem 7 *Let $g:[a,T]\times R\to R$ be a continuous function satifying*

(A1) there exists $L_g>0$ such that $|g(t,u)-g(t,v)|\le L_g|u-v|,\ t\in[a,T],\ u,v\in R$.

Then the problem (1)-(2) has a unique solution on $[a,T]$ provided that $QL_g<1$.

Proof. Consider the set $\omega_r := \{u\in C([a,T],R) : \|u\|\le r\}$ with $r\ge \frac{M_gQ}{1-LQ}$, where $M_g=\sup_{a\le t\le T}|g(t,0)|$. It is clear that

$$|g(s,u(s))|\le L_g r+M_g,\quad u\in\omega_r.$$

First we show that $G_H\omega_r\subset\omega_r$. For $u\in\omega_r$, $t\in[a,T]$ we have

$$|(G_Hu)(t)| \le \frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}\frac{|g(s,u(s))|}{s}ds + \frac{\left(\ln\frac{t}{a}\right)^{q-1}}{\Gamma(q)\left|\left(\ln\frac{T}{a}\right)^{q-1}-\beta A_1\right|}$$
$$\times\left(|\beta|\int_a^{\eta}\int_a^{s}\left(\ln\frac{s}{r}\right)^{q-1}\frac{|g(r,u(r))|}{r}drds + \int_a^T\left(\ln\frac{T}{s}\right)^{q-1}\frac{|g(s,u(s))|}{s}ds\right)$$
$$\le (L_g r+M_g)Q\le r,$$

whcih implies that $G_Hu\in\omega_r$. Next we show that the operator G_H is a contraction. Indeed, for any $u,v\in\omega_r$ we have

$$|(G_Hu)(t)-(G_Hv)(t)|$$
$$\le \frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}\frac{|g(s,u(s))-g(s,v(s))|}{s}ds + \frac{\left(\ln\frac{t}{a}\right)^{q-1}}{\Gamma(q)\left|\left(\ln\frac{T}{a}\right)^{q-1}-\beta A_1\right|}$$
$$\times\left(|\beta|\int_a^{\eta}\int_a^{s}\left(\ln\frac{s}{r}\right)^{q-1}\frac{|g(r,u(r))-g(r,v(r))|}{r}drds\right.$$
$$\left.+\int_a^T\left(\ln\frac{T}{s}\right)^{q-1}\frac{|g(s,u(s))-g(s,v(s))|}{s}ds\right)$$
$$\le L_gQ\|u-v\|.$$

Thus G_H is a contraction and by the Banach contraction mapping theorem the B.V.P has a unique solution on $[a,T]$. This completes the proof. ■

Theorem 8 *Let $g:[a,T]\times R\to R$ be a continuous function satifying (A1) and*

(A2) $|g(t,u)|\le y(t)\,,\ (t,u)\in[a,T]\times R,\ y\in C([a,T],R^{+})\,.$

If $L_gQ^{*}<1$, then there is at least one solution for BVP (1)-(2) on $[a,T]$.

Proof. We define the operators G_H^1 and G_H^2 on ω_r, with $r\ge Q\|y\|$, as follows:

$$\left(G_H^1u\right)(t)=\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}\frac{g(s,u(s))}{s}ds,$$

$$\left(G_H^2u\right)(t)=\frac{\left(\ln\frac{t}{a}\right)^{q-1}}{\Gamma(q)\left(\left(\ln\frac{T}{a}\right)^{q-1}-\beta A_1\right)}$$

$$\times\left(\beta\int_a^\eta\int_a^s\left(\ln\frac{s}{r}\right)^{q-1}\frac{g(r,u(r))}{r}drds-\int_a^T\left(\ln\frac{T}{s}\right)^{q-1}\frac{g(s,u(s))}{s}ds\right)$$

For $u,v\in\omega_r$ we have

$$\left\|G_H^1u+G_H^2v\right\|\le Q\|y\|\le r,$$

that is $G_H^1u+G_H^2v\in\omega_r$. By our assumptions, one can easily show that

$$\left\|G_H^2u-G_H^2v\right\|\le L_gQ^{*}\|u-v\|\,,$$

which implies that G_H^2 is a contraction.

In addition, the operator G_H^1 is continuous as a result of the continuity of g. Also, it is uniformly bounded as

$$\left|\left(G_H^1u\right)(t)\right|\le\frac{\|y\|}{\Gamma(q+1)}\left(\ln\frac{T}{s}\right)^{q}.$$

Moreover, for $t_1,t_2\in[a,T]$, $t_1<t_2$ we have

$$\begin{aligned}&\left|\left(G_H^1u\right)(t_1)-\left(G_H^1u\right)(t_2)\right|\\&\le\frac{\sup\{|g(t,u)|:t\in[a,T]\,,\ |u|\le r\}}{\Gamma(q)}\\&\times\left(\int_a^{t_1}\frac{1}{s}\left(\left(\ln\frac{t_2}{r}\right)^{q-1}-\left(\ln\frac{t_1}{r}\right)^{q-1}\right)ds+\int_{t_1}^{t_2}\frac{1}{s}\left(\ln\frac{t_2}{s}\right)^{q-1}ds\right)\end{aligned}$$

approaches zero as $t_2\to t_1$. Note that $|(G_H^1u)(t_1)-(G_H^1u)(t_2)|$ is independent of u implies that G_H^1 is relatively compact, by Arzela-Ascoli theorem we conclude that G_H^1 is compact. Hence, the existence of the solution of the B.V.P holds by Krasnoselskii's fixed point theorem. ■

Next we use Leray –Shauder alternative to prove the existence of the solution of the BVP.

Theorem 9 *Let $g : [a,T] \times R \to R$ be a continuous function satifying (A1) and*

(A3) there exists a function $w \in C([a,T], R^{+})$ and a nondecreasing function $\mu : R^{+} \to R^{+}$ such that $|g(t,u)| \leq w(t)\mu(|u|)$, $(t,u) \in [a,T] \times R$,

(A4) there exists a constant $M > 0$ such that $M > \|w\| \mu(M) Q$.
Then there is at least one solution for BVP (1)-(2) on $[a,T]$.

Proof. First step we show that the operator maps bounded sets into bounded sets of .

$$|(G_H u)(t)| \leq \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s}\right)^{q-1} \frac{w(s)\mu(\|u\|)}{s} ds + \frac{\left(\ln \frac{t}{a}\right)^{q-1}}{\Gamma(q)\left|\left(\ln \frac{T}{a}\right)^{q-1} - \beta A_1\right|}$$
$$\times \left(|\beta| \int_a^\eta \int_a^s \left(\ln \frac{s}{r}\right)^{q-1} \frac{w(r)\mu(\|u\|)}{r} dr ds + \int_a^T \left(\ln \frac{T}{s}\right)^{q-1} \frac{w(s)\mu(\|u\|)}{s} ds\right)$$
$$\leq \|w\| \mu(r) Q$$

■

Proof. Next we show that the operator maps bounded sets into equicontinuous sets of .

for $t_1, t_2 \in [a,T]$, $t_1 < t_2$ we have

$$\left|\left(G_H^1 u\right)(t_1) - \left(G_H^1 u\right)(t_2)\right|$$
$$\leq \frac{\|w\| \mu(r)}{\Gamma(q)}$$
$$\times \left(\int_a^{t_1} \frac{1}{s}\left(\left(\ln \frac{t_2}{r}\right)^{q-1} - \left(\ln \frac{t_1}{r}\right)^{q-1}\right) ds + \int_{t_1}^{t_2} \frac{1}{s}\left(\ln \frac{t_2}{s}\right)^{q-1} ds\right)$$
$$+ \frac{\left(\ln \frac{t_2}{a}\right)^{q-1} - \left(\ln \frac{t_1}{a}\right)^{q-1}}{\Gamma(q)\left|\left(\ln \frac{T}{a}\right)^{q-1} - \beta A_1\right|}$$
$$\times \left(|\beta| \int_a^\eta \int_a^s \left(\ln \frac{s}{r}\right)^{q-1} \frac{w(r)\mu(\|u\|)}{r} dr ds + \int_a^T \left(\ln \frac{T}{s}\right)^{q-1} \frac{w(s)\mu(\|u\|)}{s} ds\right)$$

approaches zero as $t_2 \to t_1$. Note that the right hand side of the above inequality is independent of u,by Arzela-Ascoli theorem we conclude that is completely continuous.

The last step to complete the assumptions of Leray-Schauder nonlinear alternative theorem is to show the boundedness of the set of all solution to equation $u = \delta G_H u$, $0 \le \delta \le 1$. Assume that u is a solution, then by same manner as before we show the operator is bounded:

$$|u(t)| = \delta |(G_H u)(t)| \le \|w\| \mu(\|u\|) Q,$$
$$\frac{\|u\|}{\|w\| \mu(\|u\|) Q} \le 1.$$

But by (A4) there exists a constant $M > 0$ such that $M \ne \|u\|$. Construct the set $\Omega = \{u \in C([a,T], R) : \|u\| < M\}$, it is obvious that the operator $G_H : \overline{\Omega} \to C([a,T], R)$ is continuous and completely continuous. By the constructed Ω, there is no $u \in \partial\Omega$ such that $u = \delta G_H u$ for some $0 < \delta < 1$. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that G_H has a fixed point $u \in \overline{\Omega}$ which is a solution of the BVP. This completes the proof. ■

4 Existence results for the problem (3)-(4)

In this section, some existence results are introduced for the problem (3)-(4). The proofs are omitted since they are similar to the one employed in the previous section. We define a fixed point operator $G_{CH} : C([a,T], R) \to C([a,T], R)$ associated with the problem (3)-(4) as follows

$$(G_{CH}u)(t) = \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s}\right)^{q-1} \frac{g(s,u(s))}{s} ds + \frac{\ln \frac{t}{a}}{\Gamma(q)\left(\ln \frac{T}{a} - \beta A_2\right)}$$
$$\left(\beta \int_a^\eta \int_a^s \left(\ln \frac{s}{r}\right)^{q-1} \frac{g(r,u(r))}{r} dr ds - \int_a^T \left(\ln \frac{T}{s}\right)^{q-1} \frac{g(s,u(s))}{s} ds\right).$$

For computational convenience, we let

$$R = \frac{1}{\Gamma(q+1)} \left(\ln \frac{T}{a}\right)^q + \frac{\left(\ln \frac{T}{a}\right)^{q+1} (|\beta| (\eta - a) - 1)}{\Gamma(q+1) \left|\ln \frac{T}{a} - \beta A_2\right|},$$
$$R^* = \frac{\left(\ln \frac{T}{a}\right)^{q+1} (|\beta| (\eta - a) - 1)}{\Gamma(q+1) \left|\ln \frac{T}{a} - \beta A_2\right|}.$$

Using the method of proof for the theorems obtained by the previous section and the operator we can introduce the following theorems.

Theorem 10 *Let $g : [a,T] \times R \to R$ be a continuous function satifying*

(A1) there exists $L_g > 0$ such that $|g(t,u) - g(t,v)| \le L_g |u - v|$, $t \in [a,T]$, $u, v \in R$.
Then the problem (3)-(4) has a unique solution on $[a,T]$ provided that $RL_g < 1$.

Theorem 11 *Let $g : [a,T] \times R \to R$ be a continuous function satifying (A1) and*

(A2) $|g(t,u)| \le y(t)\,,\ (t,u) \in [a,T] \times R,\ y \in C([a,T], R^+)\,.$
If $L_g R^* < 1$, then there is at least one solution for BVP (1)-(2) on $[a,T]$.

Theorem 12 *Let $g : [a,T] \times R \to R$ be a continuous function satifying (A1) and*

(A3) there exists a function $w \in C([a,T], R^+)$ and a nondecreasing function $\mu : R^+ \to R^+$ such that $|g(t,u)| \le w(t)\,\mu(|u|)\,,\ (t,u) \in [a,T] \times R,$

(A4) there exists a constant $M > 0$ such that $M > \|w\|\,\mu(M)\,R.$
Then there is at least one solution for BVP (1)-(2) on $[a,T]$.

5 Examples

Example 1: Consider the following non linear fractional Hadamard differential equation

$$\begin{cases} {}^{H}D^{3/2}u(t) = g(t,u(t))\,, \quad 1 \le t \le e, \\ u(1) = 0, \quad u(e) = \beta \int_a^{\eta} u(s)\,ds, \quad \eta = 2,\ \beta = 3. \end{cases} \tag{11}$$

For the applicability of Theorem 7 consider the function

$$g(t,u) = \frac{2\ln t}{e^t (t+4)^3}\frac{|u|}{|u|+1}, \quad 1 \le t \le e.$$

It is clear that the function g is jointly continuous and Lipschitzian wtih $L_g = \frac{2}{125e}$. One can easily compute $Q = 2.079$. Thus $L_g Q < 1$ and all conditions of the Theorem 7 satisfied which shows the existence of uniqueness solution of BVP. given by (11).

Example 2: To illustrate theorem 9 consider

$$g(t,u) = \frac{e^{-t}\tan^{-1} u}{\pi\sqrt{t^2+144}}, \quad 1 \le t \le e.$$

It is jointly continuous and

$$|g(t,u)| \le \frac{e^{-t}}{2\sqrt{t^2+144}} = w(t)\,, \quad Q = \frac{2.07}{24}.$$

Taking $M > 0.021$,we observe that all the conditions of Theorem 9 are satisfied, which implies the existence of the solution of the BVP (11)

References

[1] B. Ahmad, S.K. Ntouyas, Boundary value problems of Hadamard-type fractional differential equations and inclusions with nonlocal conditions, *Vietnam J. Math.*, **45** (2017), 409–423. https://doi.org/10.1007/s10013-016-0213-z

[2] B. Ahmad, S.K. Ntouyas, Existence and uniqueness of solutions for Caputo-Hadamard sequential fractional order neutral functional differential equations, *Electron. J. Differential Equations*, **2017** (2017), no. 36, 1-11.

[3] B. Ahmad, S.K. Ntouyas, J. Tariboon, Existence results for mixed Hadamard and Riemann-Liouville fractional integro-differential inclusions, *J. Nonlinear Sci. Appl.*, **9** (2016), 6333–6347.

[4] B. Ahmad, S.K. Ntouyas, J. Tariboon, Existence results for mixed Hadamard and Riemann-Liouville fractional integro-differential equations, *Adv. in Difference Equations*, **2015** (2015), 1-10. https://doi.org/10.1186/s13662-015-0625-1

[5] A. Alsaedi, S.K. Ntouyas, B. Ahmad, A. Hobiny, Nonlinear Hadamard fractional differential equations with Hadamard type nonlocal nonconserved conditions, *Adv. in Difference Equations*, **2015** (2015), 1-13. https://doi.org/10.1186/s13662-015-0589-1

[6] S. Abbas, M. Benchohra, J-E. Lazreg, Y. Zhou, A survey on Hadamard and Hilfer fractional differential equations: analysis and stability, *Chaos Solitons Fractals*, **102** (2017), 47–71. https://doi.org/10.1016/j.chaos.2017.03.010

[7] M. Benchohra, S. Bouriah, J.R. Graef, Boundary Value Problems for Nonlinear Implicit Caputo–Hadamard-Type Fractional Differential Equations with Impulses, *Mediterr. J. Math.*, **14** (2017), 206-216. https://doi.org/10.1007/s00009-017-1012-9

[8] P.L. Butzer, A.A. Kilbas, J.J. Trujillo, Compositions of Hadamard-type fractional integration operators and the semigroup property, *J. Math. Anal. Appl.*, **269** (2002), 387-400. https://doi.org/10.1016/s0022-247x(02)00049-5

[9] P.L. Butzer, A.A. Kilbas, J.J. Trujillo, Fractional calculus in the Mellin setting and Hadamard-type fractional integrals, *J. Math. Anal. Appl.*, **269** (2002), 1-27. https://doi.org/10.1016/s0022-247x(02)00001-x

[10] P.L. Butzer, A.A. Kilbas, J.J. Trujillo, Mellin transform analysis and integration by parts for Hadamard-type fractional integrals, *J. Math. Anal. Appl.*, **270** (2002), 1-15. https://doi.org/10.1016/s0022-247x(02)00066-5

[11] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Vol. 204, North-Holland Mathematics Studies, Elsevier Science B.V., Amsterdam, 2006.

[12] A.A. Kilbas, Hadamard-type fractional calculus, *J. Korean Math. Soc.*, **38** (2001), 1191-1204.

[13] A.A. Kilbas, J.J. Trujillo, Hadamard-type integrals as G-transforms, *Integral Transforms Spec. Funct.*, **14** (2003), 413-427. https://doi.org/10.1080/1065246031000074443

[14] M.M. Matar, O.A. Al-Salmy, Existence and Uniqueness of Solution for Hadamard Fractional Sequential Differential Equations, *IUG Journal of Natural Studies*, (2017), Special Issue, 141-147.

[15] S.K. Ntouyas, J. Tariboon, P. Thiramanus, Mixed problems of fractional coupled systems of Riemann-Liouville differential equations and Hadamard integral conditions, *J. Comput. Anal. Appl.*, **21** (2016), 813–828.

[16] S.K. Ntouyas, J. Tariboon, C. Thaiprayoon, Nonlocal boundary value problems for Riemann-Liouville fractional differential inclusions with Hadamard fractional integral boundary conditions, *Taiwanese J. Math.*, **20** (2016), 91–107. https://doi.org/10.11650/tjm.20.2016.5654

[17] J. Tariboon, S.K. Ntouyas, C. Thaiprayoon, Nonlinear Langevin equation of Hadamard-Caputo type fractional derivatives with nonlocal fractional integral conditions, *Adv. Math. Phys.*, **2014** (2014), Art. ID 372749, 1-15. https://doi.org/10.1155/2014/372749

[18] W. Yukunthorn, S. Suantai, S.K. Ntouyas, J. Tariboon, Boundary value problems for impulsive multi-order Hadamard fractional differential equations, *Bound. Value Probl.*, **148** (2015), 1-13. https://doi.org/10.1186/s13661-015-0414-5

[19] X. Zhang, T. Shu, H. Cao, Z. Liu, W. Ding, The general solution for impulsive differential equations with Hadamard fractional derivative of order $\alpha \in$(1,2), *Adv. in Difference Equations*, **2016** (2016), 1-36. https://doi.org/10.1186/s13662-016-0744-3

[20] X. Zhang, On impulsive partial differential equations with Caputo-Hadamard fractional derivatives, *Adv. in Difference Equations*, **281** (2016), 1-21. https://doi.org/10.1186/s13662-016-1008-y

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