New Fixed Point Theorems for Hybrid Kannan Type Mappings and $MT(\lambda)$-Functions

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Abstract

In this paper, we first establish a new fixed point theorem for hybrid Kannan type mappings and $MT(\lambda)$-functions which generalizes and improves Kannan’s fixed point theorem. As interesting applications of our new result, many new fixed point theorems are given. Our these new results in this paper are original and quite different from the well known generalizations in the literature.

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1. Introduction and preliminaries

Let $(X,d)$ be a metric space and $T : X \to X$ be a selfmapping. A point $x$ in $X$ is a fixed point of $T$ if $Tx = x$. The set of fixed points of $T$ is denoted by $F(T)$. Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$, the sets of positive integers and real numbers, respectively.

The celebrated Banach contraction principle [1] is one of the best known fixed point theorem in metric fixed point theory. The Banach contraction
principle plays an important role in nonlinear analysis and applied mathematical analysis and has been generalized in many various different directions; for more detail, one can refer to \([2-14]\) and references therein. In 1969, Kannan \([8]\) established his interesting fixed point theorem which is different from the Banach contraction principle. Since then a number of generalizations of Kannan’s fixed point theorem have been investigated by several authors; see, e.g., \([2, 6, 13]\) and references therein.

**Theorem 1.1. (Kannan \([8]\))** Let \((X, d)\) be a complete metric space and \(T : X \to X\) be a self-mapping on \(X\). Suppose that there exists \(\gamma \in \left[0, \frac{1}{2}\right)\) such that
\[
d(Tx, Ty) \leq \gamma (d(x, Tx) + d(y, Ty))
\]
for all \(x, y \in X\).

Then \(T\) admits a unique fixed point in \(X\).

Let \(f\) be a real-valued function defined on \(\mathbb{R}\). For \(c \in \mathbb{R}\), we recall that
\[
\limsup_{x \to c^+} f(x) = \inf_{\varepsilon > 0} \sup_{c < x < c + \varepsilon} f(x).
\]

**Definition 1.1. \([2-6]\)** A function \(\varphi : [0, \infty) \to [0, 1)\) is said to be an \(\mathcal{MT}\)-function (or \(\mathcal{R}\)-function) if \(\limsup_{s \to t^+} \varphi(s) < 1\) for all \(t \in [0, \infty)\).

It is obvious that if \(\varphi : [0, \infty) \to [0, 1)\) is a nondecreasing function or a nonincreasing function, then \(\varphi\) is an \(\mathcal{MT}\)-function. So the set of \(\mathcal{MT}\)-functions is a rich class.

**Definition 1.2. \([5]\)** A real sequence \(\{a_n\}_{n \in \mathbb{N}}\) is called

(i) **eventually strictly decreasing** if there exists \(\ell \in \mathbb{N}\) such that \(a_{n+1} < a_n\) for all \(n \in \mathbb{N}\) with \(n \geq \ell\);

(ii) **eventually strictly increasing** if there exists \(\ell \in \mathbb{N}\) such that \(a_{n+1} > a_n\) for all \(n \in \mathbb{N}\) with \(n \geq \ell\);

(iii) **eventually nonincreasing** if there exists \(\ell \in \mathbb{N}\) such that \(a_{n+1} \leq a_n\) for all \(n \in \mathbb{N}\) with \(n \geq \ell\);

(iv) **eventually nondecreasing** if there exists \(\ell \in \mathbb{N}\) such that \(a_{n+1} \geq a_n\) for all \(n \in \mathbb{N}\) with \(n \geq \ell\).
Very recently, Du [5] presented some new characterizations of MT-functions linked with eventually nonincreasing and eventually strictly decreasing sequences as follows. Although the following result was essentially proved in [5], we give its proof here for the sake of completeness and the readers convenience.

**Theorem 1.2 (see [5, Theorem 2.1]).** Let \( \varphi : [0, \infty) \rightarrow [0, 1) \) be a function. Then the following statements are equivalent.

(a) \( \varphi \) is an \( \mathcal{MT} \)-function.

(b) For each \( t \in [0, \infty) \), there exist \( r_t^{(1)} \in [0, 1) \) and \( \varepsilon_t^{(1)} > 0 \) such that \( \varphi(s) \leq r_t^{(1)} \) for all \( s \in (t, t + \varepsilon_t^{(1)}) \).

(c) For each \( t \in [0, \infty) \), there exist \( r_t^{(2)} \in [0, 1) \) and \( \varepsilon_t^{(2)} > 0 \) such that \( \varphi(s) \leq r_t^{(2)} \) for all \( s \in [t, t + \varepsilon_t^{(2)}] \).

(d) For each \( t \in [0, \infty) \), there exist \( r_t^{(3)} \in [0, 1) \) and \( \varepsilon_t^{(3)} > 0 \) such that \( \varphi(s) \leq r_t^{(3)} \) for all \( s \in (t, t + \varepsilon_t^{(3)}) \).

(e) For each \( t \in [0, \infty) \), there exist \( r_t^{(4)} \in [0, 1) \) and \( \varepsilon_t^{(4)} > 0 \) such that \( \varphi(s) \leq r_t^{(4)} \) for all \( s \in [t, t + \varepsilon_t^{(4)}] \).

(f) For any nonincreasing sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \([0, \infty)\), we have \( 0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1 \).

(g) \( \varphi \) is a function of contractive factor; that is, for any strictly decreasing sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \([0, \infty)\), we have \( 0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1 \).

(h) For any eventually nonincreasing sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \([0, \infty)\), we have \( 0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1 \).

(i) For any eventually strictly decreasing sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \([0, \infty)\), we have \( 0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1 \).

**Proof.** The equivalence of statements (a)-(g) was indeed proved in [3, Theorem 2.1]. The implications ”(h) \( \Rightarrow \) (f)” and ”(i) \( \Rightarrow \) (g)” are obvious. Let us prove ”(f) \( \Rightarrow \) (h)”. Suppose that (f) holds. Let \( \{x_n\}_{n \in \mathbb{N}} \) be an eventually nonincreasing sequence in \([0, \infty)\). Then there exists \( \ell \in \mathbb{N} \) such that \( x_{n+1} \leq x_n \) for all \( n \in \mathbb{N} \) with \( n \geq \ell \). Put \( y_n = x_{n+\ell-1} \) for \( n \in \mathbb{N} \). So \( \{y_n\}_{n \in \mathbb{N}} \) is a nonincreasing sequence in \([0, \infty)\). By (f), we obtain

\[
0 \leq \sup_{n \in \mathbb{N}} \varphi(y_n) < 1.
\]
Let $\gamma := \sup_{n \in \mathbb{N}} \varphi(y_n)$. Then
\[
0 \leq \varphi(x_{n+\ell-1}) = \varphi(y_n) \leq \gamma < 1 \quad \text{for all } n \in \mathbb{N}.
\]
Hence we get
\[
0 \leq \varphi(x_n) \leq \gamma < 1 \quad \text{for all } n \in \mathbb{N} \text{ with } n \geq \ell.
\]
Let
\[
\eta := \max\{\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_{\ell-1}), \gamma\} < 1.
\]
Then $\varphi(x_n) \leq \eta$ for all $n \in \mathbb{N}$. Hence $0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) \leq \eta < 1$ and (h) holds. Similarly, we can verify ”(g) $\Rightarrow$ (i)”. Therefore, from above, we prove that the statements (a)-(i) are all logically equivalent. The proof is completed. \(\square\)

In [5], Du introduced the concept of $\mathcal{MT}(\lambda)$-function which generalizes the concept of $\mathcal{P}$-function [11].

**Definition 1.3 [5].** Let $\lambda > 0$. A function $\mu : [0, \infty) \to [0, \lambda)$ is said to be an $\mathcal{MT}(\lambda)$-function if $\limsup_{s \to t^+} \mu(s) < \lambda$ for all $t \in [0, \infty)$.

**Remark 1.1.**

(i) Obviously, an $\mathcal{MT}$-function is an $\mathcal{MT}(1)$-function and a $\mathcal{P}$-function is an $\mathcal{MT}(\frac{1}{2})$-function.

(ii) It is easy to see that $\mu$ is an $\mathcal{MT}(\lambda)$-function if and only if $\lambda^{-1} \mu$ is an $\mathcal{MT}$-function.

The following characterizations of $\mathcal{MT}(\lambda)$-functions is an immediate consequence of Theorem 1.2.

**Theorem 1.3 (see [5, Theorem 2.4]).** Let $\lambda > 0$ and let $\mu : [0, \infty) \to [0, \lambda)$ be a function. Then the following statements are equivalent.

(1) $\mu$ is an $\mathcal{MT}(\lambda)$-function.

(2) $\lambda^{-1} \mu$ is an $\mathcal{MT}$-function.

(3) For each $t \in [0, \infty)$, there exist $\xi_t^{(1)} \in [0, \lambda)$ and $\epsilon_t^{(1)} > 0$ such that $\mu(s) \leq \xi_t^{(1)}$ for all $s \in (t, t + \epsilon_t^{(1)})$.

(4) For each $t \in [0, \infty)$, there exist $\xi_t^{(2)} \in [0, \lambda)$ and $\epsilon_t^{(2)} > 0$ such that $\mu(s) \leq \xi_t^{(2)}$ for all $s \in [t, t + \epsilon_t^{(2)}]$. 

For each \( t \in [0, \infty) \), there exist \( \xi^{(3)}_t \in [0, \lambda) \) and \( \epsilon^{(3)}_t > 0 \) such that
\[ \mu(s) \leq \xi^{(3)}_t \quad \text{for all} \quad s \in (t, t + \epsilon^{(3)}_t]. \]

For each \( t \in [0, \infty) \), there exist \( \xi^{(4)}_t \in [0, \lambda) \) and \( \epsilon^{(4)}_t > 0 \) such that
\[ \mu(s) \leq \xi^{(4)}_t \quad \text{for all} \quad s \in [t, t + \epsilon^{(4)}_t). \]

For any nonincreasing sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \([0, \infty)\), we have
\[ 0 \leq \sup_{n \in \mathbb{N}} \mu(x_n) < \lambda. \]

For any strictly decreasing sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \([0, \infty)\), we have
\[ 0 \leq \sup_{n \in \mathbb{N}} \mu(x_n) < \lambda. \]

For any eventually nonincreasing sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \([0, \infty)\), we have
\[ 0 \leq \sup_{n \in \mathbb{N}} \mu(x_n) < \lambda. \]

For any eventually strictly decreasing sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \([0, \infty)\), we have
\[ 0 \leq \sup_{n \in \mathbb{N}} \mu(x_n) < \lambda. \]

Remark 1.2. [11, Lemma 3.1] is a special case of Theorem 1.3 for \( \lambda = \frac{1}{2} \).

In this work, we first establish a new fixed point theorem for hybrid Kannan type mappings and \( \mathcal{MT}(\lambda) \)-functions which generalizes and improves Kannan’s fixed point theorem. As interesting applications of our new result, many new fixed point theorems are given. Consequently, our these new results in this paper are original and quite different from the well known generalizations in the literature.

2. Some new fixed point theorems

In this section, we first establish a new fixed point theorem for hybrid Kannan type mappings and \( \mathcal{MT}(\lambda) \)-functions which is one of the main results of this paper and generalizes Kannan’s fixed point theorem.

Theorem 2.1. Let \((X, d)\) be a complete metric space and \( T : X \to X \) be a selfmapping on \( X \). Suppose that

(H) there exists an \( \mathcal{MT}(\frac{1}{2}) \)-function \( \eta : [0, \infty) \to [0, \frac{1}{2}) \) such that
\[ \min\{d(Tx, Ty), d(x, Tx)\} \leq \eta(d(x, y))(d(x, Tx) + d(y, Ty)) \]
for all \( x, y \in X \) with \( x \neq y \).
Then $T$ admits a fixed point in $X$.

**Proof.** Given $z \in X$. If $Tz = z$, then we are done. Otherwise, if $Tz \neq z$, we define a sequence $\{x_n\}_{n \in \mathbb{N}}$ by $x_1 = z$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Since $\eta(t) < \frac{1}{2}$ for all $t \in [0, \infty)$, we can define a function $\xi : [0, \infty) \to (0, \frac{1}{2})$ by

$$\xi(t) = \frac{1}{2} \left( \frac{1}{2} + \eta(t) \right) \quad \text{for all } t \in [0, \infty).$$

Clearly, $0 \leq \eta(t) < \xi(t) < \frac{1}{2}$ for all $t \in [0, \infty)$. Since $x_2 = Tx_1 \neq x_1$, by condition (H), we have

$$d(x_3, x_2) = \min \{d(Tx_2, Tx_1), d(x_2, Tx_2)\} \leq \eta(d(x_2, x_1))(d(x_2, Tx_2) + d(x_1, Tx_1)) < \xi(d(x_2, x_1))(d(x_3, x_2) + d(x_2, x_1)).$$

If $Tx_2 = x_2$ then $x_2 \in \mathcal{F}(T)$ and the desired conclusion is proved. Assume $Tx_2 \neq x_2$. Then $x_3 \neq x_2$. By (H) again, we obtain

$$d(x_4, x_3) = \min \{d(Tx_3, Tx_2), d(x_3, Tx_3)\} < \xi(d(x_3, x_2))(d(x_4, x_3) + d(x_3, x_2)).$$

So, by induction, if $x_{n+1} \neq x_n$ then we can get

$$d(x_{n+2}, x_{n+1}) < \xi(d(x_{n+1}, x_n))(d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)) \quad \text{for all } n \in \mathbb{N}. \quad (2.1)$$

Suppose there exists $k \in \mathbb{N}$ such that $d(x_{k+1}, x_k) < d(x_{k+2}, x_{k+1})$. Thus the inequality (2.1) implies

$$d(x_{k+2}, x_{k+1}) < 2\xi(d(x_{k+1}, x_k))d(x_{k+2}, x_{k+1}) < d(x_{k+2}, x_{k+1}),$$

which leads a contradiction. Hence it must be

$$d(x_{n+2}, x_{n+1}) \leq d(x_{n+1}, x_n) \quad \text{for all } n \in \mathbb{N}. \quad (2.2)$$

By (2.2), we know that the sequence $\{d(x_{n+1}, x_n)\}_{n \in \mathbb{N}}$ is nonincreasing in $[0, \infty)$. Since $\eta$ is an $\mathcal{MT}\left(\frac{1}{2}\right)$-function, by applying Theorem 1.3, we have

$$0 \leq \sup_{n \in \mathbb{N}} \eta(d(x_{n+1}, x_n)) < \frac{1}{2}$$

and then deduces

$$0 < \sup_{n \in \mathbb{N}} \xi(d(x_{n+1}, x_n)) = \frac{1}{2} \left( \frac{1}{2} + \sup_{n \in \mathbb{N}} \eta(d(x_{n+1}, x_n)) \right) < \frac{1}{2}.$$ 

Let $\lambda := \sup_{n \in \mathbb{N}} \xi(d(x_{n+1}, x_n))$ and $\gamma := 2\lambda$. So $\gamma \in (0, 1)$. For any $n \in \mathbb{N}$, by (2.1) and (2.2), we get

$$d(x_{n+2}, x_{n+1}) < \xi(d(x_{n+1}, x_n))(d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)) \leq \gamma d(x_{n+1}, x_n)$$
and hence
\[ d(x_{n+2}, x_{n+1}) < \gamma d(x_{n+1}, x_n) < \cdots < \gamma^n d(x_2, x_1). \] 
(2.3)

Let \( c_n = \frac{\gamma^{n-1}}{1-\gamma} d(x_2, x_1) \), \( n \in \mathbb{N} \). For \( m, n \in \mathbb{N} \) with \( m > n \), we get from (2.3) that
\[ d(x_m, x_n) \leq \sum_{j=n}^{m-1} d(x_{j+1}, x_j) < c_n. \] 
(2.4)

Since \( 0 < \gamma < 1 \), \( \lim_{n \to \infty} c_n = 0 \) and hence (2.4) implies
\[ \lim_{n \to \infty} \sup \{d(x_m, x_n) : m > n\} = 0. \]

This proves that \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( X \). By the completeness of \( X \), there exists \( v \in X \) such that \( x_n \to v \) as \( n \to \infty \). We will now finish the proof by showing that \( v \in \mathcal{F}(T) \). Clearly, the condition \((H)\) still holds for \( x = y \in X \). So, by \((H)\), we have
\[
\frac{1}{2} (d(x_{n+1}, Tv) + d(v,Tv) - |d(x_{n+1}, Tv) - d(v,Tv)|) \\
= \min \{d(Tx_n, Tv), d(v,Tv)\} \\
< \frac{1}{2} (d(v, Tv) + d(x_n, x_{n+1})) \quad \text{for all } n \in \mathbb{N}.
\]

Since \( x_n \to v \) as \( n \to \infty \), taking the limit as \( n \) tends to infinity on the last inequality yields
\[ d(v, Tv) \leq \frac{1}{2} d(v, Tv) \]
and hence deduces \( d(v, Tv) = 0 \). So we obtain \( v \in \mathcal{F}(T) \). The proof is completed. \( \square \)

The following new fixed point theorem is immediate from Theorem 2.1.

**Corollary 2.1.** Let \( (X,d) \) be a complete metric space and \( T : X \to X \) be a selfmapping on \( X \). Suppose that there exists an MT(\( \frac{1}{2} \))-function \( \eta : [0,\infty) \to [0,\frac{1}{2}] \) such that
\[ d(Tx, Ty) \leq \eta(d(x,y))(d(x,Tx) + d(y,Ty)) \quad \text{for all } x, y \in X \text{ with } x \neq y. \] 
(2.5)

Then \( T \) admits a unique fixed point in \( X \).

**Proof.** Applying Theorem 2.1, \( \mathcal{F}(T) \neq \emptyset \). We claim that \( \mathcal{F}(T) \) is a singleton set. Assume that there exist \( u, v \in \mathcal{F}(T) \) with \( u \neq v \). By (2.5), we obtain
\[ 0 < d(u, v) = d(Tu, Tv) \leq \eta(d(u,v))(d(u,Tu) + d(v,Tv)) = 0, \]

and therefore \( u = v \). This proves that \( \mathcal{F}(T) \) is a singleton set. The proof is completed.
a contradiction. Therefore $\mathcal{F}(T)$ is a singleton set and $T$ has a unique fixed point in $X$. □

**Remark 2.1.**

(i) Kannan’s fixed point theorem [8] is a special case of Theorem 2.1 and Corollary 2.1;

(ii) If ”$\min\{d(Tx,Ty),d(x,Tx)\}$” is replaced with ”$\min\{d(Tx,Ty),d(y,Ty)\}$” in condition $(H)$ of Theorem 2.1, by symmetric, the conclusion of Theorem 2.1 still holds.

**Corollary 2.2.** Let $(X,d)$ be a complete metric space and $T : X \to X$ be a selfmapping on $X$. Suppose that there exists an $MT(\frac{1}{2})$-function $\eta : [0,\infty) \to [0,\frac{1}{2})$ such that

$$d(x,Tx) \leq \eta(d(x,y))(d(x,Tx) + d(y,Ty))$$

for all $x,y \in X$ with $x \neq y$. Then $T$ admits a fixed point in $X$.

**Corollary 2.3.** Let $(X,d)$ be a complete metric space and $T : X \to X$ be a selfmapping on $X$. Suppose that there exists $\gamma \in [0,\frac{1}{2})$ such that

$$d(x,Tx) \leq \gamma(d(x,Tx) + d(y,Ty))$$

for all $x,y \in X$ with $x \neq y$. Then $T$ admits a fixed point in $X$.

We can establish the following new fixed point theorems by applying Theorem 2.1 immediately.

**Theorem 2.2.** Let $(X,d)$ be a complete metric space and $T : X \to X$ be a selfmapping on $X$. Suppose that there exists an $MT(\frac{1}{2})$-function $\eta : [0,\infty) \to [0,\frac{1}{2})$ such that

$$d(Tx,Ty) + d(x,Tx) \leq 2\eta(d(x,y))(d(x,Tx) + d(y,Ty))$$

for all $x,y \in X$ with $x \neq y$. (2.6)

Then $T$ admits a fixed point in $X$.

**Remark 2.2.** The conclusion of Theorem 2.2 is still true if ”$d(Tx,Ty) + d(x,Tx)$” is replaced with ”$d(Tx,Ty) + d(y,Ty)$” in (2.6).

**Theorem 2.3.** Let $(X,d)$ be a complete metric space and $T : X \to X$ be a selfmapping on $X$. Suppose that there exists an $MT(\frac{1}{2})$-function $\eta : [0,\infty) \to [0,\frac{1}{2})$ such that

$$\sqrt{d(Tx,Ty)d(x,Tx)} \leq \eta(d(x,y))(d(x,Tx) + d(y,Ty))$$

for all $x,y \in X$ with $x \neq y$. (2.7)
Then $T$ admits a fixed point in $X$.

**Remark 2.3.** The conclusion of Theorem 2.3 is still true if $\sqrt{d(Tx,Ty)d(x,Tx)}$ is replaced with $\sqrt{d(Tx,Ty)d(y,Ty)}$ in (2.7).

In fact, we can establish a wide generalization of Theorem 2.2 as follows.

**Theorem 2.4.** Let $(X,d)$ be a complete metric space and $T : X \to X$ be a self-mapping on $X$. Suppose that there exists an MT($\frac{1}{2}$)-function $\eta : [0, \infty) \to [0, \frac{1}{2})$ such that

$$\frac{\alpha d(Tx,Ty) + \beta d(x,Tx)}{\alpha + \beta} \leq \eta(d(x,y))(d(x,Tx) + d(y,Ty)) \quad (2.8)$$

for all $x, y \in X$ with $x \neq y$,

where $\alpha$ and $\beta$ are nonnegative real numbers with $\alpha + \beta > 0$. Then $T$ admits a fixed point in $X$.

**Remark 2.4.**

(i) We can also obtain Theorem 2.2 if we take $\alpha = \beta = 1$ in Theorem 2.4;

(ii) The conclusion of Theorem 2.4 is still true if $\frac{\alpha d(Tx,Ty) + \beta d(x,Tx)}{\alpha + \beta}$ is replaced with $\frac{\alpha d(Tx,Ty) + \beta d(y,Ty)}{\alpha + \beta}$ in (2.8).

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**References**


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