Uniformly Harmonic Starlike Functions of Complex Order

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Abstract

In this paper, we introduce and investigate a new class of $p$-valent harmonic starlike functions of complex order $b$. We study various properties of this class including coefficient conditions, distortion bounds, extreme points, convex combination and find their connection with the already known classes.

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1 Introduction

A continuous function $f(z) = u(x, y) + iv(x, y)$ is harmonic in a complex domain $D$, if both $u$ and $v$ are real harmonic in $D$. Various classes of harmonic functions have been extensively investigated in the literature of the subject, for
example, see [6, 10, 13, 14] and others with references there in. In any simply connected domain, we can write \( f(z) = l(z) + k(z) \), where \( l \) and \( k \) are analytic in \( U \). We call \( l \) the analytic part and \( k \) the co-analytic part of \( f \). We note that \( f(z) = l(z) + k(z) \) reduces to \( l \) if the co-analytic part \( k \) is zero. For \( p \geq 1 \), let \( H(p) \) denote the class all multivalent harmonic functions \( f(z) = l(z) + k(z) \) defined in the open unit disk \( U = \{ z : |z| < 1 \} \), where

\[
l(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad \text{and} \quad k(z) = \sum_{n=p+1}^{\infty} b_n z^n, \quad |b_{p+t-1}| < 1, t \in N. \tag{1}
\]

Let \( F(z) = L(z) + \overline{K(z)} \) be a fixed multivalent harmonic function, where

\[
L(z) = z^p + \sum_{n=p+1}^{\infty} |A_n| z^n \quad \text{and} \quad K(z) = \sum_{n=p+1}^{\infty} |B_n| z^n, \quad |B_{p+t-1}| < 1, t \in N. \tag{2}
\]

For \( m \geq 0, b \in C \setminus \{0\} \) and \( p \in N \), the class \( MH_F(p, t, b, m) \) consists of functions \( f \in H(p) \) satisfying the condition

\[
\Re \left( p + \frac{1}{b} \left( \frac{z(f \ast F)'(z)}{z'(f \ast F)(z)} - p \right) \right) \geq m \left| \frac{1}{b} \left( \frac{z(f \ast F)'(z)}{z'(f \ast F)(z)} - p \right) \right| \quad (z \in U), \tag{3}
\]

where \( f \ast F \) is a convolution of \( f \) and \( F \) and \( z' = \frac{\partial}{\partial z}(re^{i\theta}), f'(z) = \frac{\partial}{\partial z}(re^{i\theta}) \).

From the fact \( \Re(w) > m |w - p| \) if and only if \( \Re((1 + me^{i\theta})w - mpe^{i\theta}) \geq 0 \), it follows that (3) is equivalent to:

\[
\Re \left( (1 + me^{i\theta}) \left( p + \frac{1}{b} \left( \frac{z(f \ast F)'(z)}{z'(f \ast F)(z)} - p \right) \right) - mpe^{i\theta} \right) \geq 0 \quad (z \in U). \tag{4}
\]

A function \( f \) in the class \( MH_F(p, t, b, m) \) is called \( m \)-uniformly multivalent harmonic starlike of complex order \( b \) associated with a fixed multivalent harmonic function \( F \). For \( b = 1 - \beta \), we obtain the class \( H_F(p, t, \beta, m) \) studied in [2]. In the class \( MH_F(p, t, b, m) \), if we take: \( F(z) = I(z) = z^p + \sum_{n=p+1}^{\infty} z^n + \sum_{n=p+t}^{\infty} \overline{z^n} \), we have various known classes as special cases for specific choices of parameters, for detail, see [2, 3, 4, 5, 7, 8, 9, 11, 12]. Let \( TH(p) \subset H(p) \) consisting of functions \( f(z) = l(z) + \overline{k(z)} \) such that

\[
l(z) = z^p - \sum_{n=p+t}^{\infty} |a_n| z^n \quad \text{and} \quad k(z) = \sum_{n=p+1}^{\infty} |b_n| z^n \quad (z \in U). \tag{5}
\]

We take \( MH_{TH}(p, t, b, m) = TH(p) \cap MH_F((p, t, b, m)) \).
2 Results and Discussion

Theorem 2.1. Let \( f(z) = l(z) + k(\overline{z}) \), where \( l \) and \( k \) are given by (1) and let \( F \) be a fixed \( p \)-valent harmonic function given by (2). Then \( f \in MH_F(p, t, b, m) \), if for \( m_2 = 1 + m \geq -1 \), we have

\[
\sum_{n=p+t}^{\infty} \frac{nm_2 - p(m_2 - |b|)}{|pb + b| - |pb - b|} |a_n A_n| + \sum_{n=p+t-1}^{\infty} \frac{nm_2 + p(m_2 - |b|)}{|pb + b| - |pb - b|} |b_n B_n| \leq \frac{1}{2},
\]

where \( b \in \mathbb{C} \setminus \{0\} \), and \( p, t \in \mathbb{N} \).

Proof. Let \( f = l + k \in MH_F(p, t, b, m) \) where \( l \) and \( k \) are given by (1) and also let \( F \) be a fixed \( p \)-valent harmonic function given by (2). Then for \( m_1 = 1 + me^{i\theta} \) and in view of (3) and (4), we write

\[
\Re \left[ \frac{m_1[z(l \ast L)'(z) - z(k \ast K)'(z)] - p[m_1 - b][(l \ast L)(z) + (k \ast K)(z)]}{b[(l \ast L)(z) + (k \ast K)(z)]} \right] \geq 0.
\]

Using the fact \( \Re (w) > 0 \) if and only if \( |1 + w| \geq |1 - w| \), it is sufficient to show that \( |1 + w| - |1 - w| \geq 0 \). Now, for \( \psi = p(m_1 - b) - b \), we have

\[
|1 + w| = |pb + b|z^p + \sum_{n=p+t}^{\infty} [nm_1 - \psi] a_n |A_n| z^n - \sum_{n=p+t-1}^{\infty} [nm_1 + \psi] b_n |B_n| z^n,
\]

and

\[
|1 - w| = |pb - b|z^p + \sum_{n=p+t}^{\infty} [nm_1 - \psi] a_n |A_n| z^n - \sum_{n=p+t-1}^{\infty} [nm_1 + \psi] b_n |B_n| z^n.
\]

As we take \( m_1 = 1 + m e^{i\theta} \). So for \( |m_1| \leq 1 + m |e^{i\theta}| \leq 1 + m = m_2 \), we have

\[
|1 + w| - |1 - w| \geq [(|pb + b| - |pb - b|)]|z^p| - \sum_{n=p+t}^{\infty} 2[nm_2 - p(m_2 - |b|)] a_n |A_n| |z^n| - \sum_{n=p+t-1}^{\infty} 2[nm_2 + p(m_2 - |b|)] b_n |B_n| |z^n|.
\]

Thus for \( b_1 = p(m_2 - |b|) \), we write

\[
|1 + w| - |1 - w| \geq [(|pb + b| - |pb - b|)]|z^p| + \sum_{n=p+t}^{\infty} \frac{2[nm_2 - b_1]}{|pb + b| - |pb - b|} |a_n A_n| + \sum_{n=p+t-1}^{\infty} \frac{2[nm_2 + b_1]}{|pb + b| - |pb - b|} |b_n B_n| - 1.
\]
For condition (6), we obtain the desired result. The bounds in (6) are sharp for
\[ f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{|pb+b| - |pb-b|}{2[nm_2 - p(m_2 - |b|)]} X_n z^n + \sum_{n=p+t}^{\infty} \frac{|pb+b| - |pb-b|}{2[nm_2 + p(m_2 - |b|)]} Y_n z^n, \]
where \( \sum_{n=p+t}^{\infty} |X_n| + \sum_{n=p+t-1}^{\infty} |Y_n| = 1. \)

**Corollary 2.2.** For \( p \geq \frac{1}{|b|} \), if
\[
\sum_{n=p+t}^{\infty} [nm_2 - b_1] |a_n A_n| + \sum_{n=p+t-1}^{\infty} [nm_2 + b_1] |b_n B_n| < |b|, 
\]
holds, then \( f \in MH_F(p, t, b, m) \), where \( b_1 = p(m_2 - |b|) \). For \( 1 \leq p \leq \frac{1}{|b|} \), if
\[
\sum_{n=p+t}^{\infty} [nm_2 - b_1] |a_n A_n| + \sum_{n=p+t-1}^{\infty} [nm_2 + b_1] |b_n B_n| < p |b|, 
\]
holds, then \( f \in MH_F(p, t, b, m) \) for \( b_1 = p(m_2 - |b|) \).

Now, we study the characterization of functions in \( MH_F(p, t, b, m) \).

**Theorem 2.3.** Let \( f = l + \bar{K} \) be such that \( l \) and \( k \) are given by (5) and let \( F \) be a fixed \( p \)-valent harmonic function given by (2) and \( b \in C \setminus \{0\} \). Also, assume that \( m_1 = 1 + m e^{i\theta}, m_2 = 1 + m \geq -1 \) and \( b_1 = p(m_2 - |b|) \). Then
(i) for \( 1 \leq p \leq \frac{1}{|b|} \), \( f \in MH_F(p, t, b, m) \) if and only if
\[
\sum_{n=p+t}^{\infty} [nm_2 - b_1] |a_n A_n| + \sum_{n=p+t-1}^{\infty} [nm_2 + b_1] |b_n B_n| < p |b|. \tag{7} \]
and (ii) for \( p |b| \geq 1 \), \( f \in MH_F(p, t, b, m) \) if and only if
\[
\sum_{n=p+t}^{\infty} [nm_2 - b_1] |a_n A_n| + \sum_{n=p+t-1}^{\infty} [nm_2 + b_1] |b_n B_n| < |b|. \tag{8} \]

**Proof.** Since \( MH_F(p, t, b, m) \subset MH_F(p, t, b, m) \), we only need to prove the only if. As in Corollary 2.2, we show that if the condition (7) does not hold, then \( f \notin MH_F(p, t, b, m) \), that is, we must have
\[
N = \text{Re} \left[ m_1 [z(l * L)'(z) - z(k * K)'(z)] - p[m_1 - b][(l * L)(z) + (k * K)(z)] \right] \geq 0. \tag{9} \]
Substituting the value \( z < r < 1, b = |b| \) and using \( \text{Re}(-e^{i\theta}) \geq -|e^{i\theta}| = -1 \), the inequality (9) reduces to
where \( \beta = p[m_2 - |b|][r^p - \sum_{n=p+1}^\infty b_n |B_n|^r] \). Letting \( r \to 1^- \), we obtain

\[
 R \geq \frac{p|b| - \sum_{n=p+1}^\infty [n m_2 - b_1]a_n |A_n| - \sum_{n=p+1}^\infty [n m_2 + b_1]b_n |B_n|}{|b| \left(1 - \sum_{n=p+1}^\infty a_n |A_n| + \sum_{n=p+1}^\infty b_n |B_n|\right)}
\]

If the condition (9) does not hold, then numerator in (10) is negative for \( r \) sufficiently close to 1. Hence there exist \( z_o = r_o \) in \((0, 1)\) for which (10) is negative. Therefore, \( f \notin MH_\mathbb{T}(p, t, b, m) \) and so the proof is complete.

**Theorem 2.4.** If \( f \in MH_\mathbb{T}(p, t, b, m) \), then for \( |z| = r < 1 \), \( |A_{p+t}| \leq |A_n| \leq |B_n|, m_2 = 1 + m \geq -1 \), and \( b_1 = p(m_2 - |b|), b \in C \setminus \{0\} \) and \( A_{p+t} \neq 0 \),

\[
|f(z)| \leq \begin{cases} 
 b_{\beta} r^{p+t-1} + \frac{p|b|r^{p+1}}{|(p+t)m_2-b_1||A_{p+t}|} - \frac{[(p+t-1)m_2+b_1]|b_{p+t-1}|B_{p+t-1}|r^{p+1}}{|(p+t)m_2-b_1||A_{p+t}|}, & p|b| \leq 1 \\
 b_{\beta} r^{p+t-1} + \frac{p|b|r^{p+1}}{|(p+t)m_2-b_1||A_{p+t}|} - \frac{[(p+t-1)m_2+b_1]|b_{p+t-1}|B_{p+t-1}|r^{p+1}}{|(p+t)m_2-b_1||A_{p+t}|}, & p|b| \geq 1
\end{cases}
\]

where \( b_\beta = 1 + |b_{p+t-1}| \) and

\[
|f(z)| \geq \begin{cases} 
 b_{\gamma} r^{p+t-1} - \frac{p|b|r^{p+1}}{|(p+t)m_2-b_1||A_{p+t}|} + \frac{[(p+t-1)m_2+b_1]|b_{p+t-1}|B_{p+t-1}|r^{p+1}}{|(p+t)m_2-b_1||A_{p+t}|}, & p|b| \leq 1 \\
 b_{\gamma} r^{p+t-1} - \frac{p|b|r^{p+1}}{|(p+t)m_2-b_1||A_{p+t}|} + \frac{[(p+t-1)m_2+b_1]|b_{p+t-1}|B_{p+t-1}|r^{p+1}}{|(p+t)m_2-b_1||A_{p+t}|}, & p|b| \geq 1
\end{cases}
\]

where \( b_\gamma = 1 - |b_{p+t-1}| \). These bounds are sharp.

Using (5) and Theorem 2.1, we obtain the desired proof.

**Theorem 2.5.** A function \( f \in clcoMH_\mathbb{T}(p, t, b, m) \), if and only if

\[
f(z) = \sum_{p+t-1}^{\infty} (X_n l_n(z) + Y_n k_n(z)), l_{p+t-1}(z) = z^p, z \in U,
\]

\[
l_n(z) = \begin{cases} 
 z^p - \frac{p|b|}{|nm_2-b_1||A_n|} z^n; & (n = p + t, p + t + 1, \ldots), \ p|b| \leq 1 \\
z^p - \frac{p|b|}{|nm_2-b_1||A_n|} z^n; & (n = p + t, p + t + 1, \ldots), \ p|b| \geq 1
\end{cases}
\]

and

\[
k_n(z) = \begin{cases} 
 z^p + \frac{p|b|}{|nm_2+b_1||B_n|} z^n; & (n = p + t - 1, p + t, \ldots), \ p|b| \leq 1 \\
z^p + \frac{p|b|}{|nm_2+b_1||B_n|} z^n; & (n = p + t - 1, p + t, \ldots), \ p|b| \geq 1
\end{cases}
\]
where \( X_{p+t-1} \equiv X_p = 1 - \sum_{n=p+1}^{\infty} X_n - \sum_{n=p+t-1}^{\infty} Y_n, X_n \geq 0, Y_n \geq 0, m_2 = 1 + m \geq -1 \) and \( b_1 = p(m_2 - |b|) \). In particularly, the extreme points of \( MH_\mathcal{T}(p, t, b, m) \) are \( \{l_n\} \) and \( \{k_n\} \).

**Proof.** Suppose \( p|b| \leq 1 \). For function of the form (11), we can write

\[
f(z) = z^p - \sum_{n=p+t}^{\infty} \frac{p \ |b| |X_n z^n|}{|nm_2 - b_1||A_n|} + \sum_{n=p+t-1}^{\infty} \frac{p \ |b| Y_n z^n}{|nm_2 + b_1||A_n|},
\]

On the other hand, for \( 0 \leq X_p \leq 1 \), we get

\[
\sum_{n=p+t}^{\infty} \frac{[nm_2 - b_1]|A_n||p \ |b| X_n}{|nm_2 - b_1||A_n|} + \sum_{n=p+t-1}^{\infty} \frac{[nm_2 + b_1]|B_n||p \ |b| Y_n}{|nm_2 + b_1||B_n|} = \sum_{n=p+t}^{\infty} (X_n + Y_n) + Y_{p+t-1} \leq 1.
\]

Thus, by Theorem 2.1, we have \( f \in MH_\mathcal{T}(p, t, b, m) \). Conversely, suppose \( f \in MH_\mathcal{T}(p, t, b, m) \). Then, it follows Theorem 2.1 that \( |a_n| \leq \frac{p|b|}{|nm_2 - b_1||A_n|}, |b_n| \leq \frac{p|b|}{|nm_2 + b_1||B_n|} \). Setting \( X_n = \frac{[nm_2 - b_1]|a_n| A_n}{p|b|}, Y_n = \frac{[nm_2 + b_1]|b_n| B_n}{p|b|} \) and defining \( X_p = 1 - \left( \sum_{n=p+t}^{\infty} X_n + \sum_{n=p+t-1}^{\infty} Y_n \right) \), where \( X_p \geq 0 \), we obtain

\[
f(z) = z^p - \sum_{n=p+t}^{\infty} |a_n| z^n + \sum_{n=p+t-1}^{\infty} |b_n| z^n
\]

\[
= z^p - \sum_{n=p+t}^{\infty} X_n z^n + \sum_{n=p+t-1}^{\infty} Y_n z^n + \sum_{n=p+t}^{\infty} l_n(z) X_n + \sum_{n=p+t-1}^{\infty} k_n(z) Y_n
\]

\[
= X_p z^p + \sum_{n=p+t}^{\infty} l_n(z) X_n + \sum_{n=p+t-1}^{\infty} k_n(z) Y_n.
\]

Thus \( f \) can be written as (11). The proof for the case \( p|b| \geq 1 \) is similar.

**Theorem 2.6.** The class \( MH_\mathcal{T}(p, t, b, m) \) is closed.

**Proof.** For \( j = 1, 2, \ldots \), let \( f_j(z) = z^p - \sum_{n=p+t}^{\infty} |a_{j,n}| z^n + \sum_{n=p+t-1}^{\infty} |b_{j,n}| z^n \), belong to the class \( MH_\mathcal{T}(p, t, b, m) \). For \( \sum_{j=1}^{\infty} \mu_j = 1, 0 \leq \mu_j \leq 1 \), the convex combination of \( f_j \) is expressed as

\[
\sum_{j=1}^{\infty} \mu_j f_j(z) = z^p - \sum_{n=p+t}^{\infty} \sum_{j=1}^{\infty} \mu_j |a_{j,n}| z^n + \sum_{n=p+t-1}^{\infty} \sum_{j=1}^{\infty} \mu_j |b_{j,n}| z^n.
\]

Also \( F_j : F_j(z) = z^p + \sum_{n=p+t}^{\infty} |A_{j,n}| z^n + \sum_{n=p+t-1}^{\infty} |B_{j,n}| z^n \), we have

\[
\sum_{n=p+t}^{\infty} [nm_2 - b_1] |a_{j,n} A_{j,n}| + \sum_{n=p+t-1}^{\infty} [nm_2 + b_1] |b_{j,n} B_{j,n}| \leq \begin{cases} p|b| & \text{if } p|b| \leq 1 \\ |b| & \text{if } p|b| \geq 1 \end{cases}
\].
and from (11), we write

\[
\sum_{n=p+t}^{\infty} [nm_2 - b_1] \sum_{j=1}^{\infty} \mu_j |a_{j,n}A_{j,n}| + \sum_{n=p+t-1}^{\infty} [nm_2 + b_1] \sum_{j=1}^{\infty} \mu_j |b_{j,n}B_{j,n}|
\]

\[
\leq \left\{ p \left| \sum_{j=1}^{\infty} \mu_j = p \right| \sum_{j=1}^{\infty} \mu_j \right| \begin{cases} p \left| b \right| & \text{if } p \left| b \right| \leq 1 \\
 \left| b \right| \sum_{j=1}^{\infty} \mu_j & \text{if } p \left| b \right| \geq 1 \end{cases}
\right.

Thus the coefficient estimate given by Theorem 2.1 holds. Therefore, we get \( \sum_{j=1}^{\infty} \mu_j f_j(z) \in MH_F(p, t, b, m) \).

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