Common Fixed Point Theorem for Almost Weak Contraction Maps with Rational Expression

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Abstract

In this paper we prove the existence common fixed points for selfmaps satisfying almost weak contraction condition with rational expression. Also, we prove a fixed point theorem for generalized weak contraction map in a metric space.

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1 Introduction

The development of fixed point theory is based on the generalization of contraction conditions in one direction or/and generalization of ambient space of the operator under consideration on the other. The Banach contraction mapping is one of the pivotal results of analysis. It is a very popular tool for solving existence problems in many different fields of mathematics Banach contraction principle plays an important role in solving non linear equations, and it is one of the most useful results in fixed point theory. Banach contraction principle has been generalized in various ways either by using contractive conditions or by imposing some additional conditions on the ambient space. In the direction of generalization of contraction conditions in 1975, Berinde [3] in introduced ‘weak contractions’ as a generalization of contraction maps. Berinde renamed ‘weak contractions’ as almost contraction in his later work [4] for more works on almost contractions and its generalizations, we refer to Babu, Sandhya and Kameswari [2], Abbas, Babu and Alemayehu [1] and the related references cited in this papers. In 2008, Suzuki [12] proved a fixed point theorem, which is a new type of generalization of the Banach contraction principle. The following important generalization is due to Suzuki [12].

Theorem 1.1 [12] Define a function $\theta$ from $[0, 1)$ on to $(\frac{1}{2}, 1]$ by

$$
\theta(r) = \begin{cases} 
1, & \text{if } 0 \leq r \leq \left(\frac{\sqrt{5} - 1}{2}\right); \\
\frac{1-r}{r}, & \text{if } \left(\frac{\sqrt{5} - 1}{2}\right) \leq r \leq 2\frac{1}{\sqrt{2}}; \\
\frac{1}{1+r}, & \text{if } \frac{1}{\sqrt{2}} \leq r < 1.
\end{cases}
$$

Let $(X, d)$ be a complete metric space $T$ is a mapping on $X$. If $T$ satisfy the following $\theta(r)d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$. Then $T$ has a fixed point.

Throughout this paper, we denote $R_+ = [0, \infty)$ and

$\Psi = \{\psi/\psi : R_+ \to R_+ \text{ is continuous on } R_+, \psi \text{ is nondecreasing and } \sum_{n=1}^{\infty} \psi^n(t) < \infty\}$ and $\Phi = \{\phi : R_+ \to R_+ \text{ is lower semi continuous on } R_+ \text{ with } \phi(t) = 0 \text{ if and only if } t = 0\}$.

Here we note that if $\psi \in \Psi$ then $\psi(t) < t$ for every $t > 0$ and $\psi(0) = 0$.

The Banach contraction theorem and its several extensions have been generalized using recently developed notion of Weakly contractive maps. The following basic result is due to Rhoades [10].
Theorem 1.2 [10] Let $X$ be a complete metric space and $T : X \to X$ such that for every $x, y \in X$, $d(T_x, T_y) \leq d(x, y) - \varphi(d(x, y))$, where $\varphi : R_+ \to R_+$ is a continuous and non decreasing function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Then $T$ has a unique fixed point.

Dutta and Choudhary [7] obtained by the following theorem.

Theorem 1.3 [7] Let $X$ be a complete metric space and $T : X \to X$ such that for every $x, y \in X$, $\varphi(d(T_x, T_y)) \leq \psi(d(x, y)) - \varphi(d(x, y))$, where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and monotone non decreasing function with $\psi(t) = 0$ if and only if $t = 0$ and $\varphi : R_+ \to R_+$ is lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$


Theorem 1.4 [11] Let $X$ be a complete metric space and $T : X \to X$ such that for every $x, y \in X$, $\frac{1}{2}d(x, T_x) \leq d(x, y)$

$\Rightarrow \psi(d(T_x, T_y)) \leq \psi(\max\{d(x, y), d(x, T_x), d(y, T_y), \frac{d(x, y) + d(y, T_x)}{2}, \frac{d(x, y) + d(y, T_x)}{2}, \frac{d(x, T_x) + d(y, T_y)}{2}\}) - \varphi(\max\{d(x, y), d(x, T_x), d(y, T_y), \frac{d(x, y) + d(y, T_x)}{2}, \frac{d(x, y) + d(y, T_x)}{2}, \frac{d(x, T_x) + d(y, T_y)}{2}\})$

where $\psi : R_+ \to R_+$ is a continuous and monotone non decreasing function with $\psi(t) = 0$ if and only if $t = 0$ and $\varphi : R_+ \to R_+$ is lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$. Then $T$ has a unique fixed point.

Theorem 1.5 [5] Let $(X, d)$ be a complete metric space and $T : X \to X$ a mapping such that there exists $\alpha, \beta > 0$ with $\alpha + \beta < 1$ satisfying

$$d(T_x, T_y) \leq \alpha d(x, y) + \beta \frac{d(y, T_y)[1 + d(x, T_x)]}{1 + d(x, y)}$$

for all $x, y \in X$. Then $T$ has a unique fixed point.

Dutta and Chouhdury [7] proved that every $(\psi, \varphi)$-weakly contractive map has a unique fixed point in complete metric spaces. On the other hand, Berinde [3] introduced ‘weak contractions’ as a generalization of contraction maps.

Definition 1.6 [4] Let $(X, d)$ be a metric space. A self map $T : X \to X$ is said to be a weak contraction if there exists $\delta \in (0, 1)$ and $L \geq 0$ such that for all $(x, y) \in X$,

$$d(T_x, T_y) \leq \delta d(x, y) + Ld(y, T_x).$$

Berinde [4] proved that every weak contraction has a fixed point metric space and provided an example to show that this fixed point need not be unique. In order to obtain the uniqueness of point, Berinde [4] used the following condition: there exists $\theta \in (0, 1)$ and $L_1 \geq 0$ such that

$$d(T_x, T_y) \leq \theta d(x, y) + L_1 d(x, T_x)$$

for all $x, y \in X$ \quad (1)
and proved that every weak contraction together with (1) has a unique fixed point in complete metric spaces, and further posed the following problem “Find a contractive type condition different from (1), that ensures the uniqueness of fixed point of weak contractions”.

In this context Babu, Sandhya and Kameswari [2] answered the above problem by introducing ‘condition(B)’.

**Definition 1.7** [2] Let \((X,d)\) be a metric space. A map \(T : X \rightarrow X\) is said to satisfy condition(B) if there exists \(0 < \delta < 1\) and \(L \geq 0\) such that for all \(x,y \in X\),

\[
d(Tx,Ty) \leq \delta d(x,y) + L \min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}.
\]

We use the following definitions in our subsequent discussion.

**Definition 1.8** [8] Let \(A\) and \(B\) be selfmaps of a metric space \((X,d)\). The pair \((A,B)\) is said to be a compatible pair on \(X\), if

\[
\lim_{n \to \infty} d(ABx_n,BAx_n) = 0
\]

whenever \(\{x_n\}\) is a sequence in \(X\) such that \(\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = t\) for some \(t \in X\).

**Definition 1.9** [9] Let \(A\) and \(B\) be selfmaps of a metric space \((X,d)\). The pair \((A,B)\) is said to be weakly compatible, if they commute at their coincidence points i.e., \(ABx = BAx\) whenever \(Ax = Bx, x \in X\).

Every compatible pair of maps is weakly compatible, but its converse need not true [9].

Recently, Chandok [5] established the following common fixed point result of selfmaps satisfying certain contraction condition involving rational expression.

**Theorem 1.10** [5] Let \(M\) be a subset of a metric space \((X,d)\). Suppose that \(T,f,g : M \to M\) satisfy

\[
d(Tx,Ty) \leq \alpha(\frac{d(gx,Tx)d(gy,fy)}{d(gx,gy)+d(gx,gy)+d(gy,Tx)}) + \beta(d(gx,gy))
\]

for all \(x,y \in M\) and for some \(\alpha,\beta \in (0,1)\) with \(\alpha + \beta < 1\). Suppose also that \(T(M) \cup f(M) \subseteq g(M)\) and \((g(M),d)\) is complete. Then \(T,f\) and \(g\) have a coincidence point in \(M\). Also, if the pairs \((g,T)\) and \((g,f)\) are weakly comparable, then \(T,f\) and \(g\) have a unique common fixed point in \(X\).
2 Main Results

The following is our main theorem.

**Theorem 2.1** Let $M$ be a subset of a metric space $(X, d)$. Suppose that $T, f, g : M \to M$ satisfy
\[
d(Tx, fy) \leq \phi \left( \max \left\{ \frac{d(gx, Tx) + d(gy, Ty)}{d(gx, gy)} + d(gx, Ty), d(Tx, fx) \right\} \right)
\]
for all $x, y \in M$, $L \geq 0$ and $\phi \in \Psi$. Suppose also that $T(M) \cup f(M) \subseteq g(M)$ and $(g(M), d)$ is complete. Then
(i) $T, f$ and $g$ have a coincidence point in $M$;
(ii) If the pairs $(T, g)$ and $(f, g)$ are weakly compatible, then $T, f$ and $g$ have a unique common fixed point.

**Proof:** Let $x_0 \in X$. Since $T(M) \cup f(M) \subseteq g(M)$, we can choose $x_1, x_2 \in M$ such that $gx_1 = Tx_0$ and $gx_2 = fx_1$.
By induction, we construct a sequence in $x_n \subseteq X$ such that $gx_{2n+1} = Tx_{2n}$ and $gx_{2n+2} = fx_{2n+1}$, $n \geq 0$.
By the inequality (2.1.1)
\[
d(gx_{2n+1}, gx_{2n+2}) = d(Tx_{2n}, fx_{2n+1})
\]
\[
\leq \phi \left( \max \left\{ \frac{d(gx_{2n}, Tx_{2n}) + d(gx_{2n+1}, fx_{2n+1})}{d(gx_{2n}, gx_{2n+1})} \right\} \right)
\]
\[
+ L \min \left\{ d(gx_{2n+1}, Tx_{2n}), d(gx_{2n}, fx_{2n+1}), d(gx_{2n}, gx_{2n+1}), d(Tx_{2n}, fx_{2n+1}), d(Tx_{2n}, Tx_{2n}) \right\}
\]
\[
= \phi \left( \max \left\{ \frac{d(gx_{2n}, gx_{2n+1}) + d(gx_{2n}, gx_{2n+2})}{d(gx_{2n}, Tx_{2n})} \right\} \right)
\]
\[
+ L \min \left\{ d(gx_{2n+1}, gx_{2n+1}), d(gx_{2n}, gx_{2n+2}), d(gx_{2n}, gx_{2n+1}), d(gx_{2n+1}, gx_{2n+2}), d(gx_{2n}, gx_{2n+1}) \right\}
\]
\[
\leq \phi \left( \max \left\{ \frac{d(gx_{2n}, gx_{2n+1}) + d(gx_{2n}, gx_{2n+2})}{d(gx_{2n}, gx_{2n+1})} \right\} \right)
\]
\[
\leq \phi(d(gx_{2n}, gx_{2n+1})) \leq d(gx_{2n}, gx_{2n+1}) \quad \forall \quad n = 0, 1, 2, 3, \ldots \tag{2.1.2}
\]
Thus $\{d(gx_{2n+1}, gx_{2n+2})\}$ is monotone decreasing sequence of non-negative real numbers and hence convergent. Let $\lim_{n \to \infty} d(gx_{2n+1}, gx_{2n+2}) = r \geq 0$.
On taking as $n \to \infty$ in (2.1.2), we get $r \leq \phi(r) < r$, which is a contradiction. Therefore $r = 0$. i.e., $\lim_{n \to \infty} d(gx_{2n+1}, gx_{2n+2}) = 0$.
We now prove that $\{ gx_n \}$ is Cauchy.
It is sufficient to prove that $\{ gx_n \}$ is Cauchy.
If not there is an $\epsilon > 0$ and there exist sequences $\{2m_k\}, \{2n_k\}$ with $2m_k > 2n_k > k$ such that
\[
d(gx_{2m_k}, gx_{2n_k}) \geq \epsilon \text{ and } d(gx_{2m_k-2}, gx_{2n_k}) < \epsilon. \tag{2.1.3}
\]
We now prove that (i) $\lim_{k \to \infty} d(gx_{2m_k-2}, gx_{2n_k}) = \epsilon$.
Since $\epsilon \leq d(gx_{2m_k}, gx_{2n_k})$ for all $k$, we have $\epsilon \leq \liminf_{k \to \infty} d(gx_{2m_k}, gx_{2n_k})$.
Now for each positive integer $k$, by the triangular inequality, we get
\[
d(gx_{2m_k}, gx_{2n_k}) \leq d(gx_{2m_k}, gx_{2m_k-1}) + d(gx_{2m_k-1}, gx_{2m_k-2}) + d(gx_{2m_k-2}, gx_{2n_k}).
\]
On taking limit superior as $k \to \infty$, we get
\[
\limsup_{k \to \infty} d(gx_{2m_k}, gx_{2n_k}) \leq \epsilon
\]
Therefore \( \lim_{k \to \infty} d(gx_{2n_k}, gx_{2n_k}) = \epsilon \).

In the similar way, we prove the following:

\( (i) \lim_{k \to \infty} d(gx_{2n_k}, gx_{2n_k+1}) = \epsilon; \) \( (ii) \lim_{k \to \infty} d(gx_{2n_k}, gx_{2n_k-1}) = \epsilon; \)

\( (iv) \lim_{k \to \infty} d(gx_{2n_k-1}, gx_{2n_k+1}) = \epsilon; \)

We now consider

\[
d(gx_{2n_k}, gx_{2n_k+1}) = d(Tx_{2n_k}, fx_{2n_k-1})
\]

\[
\leq \psi(\max\left\{\frac{d(gx_{2n_k},Tx_{2n_k})}{d(gx_{2n_k-1},fx_{2n_k-1})}, \frac{d(gx_{2n_k},fx_{2n_k-1})}{d(gx_{2n_k-1},Tx_{2n_k})}\right\})
\]

\[
+ L \min\{d(gx_{2n_k-1},Tx_{2n_k}), d(gx_{2n_k},fx_{2n_k-1}), d(gx_{2n_k},gx_{2n_k-1}),
\]

\[
+ L \min\{d(Tx_{2n_k},fx_{2n_k-1}), d(gx_{2n_k},Tx_{2n_k})\}
\]

On letting \( k \to \infty \), we get \( \epsilon \leq \psi(\max\{0, \epsilon\}) + L \min\{\epsilon, \epsilon, \epsilon, \epsilon, 0\} \)

implies that \( \epsilon \leq \psi(\epsilon) < \epsilon \), a contradiction.

Therefore \( \{gx_n\} \) is Cauchy. As \( (g(M), d) \) is complete, there is \( t \in M \) such that

\( gx_n \to gt \) as \( n \to \infty \).

We now prove that \( t \) is a coincidence point of \( T \), \( f \) and \( g \).

We consider

\[
d(gx_{2n+1}, ft) = d(Tx_{2n}, ft)
\]

\[
\leq \phi(\max\left\{\frac{d(gx_{2n},Tx_{2n})}{d(gx_{2n},ft)+d(gx_{2n},ft)+d(gx_{2n},ft)+d(gx_{2n},ft)\}}\right\})
\]

\[
+ L \min\{d(gx_{2n},Tx_{2n}), d(gx_{2n},ft), d(gx_{2n},gt), d(Tx_{2n},ft), d(gx_{2n},Tx_{2n})\}
\]

On letting \( n \to \infty \), we have \( d(gt, ft) \leq 0 \Rightarrow gt = ft \).

Also, we have

\[
d(Tt, gt) = d(Tt, ft)
\]

\[
\leq \phi(\max\left\{\frac{d(gx_{2n},Tx_{2n})}{d(gx_{2n},ft)+d(gx_{2n},ft)+d(gx_{2n},ft)+d(gx_{2n},ft)\}}\right\})
\]

\[
+ L \min\{d(gt, Tt), d(gt, ft), d(gt, gt), d(Tt, ft), d(gt, Tt)\} = 0.
\]

Therefore \( d(Tt, gt) \leq 0 \Rightarrow Tt = gt = ft \).

Thus \( t \) is coincidence point of \( T \), \( f \) and \( g \).

Suppose that the pairs \((g, t)\) and \((g, f)\) are weakly compatible.

Let \( z = ft = gt = Tt \).

Then we have \( gTt = Tgt \) and \( fTt = gft \), which implies that \( gz = fz = Tz \).

Suppose \( gz \neq z \). We now consider

\[
d(gz, z) = d(Tz, ft) \leq \phi(\max\left\{\frac{d(gz,Tz)}{d(gz,ft)+d(gz,ft)+d(gz,ft)+d(gz,ft)\}}\right\})
\]

\[
+ L \min\{d(Tz, dz), d(gz, ft), d(gz, gt), d(Tz, gt), d(gz, Tz)\}
\]

\[
= \phi(\max\{d(gz, z)\}) + L \min\{d(z, gz), d(gz, z), d(gz, z), d(gz, z), d(gz, Tz)\}
\]

\[
\leq \phi(d(gz, z)) < d(gz, z),
\]
which is a contradiction. Therefore \(gz = fz = Tz = z\). Hence \(z\) is common fixed point. Uniqueness follows from the inequality (2.1.1).

**Corollary 2.2** Let \(M\) be a subset of a Metric space \((X, d)\). Suppose that \(T, f, g : M \to M\) satisfy \(d(Tx, fy) \leq \phi(\max \{d(x, y), d(fy, gy), d(gy, Tx)\})\) for all \(x, y \in M\), and \(\phi \in \Psi\). Suppose also that \(T(M) \cup f(M) \subseteq g(M)\) and \((g(M), d)\) is complete. Then (i) \(T, f, g\) have a coincidence point in \(M\);
(ii) If the pairs \((T, g)\) and \((f, g)\) are weakly compatible, then \(T, f, g\) have a unique common fixed point.

**Proof:** Take \(L = 0\) in Theorem 2.1, then the proof of Corollary follows.

**Theorem 2.3** Let \(X\) be a complete metric space and \(T : X \to X\) such that for every \(x, y \in X\),
\[
\frac{1}{2}d(x, Tx) \leq d(x, y) \implies \psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y)) + LN(x, y),
\]
where \(\psi \in \Psi\), \(\phi \in \Phi\) and \(L \geq 0\). Then \(T\) has a unique fixed point.

**Proof:** Let \(x_0 \in X\). construct a sequence \(\{x_n\}\) in \(X\) such that
\(x_{n+1} = Tx_n, \quad n = 0, 1, 2, 3, \ldots\)
For any \(n\) we have
\[
\frac{1}{2}d(x_{n-1}, x_n) \leq d(x_{n-1}, x_n) \quad \text{and using (2.3.1), we get}
\]
\[
\psi(d(Tx_{n-1}, Tx_n)) \leq \psi(M(x_{n-1}, x_n) - \phi(M(x_{n-1}, x_n)) + LN(x_{n-1}, x_n)
\]
which implies that
\[
\psi(d(x_n, x_{n+1})) \leq \psi(\max \{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n+1})}{2}\})
- \varphi(\max \{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n+1})}{2}\})
+ L \min \{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n}), d(x_n, Tx_{n+1})\}
= \psi(M(x_{n-1}, x_n) - \phi(M(x_{n-1}, x_n)) + LN(x_{n-1}, x_n))
- \varphi(\max \{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})}{2}\})
+ L \min \{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_{n+1})\}
= \psi(M(x_{n-1}, x_n) - \phi(M(x_{n-1}, x_n)) + LN(x_{n-1}, x_n))
- \varphi(\max \{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_{n+1})\})
+ L \min \{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_{n+1})\}
= \psi(M(x_{n-1}, x_n) - \phi(M(x_{n-1}, x_n)) + LN(x_{n-1}, x_n))
- \varphi(\max \{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_{n+1})\})

If \(\max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})\), then
\[
\psi(d(x_n, x_{n+1})) \leq \psi(d(x_n, x_{n+1})) - \varphi(d(x_n, x_{n+1})) < \psi(d(x_n, x_{n+1}))
\]
which is a contradiction. Therefore
\[
\psi(d(x_n, x_{n+1})) \leq \psi(d(x_{n-1}, x_n)) - \varphi(d(x_{n-1}, x_n)) < \psi(d(x_{n-1}, x_n))
\]
By the property of \(\psi\), we get
\[
d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n), \quad \text{for all } n
\]
Hence the sequence \(\{d(x_n, x_{n+1})\}\) is monotone decreasing sequence of non-negative reals and bounded below by 0. So, there exists \(r \geq 0\) such that
\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = r
\]
If \(r > 0\), then taking upper limits as \(n \to \infty\) in (2.3.2) and using the property
of ϕ, we get
ψ(r) ≤ ψ(r) − ϕ(r) < ψ(r), a contradiction.
Hence, lim
\[ \lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \] (2.3.4)
We know prove that \{x_n\} is a Cauchy sequence. On the contrary suppose that \{x_n\} is not Cauchy. Then there is an \( \epsilon > 0 \) and there exists integers \( m_k \) and \( n_k \) with \( m_k > n_k > k \) such that
\[ d(x_{m_k}, x_{n_k}) \geq \epsilon \quad \text{and} \quad d(x_{m_k-1}, x_{n_k}) < \epsilon. \] (2.3.5)
(i) We prove \( \lim_{k \to \infty} d(x_{m_k}, x_{n_k}) = \epsilon \)
From (2.3.5), we have \( \epsilon \leq d(x_{m_k}, x_{n_k}) \).
Taking limit infimum as \( k \to \infty \), we get
\[ \epsilon \leq \liminf_{k \to \infty} d(x_{m_k}, x_{n_k}). \] (2.3.6)
We consider
\[ d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) < d(x_{m_k}, x_{m_k-1}) + \epsilon \]
On letting limit superior as \( k \to \infty \), we get
\[ \limsup_{k \to \infty} d(x_{m_k}, x_{n_k}) \leq \epsilon. \] (2.3.7)
From (2.3.6) and (2.3.7), we get
\[ \lim_{k \to \infty} d(x_{m_k}, x_{n_k}) = \epsilon \]
In the similar way we prove the following:
(ii) \( \lim_{k \to \infty} d(x_{m_k}, x_{n_k+2}) = \epsilon; \) (iii) \( \lim_{k \to \infty} d(x_{m_k+1}, x_{n_k}) = \epsilon. \)
(iv) \( \lim_{k \to \infty} d(x_{m_k+1}, x_{n_k+1}) = \epsilon; \) (v) \( \lim_{k \to \infty} d(x_{m_k+1}, x_{n_k+2}) = \epsilon. \)
Now, \( \frac{1}{2}d(x_{m_k}, Tx_{n_k}) = \frac{1}{2}d(x_{m_k}, x_{m_k+1}) \leq d(x_{m_k}, x_{m_k+1}) \leq d(x_{m_k}, x_{n_k+1}) \)
as \( m_k \geq n_k \) implies \( m_k+1 \geq n_k+1 \).
Therefore by (2.3.1),
\[ \psi(d(Tx_{m_k}, Tx_{n_k+1})) \leq \psi(M(x_{m_k}, x_{n_k+1})) \leq \varphi(M(x_{m_k}, x_{n_k+1}))+\ln(x_{m_k}, x_{n_k+1}) \]
which implies that
\[ \psi(d(x_{m_k+1}, x_{n_k+2})) \leq \psi(\max\{d(x_{m_k}, x_{n_k+1}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k+1}, Tx_{n_k+1}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k+1}, Tx_{n_k+1}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k+1}, Tx_{n_k+1})\}) \]
\[ - \varphi(\max\{d(x_{m_k}, x_{n_k+1}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k+1}, Tx_{n_k+1}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k+1}, Tx_{n_k+1})\}) \]
\[ + L \min\{d(x_{m_k}, x_{n_k+1}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k+1}, Tx_{n_k+1}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k+1}, Tx_{n_k+1})\} \]
\[ = \psi(\max\{d(x_{m_k}, x_{n_k+1}), d(x_{m_k}, x_{n_k+1}), d(x_{n_k+1}, x_{n_k+2}), d(x_{m_k}, x_{n_k+1}), d(x_{n_k+1}, x_{n_k+2}), d(x_{m_k}, x_{n_k+1}), d(x_{n_k+1}, x_{n_k+2})\}) \]
\[ - \varphi(\max\{d(x_{m_k}, x_{n_k+1}), d(x_{m_k}, x_{n_k+1}), d(x_{n_k+1}, x_{n_k+2}), d(x_{m_k}, x_{n_k+1}), d(x_{n_k+1}, x_{n_k+2})\}) \]
\[ + L \min\{d(x_{m_k}, x_{n_k+1}), d(x_{m_k}, x_{n_k+1}), d(x_{n_k+1}, x_{n_k+2}), d(x_{m_k}, x_{n_k+1}), d(x_{n_k+1}, x_{n_k+2})\} \]
By taking limits \( k \to \infty \), it follows that
\[ \psi(\epsilon) \leq \psi(\epsilon) - \varphi(\epsilon) < \psi(\epsilon), \]
which is contradiction with \( \epsilon > 0 \).
Therefore \( \{x_n\} \) is a Cauchy sequence in \( X \).

Since \( X \) is complete, let \( \{x_n\} \) converges to some point \( z \in X \).

We now show that \( z \) is a fixed point of \( T \).

We claim that
\[
\frac{1}{2}d(x_{2n}, Tx_{2n}) \leq d(x_{2n}, z) \quad \text{or} \quad \frac{1}{2}d(x_{2n+1}, Tx_{2n+1}) \leq d(x_{2n+1}, z)
\]

Otherwise, we have
\[
d(x_{2n}, x_{2n+1}) \leq d(x_{2n}, z) + d(z, x_{2n+1})
\]
\[
= \frac{1}{2}d(x_{2n}, Tx_{2n}) + \frac{1}{2}d(x_{2n+1}, Tx_{2n+1})
\]
\[
= \frac{1}{2}[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]
\]
\[
\leq \frac{1}{2}[d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+1})] = d(x_{2n}, x_{2n+1}),
\]
which is a contradiction.

Therefore, there exists a subsequence \( \{n_k\} \) of \( \{n\} \) such that
\[
\frac{1}{2}d(x_{n_k}, x_{n_k+1}) \leq d(x_{n_k}, z).
\]

By (2.3.1), we have
\[
\psi(d(Tx_{n_k}, Tz)) \leq \psi(M(x_{n_k}, z)) - \varphi(M(x_{n_k}, z)) + LN(x_{n_k}, z)
\]
\[
= \psi(\max\{d(x_{n_k}, z), d(x_{n_k}, Tx_{n_k}), d(z, Tz), d(x_{n_k}, Tx_{n_k}) + d(z, Tx_{n_k})\})
\]
\[
- \varphi(\max\{d(x_{n_k}, z), d(x_{n_k}, Tx_{n_k}), d(z, Tz), d(x_{n_k}, Tx_{n_k}) + d(z, Tx_{n_k})\})
\]
\[
+ L \min\{d(x_{n_k}, z), d(x_{n_k}, Tx_{n_k}), d(z, Tz), d(x_{n_k}, Tz), d(z, Tx_{n_k})\}
\]
\[
= \psi(\max\{d(x_{n_k}, z), d(x_{n_k}, x_{n_k+1}), d(z, Tz), d(x_{n_k}, Tz) + d(z, x_{n_k+1})\})
\]
\[
- \varphi(\max\{d(x_{n_k}, z), d(x_{n_k}, x_{n_k+1}), d(z, Tz), d(x_{n_k}, Tz) + d(z, x_{n_k+1})\})
\]
\[
+ L \min\{d(x_{n_k}, z), d(x_{n_k}, x_{n_k+1}), d(z, Tz), d(x_{n_k}, Tz), d(z, x_{n_k+1})\}\].

On letting \( k \to \infty \), we get
\[
\psi(d(z, Tz)) \leq \psi(d(z, Tz)) - \varphi(d(z, Tz))
\]
which implies that \( z = Tz \).

Therefore \( z \) is a fixed point of \( T \).

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**References**


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