Oscillations of Second-Order Sub-linear Delay Differential Equations with Impulses

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Abstract

By employing classical integral inequalities, the sufficient and necessary conditions for a second-order sub-linear delay differential equation with impulses to be oscillatory are established. The results of this paper contain the oscillation results in case when no impulses occur. Two examples show that the oscillation of impulsive delay differential equations can be caused by impulsive perturbations, though the corresponding delay differential equation admits a nonoscillatory solution.

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1 Introduction

Impulsive effect, likewise, exists in a wide variety of evolutionary processes in which states are changed abruptly at certain moments, involving such fields as physics, medicine and biology, economics, mechanics, electronics, telecommunications, and so forth. In the past decades, the theory of oscillatory properties of impulsive ordinary differential equation (IODE) and impulsive delay differential equation (IDDE) has been investigated by many authors (see, e.g. [1-10]). For example, Bainov et al. studied the oscillation properties of first-order impulsive differential equations with deviating arguments [3]. Especially
in [4], Chen investigated oscillations of second-order nonlinear differential with impulses, and he emphasized that the impulses may change the oscillatory behavior of an equation. Based on [4], the authors were devoted to oscillations of impulsive differential equations (see, e.g., [5-10]). However, most of these articles concern the sufficient conditions on oscillation of first order or second order IODE and IDDE. Here, we firstly devote to study the the sufficient and necessary conditions on oscillation for a second-order sub-linear delay differential equations with impulses of the form

\[
\begin{cases}
(r(t)x'(t))' + p(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) = 0, & t \neq t_k, \\
\Delta x'(t)|_{t=t_k} + q_k|x(\tau(t_k))|^{\alpha-1}x(\tau(t_k)) = 0, \\
\Delta x(t)|_{t=t_k} = 0,
\end{cases}
\]

(1)

where \(0 < \alpha < 1\), \(t \geq t_0\), and \(0 \leq t_0 < t_1 < \cdots < t_k < \cdots\) such that \(\lim_{k \to \infty} t_k = \infty\), and

\[
\Delta x^{(i)}(t)|_{t=t_k} = x^{(i)}(t_k^+) - x^{(i)}(t_k^-), \\
x^{(i)}(t_k^+) \to \lim_{t \to t_k^+} x^{(i)}(t), x^{(i)}(t_k^-) \to \lim_{t \to t_k^-} x^{(i)}(t), i = 0, 1.
\]

(2)

We will establish sufficient and necessary conditions for guaranteeing oscillation of (1), based on combinations of the following conditions:

(A) \(p(t)\) is continuous on \([t_0, \infty)\), \(r(t)\) is differentiable on \([t_0, \infty)\) and \(r'(t) \geq 0\);

(B) \(\{q_k\}\) is a sequence of real numbers;

(C) \(\tau(t)\) is continuous on \([t_0, \infty)\), and \(\tau(t) \leq t, \lim_{t \to \infty} \tau(t) = \infty\).

By a solution of (1), we mean a function \(x = x(t)\) defined on \([t_0, \infty)\) such that:

(i) \(x(t)\) is continuous on \([t_0, \infty)\);

(ii) \(x'(t)\) and \((r(t)x'(t))'\) are continuous on \([t_0, \infty)\) \(\setminus \{t_k\}\), \(x^{(i)}(t_k^+), x^{(i)}(t_k^-)\) exist and satisfy

\[
\Delta x'(t)|_{t=t_k} + q_k|x(\tau(t_k))|^{\alpha-1}x(\tau(t_k)) = 0 \text{ for any } t_k;
\]

(iii) \(x(t)\) satisfies \((r(t)x'(t))' + p(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) = 0\) at each point \(t \in [t_0, \infty) \setminus \{t_k\}\).

A solution of (1) is said to be non-oscillatory if it is eventually positive or eventually negative. Otherwise, it is said to be oscillatory.

In case \(r(t) = 1\) and without impulses effect, (1) reduces to Emden-Fowler equation with delay

\[
x''(t) + p(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) = 0, 0 < \alpha < 1,
\]

(3)

The problem of oscillation of solutions of (3) has been studied by many authors. Kusano and Onose [1] see also [2,3] proved the following necessary and sufficient conditions for oscillation of (3).
Theorem A. Assume $p(t) \geq 0$, then every solution of (3) is oscillatory if and only if
\[
\int_{\infty}^{\infty} [\tau(t)]^\alpha p(t) dt = \infty.
\] (4)

Here, we extend equation (3) to impulsive delay differential equation (1). Therefore, oscillation criteria based on our conditions will either be new or improve some, if not all, of the previous results in [11,12,13].

In this paper, the oscillation of (1) are studied. Employing classical integral inequalities, the sufficient and necessary conditions to ensure every solution of (1) to be oscillatory are obtained. Two examples both show that the impulses perturbations may play an important role on oscillation of the solutions.

2 Preparatory lemmas

To prove Theorem 1 and Theorem 2, we need the following lemmas.

Lemma 1 ([14]). Assume that each $A_k$ is continuous on $[a,b]$, then
\[
\int_{a}^{b} \sum_{s \leq t_k < b} A_k (s) ds = \sum_{a \leq t_k < b} \int_{a}^{t_k} A_k (s) ds.
\] (5)

Lemma 2 ([15]). Note $J = [a,b]$, let $u, \lambda \in C(J, \mathbb{R}_+), h \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $c \in \mathbb{R}_+$, and let $\{\lambda_k\}$ is a sequence of positive real numbers. Assume that $u(J) \subset I \subset \mathbb{R}_+$, and
\[
u(t) \leq c + \int_{a}^{t} \lambda(s) h(u(s)) ds + \sum_{a < t_k < t} \lambda_k h(u(t_k)), t \in J,
\] (6)
then
\[
u(t) \leq G^{-1}\{G(c) + \int_{a}^{t} \lambda(s) ds + \sum_{a < t_k < t} \lambda_k\}, t \in [a, \beta),
\] (7)
where
\[
G(u) = \int_{u_0}^{u} \frac{dx}{h(x)}, u, u_0 \in I,
\]
\[
\beta = \sup\{\nu \in J : G(c) + \int_{a}^{t} \lambda(s) ds + \sum_{a < t_k < t} \lambda_k \in G(\mathbb{I}), a \leq t \leq \nu\}.
\] (8)
3 Main Results

Theorem 1. Assume that the conditions (A), (B) and (C) hold, and

\[ \int_{t_0}^{\infty} [\tau(t)]^\alpha \frac{|p(t)|}{|r(t)|} dt + \sum_{k=0}^{\infty} [\tau(t_k)]^\alpha q_k < \infty, \tag{9} \]

then (1) has a solution \( x(t) \) satisfying

\[ \lim_{t \to \infty} \frac{x(t)}{t} = a \neq 0. \tag{10} \]

Proof. Let \( t_1 \geq \max\{1, t_0\} \). Integrating (1) from \( t_1 \) to \( t \), we obtain

\[
x'(t) = \frac{r(t_1)}{r(t)} x'(t_1) - \frac{1}{r(t)} \sum_{t_1 \leq t_k < t} r(t_k) q_k |x(\tau(t_k))|^{\alpha-1} x(\tau(t_k)) \]

\[
- \frac{1}{r(t)} \int_{t_1}^{t} p(s) |x(\tau(s))|^{\alpha-1} x(\tau(s)) ds, \quad t \geq t_1, \tag{11} \]

Integrating the above equality from \( t_1 \) to \( t \) and applying Lemma 1, we get

\[
x(t) = x(t_1) + x'(t_1) \int_{t_1}^{t} \frac{r(t_1)}{r(s)} ds - \sum_{t_1 \leq t_k < t} q_k |x(\tau(t_k))|^{\alpha-1} x(\tau(t_k)) \int_{t_1}^{t} \frac{r(t_k)}{r(s)} ds \]

\[
- \int_{t_1}^{t} p(s) |x(\tau(s))|^{\alpha-1} x(\tau(s)) \left( \int_{s}^{t} \frac{1}{r(v)} dv \right) ds, \quad t \geq t_1. \tag{12} \]

Set

\[
u(t) = c + \sum_{t_1 \leq t_k < t} |q_k||x(\tau(t_k))|^{\alpha} + \int_{t_1}^{t} \frac{|p(s)|}{|r(s)|} |x(\tau(s))|^{\alpha} ds, \quad t \geq t_1, \tag{13} \]

where \( c = |x(t_1)| + |x'(t_1)| \). By the condition \( r(t) \in C^1 ([t_0, \infty), \mathbb{R}_+) \) and \( r'(t) \geq 0 \), we obtain

\[
x(t) \leq |x(t_1)| + |x'(t_1)||t - t_1| + \sum_{t_1 \leq t_k < t} |q_k||x(\tau(t_k))|^{\alpha}|t - t_k| \]


\begin{align*}
+ \int_{t_1}^{t} \frac{|p(s)|}{|r(s)|} |x(\tau(s))|^{\alpha}|t-s|ds \\
\leq t|x(t_1)| + t|x'(t_1)| + t \sum_{t_1 \leq t_k < t} |q_k||x(\tau(t_k))|^{\alpha} + t \int_{t_1}^{t} \frac{|p(s)|}{|r(s)|} |x(\tau(s))|^{\alpha}ds \\
= tu(t), t \geq t_1. \quad (14)
\end{align*}

Let \( t_2 \geq t_1 \) be such that \( \tau(t) \geq t_1 \) for all \( t \geq t_2 \). Replacing \( t \) by \( \tau(t) \) in (14) and by the nondecreasing character of \( u(t) \), in view of \( \tau(t) \leq t \), we have

\[ |x(\tau(t))| \leq \tau(t)u(\tau(t)) \leq \tau(t)u(t), t \geq t_2. \quad (15) \]

It follows from (13) that

\[ u'(t) = \frac{|p(t)|}{|r(t)|} |x(\tau(t))|^{\alpha}, t \neq t_k, \quad (16) \]

\[ \triangle u(t)|_{t=t_k} = |q_k||x(\tau(t_k))|^{\alpha}, \quad (17) \]

for \( t \geq t_2 \) and \( t_k \geq t_2 \). Now, integrating (16) from \( t_2 \) to \( t \), in combination with (15) and (17), we lead to

\[ u(t) = c + \int_{t_2}^{t} \frac{|p(s)|}{|r(s)|} [\tau(s)]^\alpha[u(s)]^{\alpha}ds + \sum_{t_2 \leq t_k < t} |q_k|[\tau(t_k)]^{\alpha}[u(t_k)]^{\alpha}, \quad (18) \]

Here, we note

\[ h(x) = x^\alpha, \lambda(s) = \frac{|p(s)|}{|r(s)|} [\tau(s)]^\alpha, \lambda_k = |q_k|[\tau(t_k)]^\alpha, \quad (19) \]

Applying Lemma 2, we get

\[ u(t) \leq G^{-1}\{G(c) + \int_{t_2}^{t} \frac{|p(s)|}{|r(s)|} [\tau(s)]^\alpha ds + \sum_{t_2 \leq t_k < t} |q_k|[\tau(t_k)]^{\alpha}\}, \quad (20) \]

Because of

\[ G(u) = \frac{u^{1-\alpha}}{1-\alpha}, G^{-1}(u) = [(1-\alpha)u + u_0^{1-\alpha}]^{\frac{1}{1-\alpha}}, \quad (21) \]

then the equality (20) can be written as follow,

\[ u(t) \leq [c^{1-\alpha} + (1-\alpha) \int_{t_2}^{t} \frac{|p(s)|}{|r(s)|} [\tau(s)]^\alpha ds + (1-\alpha) \sum_{t_2 \leq t_k < t} |q_k|[\tau(t_k)]^{\alpha}] \frac{1}{1-\alpha}, \quad (22) \]

combine with (9), we obtain

\[ u(t) \leq c_1, t \geq t_2, \quad (23) \]
where
\[ c_1 = [c^{1-\alpha} + (1 - \alpha) \int_{t_2}^{t} \frac{|p(s)|}{r(s)} |\tau(s)|^\alpha ds + (1 - \alpha) \sum_{t_2 \leq t_k < t} |q_k| |\tau(t_k)|^\alpha]^{-\frac{1}{\alpha}}. \] (24)

By (14), (15) and (23), we have
\[ |x(t)| \leq c_1 t, |x(\tau(t))| \leq c_1 \tau(t), t \geq t_2. \] (25)

Next, we need to prove that \( x'(t) \) has a nonzero limit as \( t \to \infty \). To see this we integrate (1) from \( t_2 \) to \( t \), in view of \( r'(t) \geq 0 \), we obtain
\[
x'(t) = \frac{r(t_2)}{r(t)} x'(t_2) - \frac{1}{r(t)} \sum_{t_2 \leq t_k < t} r(t_k) q_k |x(\tau(t_k))|^{\alpha-1} x(\tau(t_k))
- \frac{1}{r(t)} \int_{t_2}^{t} p(s) |x(\tau(s))|^{\alpha-1} x(\tau(s)) ds
\geq \frac{r(t_2)}{r(t)} x'(t_2) - \sum_{t_2 \leq t_k < t} |r(t_k)| |q_k| |x(\tau(t_k))|^{\alpha} - \int_{t_2}^{t} \frac{|p(s)|}{|r(s)|} |x(\tau(s))|^{\alpha} ds
\geq \omega x'(t_2) - \sum_{t_2 \leq t_k < \infty} |q_k| |x(\tau(t_k))|^{\alpha} - \int_{t_2}^{\infty} \frac{|p(s)|}{|r(s)|} |x(\tau(s))|^{\alpha} ds. \] (26)

where \( \omega = \frac{r(t_2)}{r(t)} \in (0, 1) \) and \( \omega \) not tends to zero as \( t \to \infty \), employing (25), we obtain
\[ \int_{t_2}^{\infty} \frac{|p(s)|}{|r(s)|} |x(\tau(s))|^{\alpha} ds \leq c_1^\alpha \int_{t_2}^{\infty} \frac{|p(s)|}{|r(s)|} |\tau(s)|^{\alpha} ds, \\
\sum_{t_2 \leq t_k < \infty} |q_k| |x(\tau(t_k))|^{\alpha} \leq c_1^\alpha \sum_{t_2 \leq t_k < \infty} |q_k| |\tau(t_k)|^{\alpha}, \] (27)
in view of (26) and (27), we have
\[
x'(t) \geq \omega x'(t_2) - c_1^\alpha \sum_{t_2 \leq t_k < \infty} |q_k| |\tau(t_k)|^{\alpha} + \int_{t_2}^{\infty} \frac{|p(s)|}{|r(s)|} |\tau(s)|^{\alpha} ds
\geq \frac{\omega x'(t_2)}{3} \] (28)
which is always possible to hold by choosing a fixed \( t_2 \). Therefore, \( \lim_{t \to \infty} x'(t) = L > 0 \). The proof is complete.
Theorem 2. Assume that the conditions (A), (B) and (C) hold, and \( p(t) \) and \( \{q_k\} \) are nonnegative. Then every solution of (1) is oscillatory if only if

\[
\int_{t_0}^{\infty} \left[ \int_{t_0}^{\tau(t)} \frac{1}{r(s)} ds \right] \alpha p(t) dt + \sum_{k=0}^{\infty} \left[ \int_{t_0}^{\tau(t_k)} \frac{1}{r(s)} ds \right] \alpha q_k = \infty. \tag{29}
\]

Proof. If (29) does not hold. Then we have

\[
\int_{t_0}^{\infty} \left[ \int_{t_0}^{\tau(t)} \frac{1}{r(s)} ds \right] \alpha p(t) dt + \sum_{k=0}^{\infty} \left[ \int_{t_0}^{\tau(t_k)} \frac{1}{r(s)} ds \right] \alpha q_k < \infty. \tag{30}
\]

in view of \( r'(t) \geq 0 \), we have (9) hold. By Theorem 1, we know that (1) has a solution \( x(t) \) which satisfies (10). Obviously, such a solution is nonoscillatory. So the necessity is proved.

Next, we will prove the sufficiency. If there is a nonoscillatory solution \( x(t) \) of (1). We may suppose that \( x(t) \) is eventually positive, the case \( x(t) \) being eventually negative is similar. Obviously, there exists \( t_1 \geq t_0 \) such that \( x(\tau(t)) > 0 \) for all \( t \geq t_1 \). It follows from (1), we have that

\[
(r(t)x'(t))' \leq 0. \tag{31}
\]

Thus, \( r(t)x'(t) \) is decreasing on every interval of \( [t_1, \infty) \backslash \{t_k\} \). We also have \( \Delta x'(t_k) \leq 0 \) by the impulse conditions in (1). Therefore, we deduce that \( r(t)x'(t) \) is nonincreasing on \( [t_1, \infty) \).

We may claim that \( x'(t) \) is eventually positive. Because if \( x'(t) < 0 \), eventually, then \( x(t) \) becomes negative for large sufficiently \( t \). This is a contradiction. Integrate (31) from \( t_1 \) to \( t \), we get

\[
r(t)x'(t) - r(t_1)x'(t_1) + \sum_{k=0}^{\infty} q_k [x(\tau(t_k))]^\alpha \leq 0, \tag{32}
\]

it is easy to show that

\[
x'(t) \geq \frac{r(t_1)}{r(t)} x'(t_1), \tag{33}
\]

integrate (33) from \( t_1 \) to \( t \), we have

\[
x(t) \geq x(t) - x(t_1) \geq x'(t_1) \int_{t_1}^{t} \frac{r(s)}{r(t)} ds \geq x'(t) \int_{t_1}^{t} \frac{r(t_1)}{r(s)} ds, t \geq t_1, \tag{34}
\]

Let \( t_2 \geq t_1 \) such that be \( \tau(t) \geq t_1 \), using (34) and the decreasing character of \( x'(t) \), we get

\[
x(\tau(t)) \geq x'(\tau(t)) \int_{t_1}^{\tau(t)} \frac{r(t_1)}{r(s)} ds \geq x'(t) \int_{t_1}^{\tau(t)} \frac{r(t_1)}{r(s)} ds, t \geq t_2. \tag{35}
\]
and so, by (1),
\[(r(t)x'(t))' + p(t)\left[\int_{t_1}^{\tau(t)} \frac{r(t_1)}{r(s)} ds \right]^\alpha x'(t) \leq 0, \quad t \neq t_k,\] (36)
in view of the derivable character of \(r(t)\) and \(r'(t) \geq 0\), we have
\[r(t)x''(t) + p(t)\left[\int_{t_1}^{\tau(t)} \frac{r(t_1)}{r(s)} ds \right]^\alpha x'(t) \leq 0, \quad t \neq t_k,\] (37)
dividing (37) by \(r(t)[x'(t)]^\alpha\), we get
\[\frac{x''(t)}{[x'(t)]^\alpha} + \frac{p(t)}{r(t)}\left[\int_{t_1}^{\tau(t)} \frac{r(t_1)}{r(s)} ds \right]^\alpha \leq 0, \quad t \neq t_k,\] (38)
integrating (38) from \(t_2\) to \(t\), we obtain
\[\sum_{t_2 \leq t_k < t} \left\{ [x'(t_k)]^{1-\alpha} - [x'(t_k)] - q_k[x(\tau(t_k))]^{1-\alpha} \right\} + [x'(t)]^{1-\alpha} - [x'(t_2)]^{1-\alpha}\]
\[+ (1-\alpha) \int_{t_2}^{t} \frac{p(s)}{r(s)} \left[\int_{t_1}^{\tau(s)} \frac{r(t_1)}{r(v)} dv \right]^\alpha ds \leq 0,\] (39)
which implies that
\[\sum_{t_2 \leq t_k < t} b_k + (1-\alpha) \int_{t_2}^{t} \frac{p(s)}{r(s)} \left[\int_{t_1}^{\tau(s)} \frac{r(t_1)}{r(v)} dv \right]^\alpha ds \leq [x'(t_2)]^{1-\alpha},\] (40)
where
\[b_k = [x'(t_k)]^{1-\alpha} \left\{ 1 - [1 - q_k[x(\tau(t_k))]^{\alpha}]^{1-\alpha} \right\},\] (41)
Since \(1 - (1-u)^{1-\alpha} \geq (1-\alpha)u\) for \(u \in (0, \infty)\) and \(0 < \alpha < 1\), by taking
\[u = \frac{q_k[x(\tau(t_k))]^{\alpha}}{x'(t_k)},\] (42)
see from (41), we can get
\[b_k \geq (1-\alpha) \frac{q_k[x(\tau(t_k))]^{\alpha}}{[x'(t_k)]^{\alpha}}.\] (43)
From (35), we have
\[x(\tau(t_k)) \geq x'(\tau(t_k)) \int_{t_1}^{\tau(t_k)} \frac{r(t_1)}{r(s)} ds \geq x'(t_k) \int_{t_1}^{\tau(t_k)} \frac{r(t_1)}{r(s)} ds,\] (44)
an important role in the oscillatory behavior of equations under perturbing

\[ b_k \geq (1 - \alpha)q_k \left[ \int_{t_1}^{\tau(t_k)} \frac{r(t_1)}{r(s)} ds \right]^\alpha. \quad (45) \]

Finally, by (40) and (45), we obtain

\[ \sum_{t_2 \leq t_k < t} q_k \left[ \int_{t_1}^{\tau(t_k)} \frac{1}{r(s)} ds \right]^\alpha + \int_{t_2}^{t} p(s) \left[ \int_{t_1}^{\tau(s)} \frac{1}{r(v)} dv \right]^\alpha ds \leq \frac{[x'(t_2)]^{1-\alpha}}{(1-\alpha)[r(t_1)]^\alpha}, \quad (46) \]

let \( t \) tends to \( \infty \), we have

\[ \sum_{t_2 \leq t_k < \infty} q_k \left[ \int_{t_1}^{\tau(t_k)} \frac{1}{r(s)} ds \right]^\alpha + \int_{t_2}^{\infty} p(s) \left[ \int_{t_1}^{\tau(s)} \frac{1}{r(v)} dv \right]^\alpha ds < \infty, \quad (47) \]

which contradicts with (29). The proof is complete.

**Example 1.** Consider the delay equation

\[
\begin{cases}
(t^{\frac{1}{2}}x)' + \frac{1}{8}t^{-\frac{7}{4}}(t-1)^{-\frac{1}{4}}|x(t-1)|^{-\frac{3}{2}}x(t-1) = 0, t \neq k, \\
\Delta x'(t) |_{t=k} + (k-1)^{-1}|x(k-1)|^{-\frac{1}{2}}x(k-1) = 0, \\
\Delta x(t) |_{t=k} = 0,
\end{cases}
\]

where \( t \geq 2 \) and \( k \geq 2 \).

We see that \( \alpha = \frac{1}{2}, r(t) = t^{\frac{1}{2}}, \tau(t) = t-1, p(t) = \frac{1}{8}t^{-\frac{7}{4}}(t-1)^{-\frac{1}{4}}, q_k = (k-1)^{-1} \) and \( t_k = k \). Since

\[
\int_{t_2}^{\infty} \left[ \int_{2}^{t-1} \frac{1}{s^{\frac{3}{4}}} ds \right]^{\frac{1}{2}} \frac{1}{t^{\frac{7}{4}}(t-1)^{-\frac{1}{4}}} dt + \sum_{2}^{\infty} \left[ \int_{2}^{k-1} \frac{1}{s^{\frac{3}{4}}} ds \right]^{\frac{1}{2}}(k-1)^{-1} = \infty.
\]

applying Theorem 2 we conclude that every solution of (48) is oscillatory. But

the delay differential equation

\[
(t^{\frac{1}{2}}x)' + \frac{1}{8}t^{-\frac{7}{4}}(t-1)^{-\frac{1}{4}}|x(t-1)|^{-\frac{3}{2}}x(t-1) = 0, t \geq 2,
\]

has a nonnegative solution \( x = \sqrt{t} \). This example shows that impulses play

an important role in the oscillatory behavior of equations under perturbing

impulses.

**Example 2.** Consider the equation

\[
\begin{cases}
x''(t) + (t-1)^{-2}|x(t-1)|^{-\frac{1}{2}}x(t-1) = 0, t \neq k, \\
\Delta x'(t) |_{t=k} + (k-1)^{-1}|x(k-1)|^{-\frac{1}{2}}x(k-1) = 0, \\
\Delta x(t) |_{t=k} = 0,
\end{cases}
\]

}\]
where \( t \geq 2 \) and \( k \geq 2 \).

We see that \( \alpha = \frac{1}{4}, r(t) = 1, \tau (t) = t - 1, p(t) = (t - 1)^{-2}, q_k = (k - 1)^{-1} \) and \( t_k = k \). Since

\[
\int_{2}^{\infty} (t - 1)^{-\frac{7}{4}} dt + \sum_{2}^{\infty} (k - 1)^{-\frac{3}{4}} = \infty,
\]

applying Theorem 2 we conclude that every solution of (49) is oscillatory.

We note that if the equation is not subject to any impulse condition, then, since

\[
\int_{2}^{\infty} (t - 1)^{-\frac{7}{4}} dt < \infty,
\]

The equation

\[ x''(t) + (t - 1)^{-2}|x(t - 1)|^{-\frac{1}{4}} x(t - 1) = 0 \]

has a nonoscillatory solution by Theorem A.

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References


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