Necessary and Sufficient Conditions for the Existence of Globally Asymptotic Stability of Quasi-periodic Dynamical Systems

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Abstract

The paper deals with systems of differential equations whose right-hand sides are quasi-periodic functions with respect to an independent variable. General properties of solutions and the structure of invariant sets of the systems are examined. In this paper we present necessary and sufficient conditions for the existence of globally asymptotic stability of periodic and quasi-periodic solutions. Characteristics of some general properties of integrated curves of systems of differential equations with quasi-periodic with respect to independent argument right-
hand sides are given. Necessary and sufficient conditions for existence of quasi-periodic oscillations and globally asymptotic stability are provided. Qualitative methods have been applied to obtain the results presented in the paper.

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**Keywords:** invariant set, minimal set, convergence, recurrent motion, quasi-periodic dynamical system, continuity properties, limit point, globally asymptotic stability

1 **Introduction**

The description of real dynamical processes is carried out often by means of non-autonomous systems of differential equations or reduced to such systems.

Therefore, oscillations in various mechanical, electro-physical, radio-technical, social and economic systems are studied by means of mathematical apparatus of the differential equations with the right-hand sides being quasi-periodic with respect to independent argument [1], [12].

2 **Problem statement of stability of invariant sets**

Let us consider the system of differential equations

\[
\frac{dx}{dt} = F(t, x),
\]

such that the right-hand sides of the system satisfy the following conditions:

1). Vector function

\[
F(t, x) = F(\Omega t, x), \Omega = (\omega_1, \ldots, \omega_m), \omega_i > 0, i = 1, \ldots, m,
\]

where \( t \in (-\infty, +\infty), \ X \in \mathbb{E}^n \) is real and continuous with respect to all of its arguments;

2). The function \( F(Y, X) \) is \( 2\pi \)-periodic with respect to all the components of the vector \( Y = (y_1, \ldots, y_m) \), and the components of the vector \( \Omega = (\omega_1, \ldots, \omega_m) \) are incommensurable;

3). In some bounded region \( G \in \mathbb{E}^n \) the function \( F(t, x) \) satisfies the Lipschitz condition
\[ \|F(t,x_1) - F(t,x_2)\| < L_g \|X_1 - X_2\|, t \in (-\infty, +\infty), \]

where \( L_g \) is a positive constant, that may depend on region \( G \).

When these conditions are met, each \( X_0 \in \mathbb{E}^n \) and \( t_0 \in (-\infty, +\infty) \) corresponds to only one vector function \( X(t,t_0,x_0) \) satisfying system (1) and such that \( X(t_0,t_0,x_0) = x_0 \). This vector function is continuous with respect to all its arguments. We assume that it is defined for all \( t \in (-\infty, +\infty) \).

Let us give \( t = t_0 + \tau \) and denote by \( Z(\tau,Y_0,Z_0) \) the solution of the system

\[
\frac{dz}{dt} = F(\Omega t + Y_0, Z),
\]

(2)

\( Y_0 \in \mathbb{E}^m, Z \in \mathbb{E}^n \), such that \( Z(0,Y_0,Z_0) = Z_0 \).

Note the following properties of the function \( Z(\tau,Y_0,Z_0) \), see [9], [13]:
1). for any \( Z_0 \in \mathbb{E}^n \) and \( Y_0 \in \mathbb{E}^m \) the only curve in \( \mathbb{E}^n \), is defined for \( \tau \in (-\infty, +\infty) \), such that

\[ Z(0,Y_0,Z_0) = Z_0; \]

2). The function \( Z(\tau,Y_0,Z_0) \) is continuous in all of its arguments;
3). for any \( \tau_1, \tau_2 \in (-\infty, +\infty) \)

\[ Z(\tau_1 + \tau_2,Y_0,Z_0) = Z(\tau_2,Y_0 + \Omega \tau_1,Z(\tau_1,Y_0,Z_0)); \]

4). for any \( Z_0 \in \mathbb{E}^n \) and \( \tau \in (-\infty, +\infty) \)

\[ Z(\tau,Y_0 + 2\pi P,Z_0) = Z(\tau,Y_0,Z_0), \]

\( P \) - is the vector space \( \mathbb{E}^m \) with integer components.

**Definition 1 .** We shall say that a quasi-periodic dynamical system \( \mathcal{D}_q \) is defined in \( \mathbb{E}^n \) if a vector function \( Z(\tau,Y_0,Z_0) \in \mathbb{E}^n \) for \( \tau \in (-\infty, +\infty) \), \( Y_0 \in \mathbb{E}^m, Z_0 \in \mathbb{E}^n \) satisfies conditions 1 – 4 [3].

**Definition 2 .** A set \( A \in \mathbb{E}^n \) is called an invariant set \( \mathcal{D}_q \) if for any \( Z_0 \in A \) it is possible to specify \( Y_0 \in \mathbb{E}^m \) such that the trajectory \( Z((\tau_0, +\infty),Y_0,Z_0) \) of motion \( Z(\tau,Y_0,Z_0) \) is contained in \( A \) [3], [8], [11].

3 Qualitative analysis of invariant sets

**Definition 3 .** Bounded closed non-empty invariant set \( \mathcal{D}_q \) that contains no proper subset with the same properties is called the minimal set [4].
Theorem 1 : Any motion $Z(\tau, Y_0, Z_0)$ that belongs to the minimal set is recurrent.

Proof : Suppose that this is not the case, then there is at least one $\varepsilon > 0$ such that there exist three sequences $\alpha_k$, $\tau_k$, $T_k$, where $T_k \to +\infty$ as $k \to +\infty$, and the point $Z(\tau_k, Y_0, Z_0)$ does not belong to $\varepsilon$-neighbourhood of set $Z([\alpha_k, (\alpha_k + T_k)], Y_0, Z_0)$.

Let us select sub-sequences from these sequences such that

$Z(\tau_n, Y_0, Z_0) \to Z_0$, $Y(\tau_n, Y_0, Z_0) \to Y_0 (mod 2\pi)$,

$Z(\alpha_n, Y_0, Z_0) \to Z_1$, $Y(\alpha_n, Y_0, Z_0) \to Y_1 (mod 2\pi)$ for $n \to +\infty$.

The trajectory of $Z(\tau_k, Y_1, Z_1)$ belongs to the considered minimal set $F$ because it is invariant and closed. Also, $Z_0 \in F$ but

$$\| Z_0 - Z(\tau, Y_1, Z_1) \| \geq \varepsilon$$ for $t \in (-\infty, +\infty)$,

since if there exists $\tau \in (-\infty, +\infty)$ such that

$$\| Z_0 - Z(\tau, Y_1, Z_1) \| < \varepsilon,$$

then starting from a certain $n$ we would have

$$\| Z(\tau_n, Y_0, Z_0) - Z(\tau + \alpha_n, Y_0, Z_0) \| < \varepsilon,$$

which contradicts the choice of sequences $\alpha_k$, $\tau_k$, $T_k$. Consequently, the set of $\omega$-limit points of the motion $Z(\tau, Y_1, Z_1)$ is a closed invariant proper subset of $F$, since $Z_0$ cannot be $\omega$-limit point of the motion $Z(\tau, Y_1, Z_1)$, which means that $F$ is the minimal set. This contradiction proves the theorem.

Theorem 2 : If $Z(\tau, Y_0, Z_0)$ is a recurrent motion, the closure of its trajectory is the minimal set.

Proof : Let $\overline{Z((-\infty, +\infty), Y_0, Z_0)} = F$.

If there is the minimal set of $M \subset F$, then for some $\tau \in (-\infty, +\infty)$,

$Z(\tau, Y_0, Z_0) = Z \in M$, $Y(\tau, Y_0, Z_0) = Y$ and $R(Z, M) = a > 0$. 

Consider $Z_1 \in M$, there exists a sequence $\tau_k$ such that
\[ Z(\tau_k, Y, Z) \to Z_1 \text{ and } Y(\tau_k, Y, Z) \to Y_1(mod2\pi) \text{ as } k \to +\infty. \]

Let $\varepsilon = \frac{a}{2}$, find the definition of recurrence $T_\varepsilon$ and choose $\delta > 0$ so small that if
\[ \| Z_1 - Z(\tau_k, Y, Z) \| < \delta, \| Y_1 - Y(\tau_k, Y, Z) \| < \delta(mod2\pi) \]
then
\[ \| Z(\tau, Y_1, Z_1) - Z(\tau + \tau_k, Y, Z) \| < \frac{a}{2} \text{ for } \tau \in [-T_\varepsilon, T_\varepsilon], \]
but
\[ \| Z - Z(\tau, Y_1, Z_1) \| \geq a \text{ for all } \tau \in (-\infty, +\infty) \]
therefore
\[ \| Z - Z(\tau + \tau_k, Y, Z) \| \geq \frac{a}{2} \text{ for } \tau \in [-T_\varepsilon, T_\varepsilon]. \]

This contradicts the choice of $T_\varepsilon$, so the theorem is proven.

**Theorem 3 . :** For function $f(\tau)$ defined and continuous on $\tau \in (-\infty, +\infty)$ to be represented in the form $f(\tau) = F(\Omega \tau)$ it is necessary and sufficient that for any sequence $\tau_n$ such that
\[ \Omega \tau_n \to Y(mod2\pi) \text{ as } n \to +\infty, \]
there exists
\[ \lim_{n \to \infty} f(\tau_n) [4]. \]

**Definition 4 . :** We will say that a quasi-periodic dynamical system $D_q$ has the convergence property if it has unique quasi-periodic motion
\[ Z(\tau, Y_0, Z_0) = F(\Omega \tau + Y_0), \]
where $F(Y)$ is a vector function defined and continuous for all $Z \in E^m$, $2\pi$-periodic in all the components of the vector $Z$ such that
\[ 1). \text{ for any } \varepsilon > 0 \text{ there exists } \delta(\varepsilon) > 0 \text{ such that} \]
\[ \| Z(\tau, Y_0, Z_0) - F(\Omega \tau + Y_0) \| < \varepsilon \text{ for } \tau \geq 0 \]
and
\[ \| Z_0 - F(Y_0) \| < \delta(\varepsilon); \]
\[
\|Z(\tau, Y_0, Z_0) - F(\Omega \tau + Y_0)\| \to 0 \text{ as } \tau \to \infty
\]
uniformly for \( Y_0 \in \mathbb{E}^m \), if \( \| Z_0 \| \leq r \) \[2\], \[10\], \[14\].

4 Main Results

Theorem 4. (Necessary and sufficient conditions for the existence of globally asymptotic stability quasi-periodic dynamical systems):
For quasi-periodic dynamical system \( D_q \) to have the properties of convergence, it is necessary and sufficient that:

(i) any motion \( Z(\tau, Y_0, Z_0) \) is bounded for \( \tau \geq 0 \);
(ii) for any \( r > 0 \) and \( \varepsilon > 0 \) there exists \( \delta(r, \varepsilon) > 0 \) such that

\[
\|Z_0 - Z_0\| < \delta(r, \varepsilon) \text{ and } \tau \geq 0,
\]
\[
\|Z(\tau, Y_0, Z_0) - Z(\tau, Y_0, Z_0)\| < \varepsilon,
\]
then
\[
\|Z(\tau, Y_0, Z_0) - Z(\tau, Y_0, Z_0)\| \to 0 \text{ as } \tau \to +\infty
\]
uniformly for \( Y_0 \in \mathbb{E}^m \) and \( \| Z_0 \| \leq r, \| Z_0 \| \leq r \);
(iii) for any \( Z_0 \in \mathbb{E}^n \) and \( Y_0 \in \mathbb{E}^m \) there exists \( \lim_{\tau_n \to +\infty} Z(\tau_n, Y_0, Z_0) \text{ if } \Omega \tau_n \to \overline{Y} (mod 2\pi) \text{ as } \tau_n \to +\infty. \)

Proof: Necessity. Let quasi-periodic dynamical system \( D_q \) have the properties of convergence and let

\[
Z(\tau, Y_0, Z_0) = F(\Omega \tau + Y_0)
\]
be a quasi-periodic motion, then there exists a real number \( M_1 > 0 \) such that

\[
\|F(\Omega t + Y_0)\| \leq M_1 \text{ for any } t \in (-\infty, +\infty).
\]
Consider the motion \( Z(\tau, Y_0, Z_0) \):

\[
\|Z(\tau, Y_0, Z_0)\| \leq \|F(\Omega \tau + Y_0)\| + \|Z(\tau, Y_0, Z_0) - F(\Omega \tau + Y_0)\|.
\]
O fulfil condition (iii) of the Theorem and continuity on \( \tau \) there exists a real number \( M_2 > 0 \) such that

\[
\|Z(\tau, Y_0, Z_0) - F(\Omega \tau + Y_0)\| \leq M_2 \text{ for } \tau \geq 0.
\]
Consequently,

\[ \|Z(\tau, Y_0, Z_0)\| \leq M_1 + M_2 \text{ for } \tau \geq 0. \]

Because \( Z(\tau, Y_0, Z_0) \) is continuous with respect to all of its arguments and 2\( \pi \)-periodic in all the components of the vector \( Y_0 \), then for any \( \delta > 0 \) and for any \( r > 0 \) there exists \( T > 0 \) such that

\[ \|Z(\tau, Y_0, Z_0) - F(\Omega \tau + Y_0)\| < \delta \text{ for } \tau \geq T \text{ and } \|Z_0\| \leq r, \]

because of the continuity properties of the motion \( Z(\tau, Y_0) \) [5], [7] condition (ii) of the Theorem is met.

Condition (iii) of the Theorem follows from Definition 4 and Theorem 3.

**Sufficiency.** Suppose that the conditions of the theorem are fulfilled. Then, as shown in the proof of Theorem 3 [4], [6], for the space \( \mathbb{E}^n \) there exists an invariant set

\[ I = \{ Z \in \mathbb{E}^n : Z = F(Y), Y \in \mathbb{E}^m \}, \]

where \( F(Y) \) is continuous, 2\( \pi \)-periodic function in all of its arguments.

All the motions which are contained in the invariant set, are quasi-periodic motions. From the conditions (i, ii, iii) of the Theorem it follows that these motions satisfy all of the properties of convergence.

This completes the proof of theorem.

**References**

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