The Iterative Procedure with Errors for Approximating Fixed Points of Multivalued Quasi-Nonexpansive Mappings

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Abstract

In this paper, motivated by Khan et. al.[11], we use three-step iterative procedures with errors to approximate fixed point of multivalued quasi- nonexpansive mappings in uniformly Banach space and establish strong and weak convergence theorems for the proposed process. Our results extend important results.

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1. Introduction and Preliminaries

Throughout this paper, let $E$ be a Banach space with the norm $|| \cdot ||$ and $\mathbb{N}$ denote the set of all positive integers.

Let $K$ be a nonempty subset of $E$. The set $K$ is said to be proximinal if for each $x \in E$, there exists an element $y \in K$ such that $|| x - y || = d(x, K)$, where $d(x, K) = \inf \{ || x - z || : z \in K \}$. It is known that a weakly compact convex subset of a Banach space and closed convex subsets of a uniformly convex Banach space are proximinal[3] We shall denote $CB(K)$, $C(K)$ and $P(K)$ by the families of nonempty closed and bounded subsets, nonempty compact subsets and nonempty proximinal bounded subsets of $K$, respectively. Let $H$ be the Hausdorff metric induced by the
metric $d$ of $E$ and given by $H(A, B) = \max \{\sup_{x \in A}d(x, B), \sup_{y \in B}d(y, A)\}$ for $A, B \in CB(E)$.

A multivalued mapping $T : K \to P(K)$ is said to be contraction if there exists a constant $k \in [0, 1)$ such that for all $x, y \in K$, $H(Tx, Ty) \leq k \|x - y\|$, and nonexpansive if $H(Tx, Ty) \leq \|x - y\|$ for all $x, y \in D$ and quasi-nonexpansive if $F(T) \neq \emptyset$ and $H(Tx, Ty) \leq \|x - y\|$ for all $x, y \in K$ and all $p \in F(T)$ [20]. A point $x \in D$ is called a fixed point of a multivalued mapping $T$ if $x \in Tx$. Denote by $F(T)$ the set of fixed points of $T$, that is, $F(T) = \{x \in K : Tx = x\}$. It is clear that every nonexpansive multi-valued map $T$ with $F(T) \neq \emptyset$ is quasi-nonexpansive. But there exist quasi-nonexpansive mappings that are not nonexpansive. It is known that if $T$ is a quasi-nonexpansive multi-valued map, then $F(T)$ is closed [22].

A multivalued nonexpansive mapping $T : K \to CB(K)$ where $K$ a subset of $E$, is said to satisfy condition (I) if there exists a nondecreasing function $f : (0, \infty) \to \{0, \infty\}$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in K$ (see [19]). The mapping $T : K \to CB(K)$ is said to be hemi-compact if, for any sequence $\{x_n\}$ in $K$ such that $\lim_{n \to \infty}d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{n \to \infty}x_{n_k} = x \in D$. A multivalued mapping $T : K \to P(E)$ is called demiclosed at $y \in K$ if for any sequence $\{x_n\}$ in $K$ weakly convergent to an element $x$ and $y_n \in Tx_n$ strongly convergent to $y$, we have $y \in Tx$.

A Banach space $E$ is said to satisfy Opial’s condition [16] if for any sequence $\{x_n\}$ in $E$, $x_n \rightharpoonup x$ implies that

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$. Examples of Banach spaces satisfying this condition are Hilbert spaces and $L^p$ spaces $(1 < p < \infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial’s condition.

The study of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff metric was initiated by Markin [14] (see also [15]). Later, an interesting and rich fixed point theory for such maps was developed which has applications in control theory, convex optimization, differential inclusion and economics (see [7]). Moreover, the existence of fixed points for multivalued nonexpansive mappings in uniformly convex Banach spaces was proved by Lim [13]. The theory of multivalued nonexpansive mappings is harder than the corresponding theory of single valued nonexpansive mappings. Different iterative processes have been used to approximate fixed points of multivalued nonexpansive mappings; in particular, Sastry and Babu [18] considered the following:

Let $K$ be a nonempty convex subset of $E$, $T : K \to P(K)$ be a multivalued mapping with $p \in Tp$.

The sequences of Mann iterates is defined by $x_1 \in K$

$$x_{n+1} = (1 - a_n)x_n + a_n y_n, \quad n \in \mathbb{N}$$

where $y_n \in Tx_n$ is such that $\|y_n - p\| = d(p, Tx_n)$ and $\{a_n\}$ is sequence in $(0, 1)$ satisfying $\sum a_n = \infty$.

The sequence of Ishikawa iterates is defined by $x_1 \in D$
Three-step iterative process with errors

\[
\begin{align*}
    y_n &= (1 - b_n) x_n + b_n z_n \\
    x_{n+1} &= (1 - a_n) x_n + a_n u_n, \quad n \in \mathbb{N}
\end{align*}
\]

where \( z_n \in T x_n, u_n \in T y_n \) are such that \( \| z_n - p \| = d(p, Tx_n) \) and \( \| u_n - p \| = d(p, T y_n) \), and \( \{a_n\}, \{b_n\} \) are real sequences of numbers with \( 0 \leq a_n, b_n < 1 \) satisfying \( \lim_{n \to \infty} b_n = 0 \) and \( \sum a_n b_n = \infty \).

The following is a useful Lemma due to Nadler [15].

**Lemma 1.** Let \( A, B \in CB(E) \) and \( a \in A \). If \( \eta > 0 \), then there exists \( b \in B \) such that \( d(a, b) \leq H(A, B) + \eta \).

Based on the above lemma, Song and Wang [22] modified the iteration scheme used in [11] and improved the results presented therein. This scheme reads as follows:

The sequence of Ishikawa iterates is defined \( x_1 \in D \)

\[
\begin{align*}
    y_n &= (1 - b_n) x_n + b_n z_n \\
    x_{n+1} &= (1 - a_n) x_n + a_n u_n, \quad n \in \mathbb{N}
\end{align*}
\]

where \( z_n \in T x_n, u_n \in T y_n \) are such that \( \| z_n - u_n \| \leq H(T x_n, T y_n) + \eta_n \) and \( \| z_{n+1} - u_n \| \leq H(T x_{n+1}, T y_n) + \eta_n, \quad \eta_n \in (0, \infty) \) and \( \{a_n\}, \{b_n\} \) are real sequences of numbers with \( 0 \leq a_n, b_n \leq 1 \) satisfying \( \lim_{n \to \infty} b_n = 0 \) and \( \sum a_n b_n = \infty \).

It is to be noted that Song and Wang [22] need the condition \( T p = \{p\} \) in order to prove their Theorem 1. Actually, Panyanak [17] proved some results using Ishikawa type iteration process without this condition. Song and Wang [22] showed that without this condition his process was not well-defined. They reconstructed the process using the condition \( T p = \{p\} \) which made it well-defined. Such a condition was also used by Jung [8]. They defined \( P_T(x) = \{y \in Tx : \|x - y\| = d(x, Tx)\} \) for a multivalued mapping \( T : K \to P(K) \). They also proved a couple of strong convergence results using Ishikawa type iteration process.

In 2000, Noor [1] introduced a three-step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces. Suantai [23] defined a new three-step iterative which is an extension of Noor iterations and gave some weak and strong convergence theorems of such iterations for asymptotically nonexpansive mappings in uniformly convex Banach spaces.


Kaplan and Kopuzlu [9] introduced new three-step iterative procedures and proved strong convergence theorems of the proposed iterations in uniformly convex Banach spaces. Kaplan [10] also introduced new three-step iterative scheme with errors and proved some strong and weak convergence theorems of proposed in uniformly convex Banach space. Recently, Khan and et. al. [11] introduced new three iterative procedures and proved some strong and weak convergence theorems for quasi nonexpansive multivalued mappings in Banach spaces. They showed that
the iterative process used is independent of Ishikawa iterative process and converges faster.

In this paper, motivated by Khan and et. al. [11], we use the iterative schemes with errors to approximate fixed points of a multivalued quasi-nonexpansive mappings in Banach spaces and prove strong and weak convergence theorems of the proposed iteration.

We constitute our iteration scheme as follow:

\[
\begin{aligned}
x_{n+1} &= \alpha_n x_n + \beta_n y_n + \gamma_n s_n \\
y_n &= \alpha_n u_n + \beta_n' w_n + \gamma_n' r_n \\
z_n &= \alpha_n'' x_n + \beta_n'' u_n + \gamma_n'' t_n
\end{aligned}
\]

where \(u_n \in P_T(x_n), v_n \in P_T(y_n), w_n \in P_T(z_n)\) and \(\alpha_n, \alpha_n', \alpha_n'', \beta_n, \beta_n', \beta_n'', \gamma_n, \gamma_n', \gamma_n'' \in [a, b] \subset (0, 1)\) and \(\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = \alpha_n'' + \beta_n'' + \gamma_n'' = 1\). Also \(\{s_n\}, \{r_n\}\) and \(\{t_n\}\) are bounded in \(K\).

Now we state some useful lemmas.

**Lemma 2.** [24] Let \(\{s_n\}\) and \(\{t_n\}\) be sequences of nonnegative real numbers satisfying the inequality \(s_{n+1} \leq s_n + t_n\), \(\forall n \geq 1\). If \(\sum_{n=1}^{\infty} t_n < \infty\), then \(\lim_{n \to \infty} s_n\) exists.

**Lemma 3.** [3] Let \(E\) be a uniformly convex Banach space and \(0 < p \leq t_n \leq q < 1\) for all \(n \in \mathbb{N}\). Suppose that \(\{x_n\}\) and \(\{y_n\}\) are two sequences of \(E\) such that \(\limsup_{n \to \infty} \|x_n\| \leq r\), \(\limsup_{n \to \infty} \|y_n\| \leq r\) and \(\lim_{n \to \infty} \|t_n x_n + (1 - t_n) y_n\| = r\) hold for some \(r \geq 0\). Then \(\lim_{n \to \infty} \|x_n - y_n\| = 0\).

**Lemma 4.** [22] Let \(T : K \to P(K)\) be a multivalued mapping and \(P_T(x) = \{y \in Tx : \|x - y\| = d(x, Tx)\}\). Then the following are equivalent.

1. \(x \in F(T)\)
2. \(P_T(x) = \{x\}\)
3. \(x \in F(P_T)\). Moreover, \(F(T) = F(P_T)\).

**2. Main results**

**Lemma 5.** Let \(E\) be a normed space and \(K\) be a nonempty closed convex subset of \(E\). Let \(T : K \to P(K)\) be a multivalued mapping such that \(F(T) \neq \emptyset\) and \(P_T\) is a quasi-nonexpansive mapping. Let \(\{x_n\}\) be the sequence as defined in (1.1). Assume that \(0 < l \leq \alpha_n, \alpha_n', \alpha_n'' \leq k < 1\) and \(\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma_n' < \infty\) and \(\sum_{n=1}^{\infty} \gamma_n'' < \infty\). Then

1. \(\lim_{n \to \infty} \|x_n - p\|\) exists for all \(p \in F\).
2. For any \(p \in F\) \(\lim_{n \to \infty} d(x_n, Tx_n) = 0\).

**Proof.** Let \(p \in F(T)\). Then \(p \in P_T(p) = \{p\}\) by Lemma 4. Since \(s_n, r_n\) and \(t_n\) are bounded, there exists \(M > 0\) such that

\[
\max \left\{ \sup_{n \in N} \|t_n - p\|, \sup_{n \in N} \|s_n - p\|, \sup_{n \in N} \|r_n - p\| \right\} \leq M.
\]
It follows from (1.1) that

\[
\| z_n - p \| = \left\| \alpha_n x_n + \beta_n u_n + \gamma_n t_n - p \right\|
\]

\[
\leq \alpha_n^* \| x_n - p \| + \beta_n^* \| u_n - p \| + \gamma_n^* \| t_n - p \|
\]

\[
\leq \alpha_n^* \| x_n - p \| + \beta_n^* d(u_n, P_T(p)) + \gamma_n^* M
\]

\[
\leq \alpha_n^* \| x_n - p \| + \beta_n^* H(P_T(x_n), P_T(p)) + \gamma_n^* M
\]

\[
\leq \alpha_n^* \| x_n - p \| + \beta_n^* \| x_n - p \| + \gamma_n^* M
\]

\[= \left( \alpha_n^* + \beta_n^* \right) \| x_n - p \| + \gamma_n^* M
\]

\[
\leq \| x_n - p \| + \gamma_n^* M
\]

(2.1)

Also,

\[
\| y_n - p \| = \left\| \alpha_n' u_n + \beta_n' w_n + \gamma_n' r_n - p \right\|
\]

\[
\leq \alpha_n' \| u_n - p \| + \beta_n' \| w_n - p \| + \gamma_n' \| r_n - p \|
\]

\[
\leq \alpha_n' d(u_n, P_T(p)) + \beta_n' d(w_n, P_T(p)) + \gamma_n' M
\]

\[
\leq \alpha_n' H(P_T(x_n), P_T(p)) + \beta_n' H(P_T(z_n), P_T(p)) + \gamma_n' M
\]

\[
\leq \alpha_n' \| x_n - p \| + \beta_n' \| z_n - p \| + \gamma_n' M
\]

\[
\leq \alpha_n' \| x_n - p \| + \beta_n' \| x_n - p \| + \gamma_n' M + \beta_n' \gamma_n^* M
\]

\[
\leq \| x_n - p \| + \gamma_n' M + \beta_n' \gamma_n^* M
\]

(2.2)

and

\[
\| x_{n+1} - p \| = \left\| \alpha_n v_n + \beta_n w_n + \gamma_n s_n - p \right\|
\]

\[
\leq \alpha_n \| v_n - p \| + \beta_n \| w_n - p \| + \gamma_n \| s_n - p \|
\]

\[
\leq \alpha_n d(v_n, P_T(p)) + \beta_n d(w_n, P_T(p)) + \gamma_n M
\]

\[
\leq \alpha_n H(P_T(x_n), P_T(p)) + \beta_n H(P_T(z_n), P_T(p)) + \gamma_n M
\]

\[
\leq \alpha_n \| y_n - p \| + \beta_n \| z_n - p \| + \gamma_n M
\]

\[
\leq \alpha_n \| x_n - p \| + \beta_n \| x_n - p \| + \gamma_n M + \alpha_n \gamma_n' M + \alpha_n \beta_n' \gamma_n^* M
\]

\[
\leq \| x_n - p \| + \gamma_n M + \beta_n \| x_n - p \| + \gamma_n M
\]

\[= \| x_n - p \| + \gamma_n M
\]

(2.3)

where \( \varepsilon_n = \left( \gamma_n + \alpha_n \gamma_n' + \beta_n \gamma_n'' + \alpha_n \beta_n' \gamma_n^* \right) M \). From hipotez(ii), \( \sum_{n=1}^{\infty} \varepsilon_n < \infty \). By Lemma 2, we have \( \lim_{n \to \infty} \| x_n - p \| \) exists for each \( p \in F(T) \). By hipotez (1), since \( \lim_{n \to \infty} \| x_n - p \| \) exists and therefore \( \{ x_n \}, \{ y_n \} \) ve \( \{ z_n \} \) sequences are bounded. Suppose that

\[
\lim_{n \to \infty} \| x_n - p \| = c,
\]
where $c \geq 0$. We now prove that
\[ \lim_{n \to \infty} d (x_n, Tx_n) = 0. \]
The case when $c = 0$ is obvious. We thus assume that $c > 0$; inasmuch as
\[ \lim_{n \to \infty} d(x_n, T(x_n)) \leq \lim_{n \to \infty} d(x_n, P_T(x_n)) \leq \lim_{n \to \infty} \|x_n - u_n\| = 0. \]
It suffices to prove that $\lim_{n \to \infty} \|x_n - u_n\| = 0$. Let
\[ S = \left\{ \sup_{n \in N} \|s_n - w_n\|, \sup_{n \in N} \|r_n - w_n\|, \sup_{n \in N} \|t_n - u_n\| \right\}. \]
Now
\[ \limsup_{n \to \infty} \|u_n - p\| \leq \limsup_{n \to \infty} H (P_T(\{x_n\}), P_T(p)) \leq \limsup_{n \to \infty} \|x_n - p\| = c, \]
implies that
\[ \limsup_{n \to \infty} \|u_n - p\| \leq c. \quad (2.4) \]
Similarly,
\[ \limsup_{n \to \infty} \|v_n - p\| \leq c, \]
\[ \limsup_{n \to \infty} \|w_n - p\| \leq c. \]
Also taking $\limsup$ on both sides of (2.2) and (2.1), respectively, we obtain
\[ \limsup_{n \to \infty} \|z_n - p\| \leq c. \quad (2.5) \]
\[ \limsup_{n \to \infty} \|y_n - p\| \leq c. \]
Next, we consider
\[ \|v_n - p + \gamma_n (s_n - w_n)\| \leq \|v_n - p\| + \gamma_n \|s_n - w_n\| \]
\[ \leq d(v_n, P_T(p)) + \gamma_n S \]
\[ \leq H (P_T(y_n), P_T(p)) + \gamma_n S \]
\[ \leq \|y_n - p\| + \gamma_n S \]
It follows that
\[ \limsup_{n \to \infty} \|u_n - p + \gamma_n (s_n - v_n)\| \leq c. \quad (2.6) \]
Similarly,
\[ \|w_n - p + \gamma_n (s_n - w_n)\| \leq \|w_n - p\| + \gamma_n \|s_n - w_n\| \]
\[ \leq d(w_n, P_T(p)) + \gamma_n S \]
\[ \leq H (P_T(z_n), P_T(p)) + \gamma_n S \]
\[ \leq \|z_n - p\| + \gamma_n S \]
This implies that
\[ \limsup_{n \to \infty} \|w_n - p + \gamma_n (s_n - w_n)\| \leq c. \]
Since
\[ \lim_{n \to \infty} \|\alpha_n(v_n - p + \gamma_n (s_n - w_n)) + (1 - \alpha_n)(w_n - p + \gamma_n (s_n - w_n))\| = \lim_{n \to \infty} \|x_{n+1} - p\| = c \]
from Lemma 3, we obtain that
\[ \lim_{n \to \infty} \|v_n - w_n\| = 0. \quad (2.7) \]
Also,
\[ \|x_{n+1} - p\| = \|\alpha_n v_n + \beta_n w_n + \gamma_n s_n - p\| \
= \|w_n - p + \alpha_n(v_n - w_n) + \gamma_n (s_n - w_n)\| \\
\leq \|w_n - p\| + \alpha_n \|v_n - w_n\| + \gamma_n \|s_n - w_n\| \\
\leq \|w_n - p\| + \alpha_n \|v_n - w_n\| + \gamma_n S. \]
this implies that \( c \leq \lim \inf_{n \to \infty} \|w_n - p\| \) and thus together with \((2.4)\) inequality
\[ c \leq \lim \inf_{n \to \infty} \|w_n - p\| \leq \lim \sup_{n \to \infty} \|w_n - p\| \leq c \]
we have \( \lim_{n \to \infty} \|w_n - p\| = c. \) Also,
\[ \|w_n - p\| \leq \|w_n - v_n\| + \|v_n - p\| \\
\leq \|w_n - v_n\| + d(v_n, P_T (p)) \\
\leq \|w_n - v_n\| + H(P_T (y_n), P_T (p)) \\
\leq \|w_n - v_n\| + \|y_n - p\| \]
It implies that \( c \leq \lim \inf_{n \to \infty} \|y_n - p\| \).
Thus \( c \leq \lim \inf_{n \to \infty} \|y_n - p\| \leq \lim \sup_{n \to \infty} \|y_n - p\| \leq c, \) it follows that
\[ \lim_{n \to \infty} \|y_n - p\| = c. \]
if \( y_n - p = \alpha'_n \left(u_n - p + \gamma'_n (r_n - w_n)\right) + (1 - \alpha'_n) \left(w_n - p + \gamma'_n (r_n - w_n)\right)\) can be written, we have
\[ \lim_{n \to \infty} \|\alpha'_n \left(u_n - p + \gamma'_n (r_n - w_n)\right) + (1 - \alpha'_n) \left(w_n - p + \gamma'_n (r_n - w_n)\right)\| = c. \]
Moreover, we get
\[ \|u_n - p + \gamma'_n (r_n - w_n)\| \leq \|u_n - p\| + \gamma'_n \|r_n - w_n\| \\
\leq d(u_n, P_T (p)) + \gamma'_n S \\
\leq H(P_T (x_n), P_T (p)) + \gamma'_n S \\
\leq \|x_n - p\| + \gamma'_n S. \]
This yields that
\[ \lim \sup_{n \to \infty} \|u_n - p + \gamma'_n (r_n - w_n)\| \leq c. \]
Similarly,
\[
\| w_n - p + \gamma_n^\prime (r_n - w_n) \| \leq \| w_n - p \| + \gamma_n^\prime \| r_n - w_n \| \\
\leq d(w_n, P_T (p)) + \gamma_n^\prime S \\
\leq H(P_T (z_n), P_T (p)) + \gamma_n^\prime S \\
\leq \| z_n - p \| + \gamma_n^\prime S.
\]
This implies that
\[
\limsup_{n \to \infty} \| w_n - p + \gamma_n^\prime (r_n - w_n) \| \leq c.
\]
Again by Lemma 3,
\[
\lim_{n \to \infty} \| u_n - w_n \| = 0. \tag{2.8}
\]
With (2.8), we have
\[
\| u_n - p \| \leq \| u_n - p \| + \| w_n - p \| \\
\leq \| u_n - w_n \| + d(w_n, P_T (p)) \\
\leq \| u_n - w_n \| + H(P_T (z_n), P_T (p)) \\
\leq \| u_n - w_n \| + \| z_n - p \|.
\]
It implies that \( c \leq \liminf_{n \to \infty} \| z_n - p \| \) and thus together with (2.5)
\[
c \leq \liminf_{n \to \infty} \| z_n - p \| \leq \limsup_{n \to \infty} \| z_n - p \| \leq c,
\]
this implies that \( \lim_{n \to \infty} \| z_n - p \| = c \). Moreover,
\[
z_n - p = \alpha_n^\prime \left( x_n - p + \gamma_n^\prime (t_n - u_n) \right) + \left( 1 - \alpha_n^\prime \right) \left( u_n - p + \gamma_n^\prime (t_n - u_n) \right)
\]
can be written, then we have
\[
\lim_{n \to \infty} \| \alpha_n^\prime \left( x_n - p + \gamma_n^\prime (t_n - u_n) \right) + \left( 1 - \alpha_n^\prime \right) \left( u_n - p + \gamma_n^\prime (t_n - u_n) \right) \| = c.
\]
Also
\[
\| x_n - p + \gamma_n^\prime (t_n - u_n) \| \leq \| x_n - p \| + \gamma_n^\prime \| t_n - u_n \| \\
\leq \| x_n - p \| + \gamma_n S
\]
which implies that
\[
\limsup_{n \to \infty} \| x_n - p + \gamma_n^\prime (t_n - u_n) \| \leq c.
\]
Similarly, we have
\[
\| u_n - p + \gamma_n^\prime (t_n - u_n) \| \leq \| u_n - p \| + \gamma_n^\prime \| t_n - u_n \| \\
\leq d(u_n, P_T (p)) + \gamma_n^\prime S \\
\leq H(P_T (x_n), P_T (p)) + \gamma_n^\prime S \\
\leq \| x_n - p \| + \gamma_n^\prime S.
\]
This implies that \( \limsup_{n \to \infty} \| u_n - p + \gamma_n' (t_n - u_n) \| \leq c \). Hence by Lemma 3, we have

\[
\lim_{n \to \infty} \| x_n - u_n \| = 0
\]  

(2.9)

which yields \( \lim_{n \to \infty} d(x_n, P_T(x_n)) = 0 \) as desired. \( \square \)

We now give some strong convergence theorems.

**Theorem 1.** Let \( E \) be a uniformly convex Banach space and \( K \) be a nonempty compact closed convex subset of \( E \). Let \( T : K \to P(K) \) be a nonexpansive multivalued mapping such that \( F(T) \neq \emptyset \) and \( P_T \) is a quasi-nonexpansive mapping. Assume that \( \sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma_n' < \infty, \sum_{n=1}^{\infty} \gamma_n'' < \infty \) and \( 0 < l \leq \alpha_n, \alpha_n', \alpha_n'' \leq k < 1 \). Let \( \{ x_n \} \) be the sequence as defined in (1.1). Then the sequence \( \{ x_n \} \) converges strongly to a fixed point of \( T \).

**Proof.** We have proved in Lemma 5 that \( \lim_{n \to \infty} \| x_n - p \| \) exists for all \( p \in F(T) \). Now from the compactness of \( K \), there exists a subsequence \( \{ x_{n_k} \} \) of \( \{ x_n \} \) such that \( \lim_{k \to \infty} \| x_{n_k} - q \| = 0 \) for some \( q \in K \). Then

\[
d(q, T(q)) \leq d(q, P_T(q)) \leq \| q - x_{n_k} \| + \| x_{n_k} - u_{n_k} \| + d(u_{n_k}, P_T(q)) \\
\leq \| q - x_{n_k} \| + \| x_{n_k} - u_{n_k} \| + H(P_T(x_{n_k}), P_T(q)) \\
\leq 2 \| q - x_{n_k} \| + \| x_{n_k} - u_{n_k} \| \to 0
\]

by Lemma 5, we have \( \lim_{n \to \infty} \| x_{n_k} - u_{n_k} \| = 0 \). This implies that \( d(q, P_T(q)) = 0 \) and hence, \( q \) is a fixed point of \( P_T \). Since the set of fixed of \( P_T \) is the same as that of \( T \) by Lemma 4, therefore \( \{ x_n \} \) converges strongly to a fixed point of \( T \). \( \square \)

**Theorem 2.** Let \( E \) be a uniformly convex Banach space and \( K \) be a nonempty closed convex subset of \( E \). Let \( T : K \to P(K) \) be a multivalued mapping satisfying Condition (I) such that \( F(T) \neq \emptyset \) and \( P_T \) is a quasi-nonexpansive mapping. Assume that

\[
\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma_n' < \infty \quad \text{ve} \quad \sum_{n=1}^{\infty} \gamma_n'' < \infty \quad \text{and} \quad 0 < l \leq \alpha_n, \alpha_n', \alpha_n'' \leq k < 1.
\]

Then the sequence \( \{ x_n \} \) as defined in (1.1) converges strongly to a fixed point of \( T \).

**Proof.** Let \( p \in F(T) = F(P_T) \). By the Lemma 5, \( \lim_{n \to \infty} \| x_n - p \| \) exists for all \( p \in F(T) \) and it can be shown that \( d(x_n, T(x_n)) \to 0 \) as \( n \to \infty \). Now \( \| x_{n+1} - p \| \leq \| x_n - p \| \) gives

\[
\inf_{p \in F(T)} \| x_{n+1} - p \| \leq \inf_{p \in F(T)} \| x_n - p \| ,
\]

which implies that \( d(x_{n+1}, F(T)) \leq d(x_n, F(T)) \) and so \( \lim_{n \to \infty} d(x_n, F(T)) \) exists. By using Condition (I) and Lemma 5, we have

\[
\lim_{n \to \infty} f(d(x_n, F(T))) \leq \lim_{n \to \infty} d(x_n, T x_n) = 0.
\]

That is,

\[
\lim_{n \to \infty} f(d(x_n, F(T))) = 0.
\]
We have \( d(x_n, F(T)) = 0 \). Thus there is a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) and a sequence \( \{p_k\} \subset F(T) \) such that \( \|x_{n_k} - p_k\| < \frac{1}{2^k} \) for all \( k \). From (2.3), we obtain
\[
\|x_{n_{k+1}} - p\| \leq \|x_{n_{k+1-1}} - p\| + \varepsilon_{n_{k+1-1}} \\
\leq \|x_{n_{k+1-2}} - p\| + \varepsilon_{n_{k+1-2}} + \varepsilon_{n_{k+1-1}} \\
\quad \vdots \\
\leq \|x_{n_k} - p\| + \sum_{i=0}^{n_{k+1}-n_k-1} \varepsilon_{n_k+i}
\]
for all \( p \in F(T) \). This implies that
\[
\|x_{n_{k+1}} - p\| \leq \|x_{n_k} - p\| + \sum_{i=0}^{n_{k+1}-n_k-1} \varepsilon_{n_k+i} < \frac{1}{2^k} + \sum_{i=0}^{n_{k+1}-n_k-1} \varepsilon_{n_k+i}
\]
Now, we small show that \( \{p_k\} \) is a Cauchy sequence in \( K \). Noted that.
\[
\|p_{k+1} - p_k\| \leq \|p_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - p_k\| \\
< \frac{1}{2^{k+1}} + \frac{1}{2^k} + \sum_{i=0}^{n_{k+1}-n_k-1} \varepsilon_{n_k+i} \\
< \frac{1}{2^{k-1}} + \sum_{i=0}^{n_{k+1}-n_k-1} \varepsilon_{n_k+i}
\]
This implies that \( \{p_k\} \) is a Cauchy sequence in \( K \) and thus \( q \in K \). Now we show that \( q \in F \). Therefore
\[
d(p_k, T(q)) \leq d(p_k, P_T(q)) \leq H(P_T(p_k), P_T(q)) \leq \|p_k - q\|
\]
p_k \to q as \( n \to \infty \), it follows that \( d(q, Tq) = 0 \) and thus \( q \in F \). \( P_T \) is a quasi-nonexpansive mapping, \( F(P_T) \) is closed. Therefore, \( p \in F(T) = F(P_T) \). It implies by \( \|x_{n_k} - p_k\| < \frac{1}{2^k} \) that \( \{x_n\} \) converges strongly to \( q \). Since \( \lim_{n \to \infty} \|x_n - q\| \) exists, it follows that \( \{x_n\} \) converges strongly to \( q \).

Now we approximate fixed points of the mapping \( T \) through weak convergence of the sequence \( \{x_n\} \) defined in (1.1).

**Theorem 3.** \( L E \) be a uniformly convex Banach space satisfying Opial’s condition and \( K \) a nonempty closed convex subset of \( E \). Let \( T : K \to P(K) \) be a multivalued mapping such that \( F(T) \neq \emptyset \) and \( P_T \) is a quasi-nonexpansive mapping. Let \( \{x_n\} \) be the sequence as defined in (1.1). Assume that \( \sum_{n=1}^{\infty} \gamma_n < \infty \), \( \sum_{n=1}^{\infty} \gamma'_n < \infty \) and \( \sum_{n=1}^{\infty} \gamma''_n < \infty \) and \( 0 < l \leq \alpha_n, \alpha'_n, \alpha''_n \leq k < 1 \). Let \( I - P_T \) be demiclosed with respect to zero, Then \( \{x_n\} \) converges weakly to fixed point of \( T \).

**Proof.** Let \( p \in F(T) = F(P_T) \). As in the proof of Lemma 5, \( \lim_{n \to \infty} \|x_n - p\| \) exists. We now prove that \( \{x_n\} \) has a unique weak subsequently limit in \( F(T) \). To prove this, let \( z_1 \) and \( z_2 \) be weak limits of the subsequences \( \{x_{n_i}\} \) and \( \{x_{n_j}\} \) of \( \{x_n\} \), respectively. By (2.9), there exists \( u_n \in Tx_n \) such that \( \lim_{n \to \infty} \|x_n - u_n\| = 0 \). Since \( I - P_T \)
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is demiclosed with respect to zero, therefore we obtain \( z_1 \in F(P_T) = F(T) \). In the same way, we can prove that \( z_2 \in F(T) \). Next, we prove uniqueness. For this, suppose that \( z_1 \neq z_2 \). Then by Opial’s condition, we have

\[
\lim_{n \to \infty} \|x_n - z_1\| = \lim_{n_i \to \infty} \|x_{n_i} - z_1\| < \lim_{n_i \to \infty} \|x_{n_i} - z_2\| = \lim_{n \to \infty} \|x_n - z_2\| = \lim_{n_j \to \infty} \|x_{n_j} - z_2\| < \lim_{n_i \to \infty} \|x_{n_i} - z_1\| = \lim_{n \to \infty} \|x_n - z_1\|.
\]

This is a contradiction. Hence \( \{x_n\} \) converges weakly to a point in \( F(T) \). □

REFERENCES


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