Common Fixed Point Results in Dislocated Metric Spaces

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Abstract

The main purpose of this paper is to generalize a common fixed theorem established by D. Panthi [D. Panthi, International Journal of Mathematical Analysis Vol. 9, 2015, no. 45, 2235 - 2242] on dislocated metric.

Keywords: Fixed point, Common fixed point, Dislocated metric space, Weak compatibility

1 Introduction

During the recent years, a number of fixed point results have been established by different authors for single and pair of mappings in dislocated metric spaces. Recall that The notion of dislocated metric, introduced in 2000 by P. Hitzler and A.K. Seda, is characterized by the fact that self distance of a point need not be equal to zero and has useful applications in topology, logical programming and in electronics engineering. For further details on dislocated metric spaces, see, for example [2]-[4].

In 2015, D. Panthi [3] have established the following result
Theorem 1.1  Let \((X, d)\) be a complete dislocated metric space. Let \(A, B, T, S : X \rightarrow X\) be mappings satisfying

1. \(TX \subset AX\) and \(SX \subset BX\)

2. The pairs \((S, A)\) and \((T, B)\) are compatible and

3. For all \(x, y \in X\) and \(\alpha, \beta, \gamma, \delta \geq 0\) satisfying \(2\alpha + \beta + 2\gamma + 2\delta < 1\), we have

\[
d(Tx, Sy) \leq \alpha [d(Bx, Tx) + d(Ay, Sy)] + \beta d(Bx, Ay) + \gamma d(Bx, Sy) + \delta d(Tx, Ay)
\]

(1)

If one of let \(A, B, T\) and \(S\) is continuous, then \(A, B, T\) and \(S\) have a unique common fixed point in \(X\).

Our purpose here is to prove that these assumptions (continuity and compatibility) are still too strong. We begin by recalling some basic concepts of the theory of dislocated metric spaces.

Definition 1.2 Let \(X\) be a non empty set and let \(d : X \times X \rightarrow [0, \infty)\) be a function satisfying the following conditions

1. \(d(x, y) = d(y, x)\)

2. \(d(x, y) = d(y, x) = 0\) implies \(x = y\)

3. \(d(x, y) \leq d(x, z) + d(z, y)\) for all \(x, y, z \in X\)

Then \(d\) is called dislocated metric(or simply d-metric) on \(X\).

Definition 1.3 A sequence \(\{x_n\}\) in a d-metric space \((X, d)\) is called a Cauchy sequence if for given \(\epsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that for all \(m, n \geq n_0\), we have \(d(x_m, x_n) < \epsilon\).

Definition 1.4 A sequence in a d-metric space converges if there exists \(x \in X\) such that \(d(x_n, x) \rightarrow 0\).

Definition 1.5 A d-metric space \((X, d)\) is called complete if every Cauchy sequence is convergent.

Remark 1.6 It is easy to verify that in a dislocated metric space, we have the following technical properties

- A subsequence of a Cauchy sequence in d-metric space is a Cauchy sequence.
Main results

Theorem 2.1 Let $A, B, T$ and $S$ be four self-mappings of a $d$-metric space $(X, d)$ such that

1. $TX \subset AX$ and $SX \subset BX$

2. The pairs $(S, A)$ and $(T, B)$ are weakly compatible and

3. For all $x, y \in X$ and $\alpha, \beta, \gamma, \delta \geq 0$ satisfying $2\alpha + \beta + 2\gamma + 2\delta < 1$, we have

$$d(Tx, Sy) \leq \alpha [d(Bx, Tx) + d(Ay, Sy)] + \beta d(Bx, Ay) + \gamma d(Bx, Sy) + \delta d(Tx, Ay)$$

(2)

4. The range of one of the mappings $A, B, S$ or $T$ is a complete subspace of $X$.

Then $A, B, T$ and $S$ have a common fixed point in $X$. Furthermore, if $4\alpha + \beta + \gamma + \delta < 1$, then the common fixed point is unique.

Proof. Let $x_0$ be an arbitrary point in $X$. Choose $x_1 \in X$ such that $Bx_1 = Sx_0$. Choose $x_2 \in X$ such that $Ax_2 = Tx_1$. Continuing in this fashion, choose $x_n \in X$ such that $Sx_{2n} = Bx_{2n+1}$ and $Tx_{2n+1} = Ax_{2n+2}$ for $n = 0, 1, 2, ...$.

To simplify, we consider the sequence $(y_n)$ defined by $y_{2n} = Sx_{2n}$ and $y_{2n+1} = Tx_{2n+1}$ for $n = 0, 1, 2, ...$.

We claim that $(y_n)$ is a Cauchy sequence. Indeed, by using (2) for $n \geq 1$, we have

$$d(y_{2n+1}, y_{2n}) = d(Tx_{2n+1}, Sx_{2n})$$

$$\leq \alpha [d(Bx_{2n+1}, Tx_{2n+1}) + d(Ax_{2n}, Sx_{2n})] + \beta d(Bx_{2n+1}, Ax_{2n}) + \gamma d(Bx_{2n+1}, Sx_{2n}) + \delta d(Tx_{2n+1}, Ax_{2n})$$

$$\leq \alpha [d(y_{2n+1}, y_{2n+1}) + d(y_{2n-1}, y_{2n})] + \beta d(y_{2n}, y_{2n-1}) + \gamma d(y_{2n}, y_{2n}) + \delta d(y_{2n+1}, y_{2n-1})$$

$$\leq \alpha [d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] + \beta d(y_{2n-1}, y_{2n}) + 2\gamma d(y_{2n-1}, y_{2n}) + \delta [d(y_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n})]$$

$$\leq (\alpha + \beta + 2\gamma + \delta) d(y_{2n-1}, y_{2n}) + (\alpha + \delta) d(y_{2n}, y_{2n+1}).$$
Therefore
\[ d(y_{2n}, y_{2n+1}) \leq h \cdot d(y_{2n-1}, y_{2n}) \]
where \( h = \frac{\alpha + \beta + 2\gamma + \delta}{1 - \alpha - \delta} \in [0, 1[. \) Hence \((y_n)\) is a Cauchy sequence in \( X \) and therefore, according to Remarks 1.6, \((Sx_{2n}), (Bx_{2n+1}), (Tx_{2n+1})\) and \((Ax_{2n+2})\) are also Cauchy sequence.

Suppose that \( SX \) is a complete subspace of \( X \), then the sequence \((Sx_{2n})\) converges to some \( Sa \) such that \( a \in X \). According to Remark 1.6, \((y_n), (Bx_{2n+1}), (Tx_{2n+1})\) and \((Ax_{2n+2})\) converge to \( Sa \). Since \( SX \subset BX \), there exists \( u \in X \) such that \( Sa = Bu \). Since \( d(Bu, Bu) \leq d(Bu, Sx_{2n}) + d(Sx_{2n}, Bu) \), we get, on letting \( n \) to infty, \( d(Bu, Bu) = 0 \). Now we show that \( Bu = Tu \). In fact, by using (2), we have
\[ d(Tu, Sx_{2n}) \leq \alpha[d(Bu, Tu) + d(Ax_{2n}, Sx_{2n})] + \beta d(Bu, Ax_{2n}) + \gamma d(Bu, Sx_{2n}) + \delta d(Tu, Ax_{2n}) \]
and therefore, on letting \( n \) to infty, we get
\[ d(Tu, Bu) \leq \alpha[d(Bu, Tu) + d(Bu, Bu)] + \beta d(Bu, Bu) + \gamma d(Bu, Bu) + \delta d(Tu, Bu) \]
which implies that \( d(Bu, Tu) = 0 \), since \((1 - \alpha - \delta) < 0 \), which implies that \( Tu = Bu \). Since \( TX \subset AX \), there exists \( v \in X \) such that \( Tu = Av \). We show that \( Sv = Av \). Indeed, by using (2), we have
\[ d(Av, Sv) = d(Tu, Sv) \]
\[ \leq \alpha[d(Bu, Tu) + d(Av, Sv)] + \beta d(Bu, Av) + \gamma d(Bu, Sv) + \delta d(Tu, Av) \]
\[ \leq \alpha d(Av, Sv) + \gamma d(Av, Sv) \]
and therefore \( d(Av, Sv) = 0 \), since \((1 - \alpha - \gamma) < 0 \), which implies that \( Av = Sv \). Hence \( Bu = Tu = Av = Sv \).

The weak compatibility of \( S \) and \( A \) implies that \( ASv = SAv \), from which it follows that \( AAu = ASv = SAv = SSv \).

The weak compatibility of \( B \) and \( T \) implies that \( BTu = TBu \), from which it follows that \( BBu = BTu = TBu = TTu \).

Let us show that \( Bu \) is a fixed point of \( T \). Indeed, from (2), we get
\[ d(Bu, TBu) = d(TBu, Sv) \]
\[ \leq \alpha[d(BBu, TBu) + d(Av, Sv)] + \beta d(BBu, Av) + \gamma d(BBu, Sv) + \delta d(TBu, Av) \]
\[ \leq \alpha d(TBu, TBu) + \beta d(Bu, TBu) + \gamma d(Bu, TBu) + \delta d(Bu, TBu) \]
\[ \leq (2\alpha + \beta + \gamma + \delta) d(Bu, TBu) \]
and therefore \( d(Bu, TBu) = 0 \), since \((1 - 2\alpha - \beta - \gamma - \delta) < 0 \), which implies that \( TBu = Bu \). Hence \( Bu \) is a fixed point of \( T \). It follows that \( BBu = TBu = Bu \),
which implies that $Bu$ is a fixed point of $B$.

On the other hand, in view of (2), we have

$$d(SBu, Bu) = d(TBu, SBu)$$

$$\leq \alpha [d(BBu, TBu) + d(ABu, SBu)] + \beta d(BBu, ABu) + \gamma d(BBu, SBu) + \delta d(TBu, ABu)$$

$$\leq \alpha [d(Bu, Bu) + d(SBu, SBu)] + \beta d(Bu, SBu) + \gamma d(Bu, SBu) + \delta d(Bu, SBu)$$

$$\leq (2\alpha + \beta + \gamma + \delta) d(Bu, SBu)$$

and therefore $d(Bu, SBu) = 0$, since $1 - 2\alpha - \beta - \gamma - \delta < 0$, which implies that $SBu = Bu$. Hence $Bu$ is a fixed point of $S$. It follows that $ABu = SBu = Bu$, which implies that $Bu$ is also a fixed point of $S$. Thus $Bu$ is a common fixed point of $S$, $T$, $A$ and $B$.

Finally to prove uniqueness, suppose that there exists $u, v \in X$ such that $Su = Tu = Au = Bu = u$ and $Sv = Tv = Av = Bv = v$. If $d(u, v) \neq 0$, then, by using (2), we get

$$d(u, v) = d(Tu, Sv)$$

$$\leq \alpha [d(Bu, Tu) + d(Av, Sv)] + \beta d(Bu, Av) + \gamma d(Bu, Sv) + \delta d(Tu, Av)$$

$$\leq \alpha [d(u, u) + d(v, v)] + \beta d(u, v) + \gamma d(u, v) + \delta d(u, v)$$

$$\leq (4\alpha + \beta + \gamma + \delta) d(u, v)$$

from which it follows that $(1 - 4\alpha - \beta - \gamma - \delta) d(u, v) \leq 0$. Therefore, if $2\alpha + \beta + \gamma + \delta < 1$, the we get $d(u, v) = 0$ and therefore $u = v$.

The proof is similar when $TX$ or $AX$ or $Bx$ is a complete subspace of $X$. This completes the proof of the Theorem.

**Remark 2.2** We note that there is no connection between the inequalities $2\alpha + \beta + 2\gamma + 2\delta < 1$ and $4\alpha + \beta + \gamma + \delta < 1$. Indeed, we have

1. For $\alpha = \frac{1}{4}$, $\beta = \frac{1}{8}$ and $\gamma = \delta = \frac{1}{24}$, we have $2\alpha + \beta + 2\gamma + 2\delta < 1$ and $4\alpha + \beta + \gamma + \delta > 1$.

2. For $\alpha = \frac{1}{48}$, $\beta = \frac{2}{3}$ and $\gamma = \delta = \frac{1}{12}$, we have $2\alpha + \beta + 2\gamma + 2\delta > 1$ and $4\alpha + \beta + \gamma + \delta < 1$.

For $A = B$ and $S = T$ in Theorem 2.1, we have the following result

**Corollary 2.3** Let $A$ and $S$ be two self-mappings of a $d$-metric space $(X, d)$ such that
1. $SX \subseteq AX$

2. The pairs $(S, A)$ is weakly compatible and

3. For all $x, y \in X$ and $\alpha, \beta, \gamma, \delta \geq 0$ satisfying $2\alpha + \beta + 2\gamma + 2\delta < 1$, we have

$$d(Sx, Sy) \leq \alpha [d(Ax, Sx) + d(Ay, Sy)] + \beta d(Ax, Ay) + \gamma d(Ax, Sy) + \delta d(Sx, Ay)$$

(3)

4. The range of one of the mappings $A$ or $S$ is a complete subspace of $X$.

Then $A$ and $S$ have a common fixed point in $X$. Furthemore, if $4\alpha + \beta + \gamma + \delta < 1$, then the common fixe point is unique.

For $A = B = Id_X$ in Theorem 2.1, we get the following corollary

**Corollary 2.4** Let $T$ and $S$ be two self-mappings of a $d$-metric space $(X,d)$ such that

1. For all $x, y \in X$ and $\alpha, \beta, \gamma, \delta \geq 0$ satisfying $2\alpha + \beta + 2\gamma + 2\delta < 1$, we have

$$d(Tx, Sy) \leq \alpha [d(x, Tx) + d(y, Sy)] + \beta d(x, y) + \gamma d(x, Sy) + \delta d(Tx, y)$$

(4)

2. The range of one of the mappings $S$ or $T$ is a complete subspace of $X$.

Then $T$ and $S$ have a common fixed point in $X$. Furthemore, if $4\alpha + \beta + \gamma + \delta < 1$, then the common fixe point is unique.

For $S = T = Id_X$ in Theorem 2.1, we have the following result

**Corollary 2.5** Let $A$ and $B$ be two surjective self-mappings of a complete $d$-metric space $(X,d)$ such that

1. For all $x, y \in X$ and $\alpha, \beta, \gamma, \delta \geq 0$ satisfying $2\alpha + \beta + 2\gamma + 2\delta < 1$, we have

$$d(x, y) \leq \alpha [d(Bx, x) + d(Ay, y)] + \beta d(Bx, Ay) + \gamma d(Bx, y) + \delta d(x, Ay)$$

(5)

Then $A$ and $B$ have a common fixed point in $X$. Furthemore, if $4\alpha + \beta + \gamma + \delta < 1$, then the common fixe point is unique.
References


Received: July 10, 2016; Published: August 15, 2016