Conservation Laws and Multiple Simplest Equation Methods to an Extended Quantum Zakharov-Kuznetsov Equation

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Abstract
The extended quantum Zakharov-Kuznetsov (QZK) equation, which arises in the study of astrophysics is investigated. Exact traveling wave solutions are obtained by using distinct types of the simplest equation method namely, the simplest equation method, the extended simplest equation method and the modified simplest equation method. In addition conservation laws via a new conservation theorem are constructed for the underlying equation.

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1. Introduction

It is of great importance to extract exact solutions of nonlinear partial differential equations (NPDEs) arising in mathematical physics, because many physical phenomena are described by NPDEs. Great efforts have been developed to different methods to solve NPDEs in the literature [1-9]. Also, Conservation laws
have a significant role in the solution processes of NPDEs. By using the relationship between Lie point symmetries and conservation laws, we can further generate other conservation laws. Conservation laws describe physical conserved quantities such as mass, energy, momentum and angular momentum as well as charge and other constants of motion [10, 11]. A variety of powerful techniques have been used to derive conservation laws [12-14]. The role of conservation laws have been recently used combined with double reduction theory to obtain exact solutions of NPDEs [15-19], therefore it is very important to investigate conservation laws for PDEs.

In this work, we consider the extended quantum Zakharov-Kuznetsov (QZK) equation [20, 21] as

\[ u_t + auu_x + b(u_{xxx} + u_{yyy}) + c(u_{xxy} + u_{xyy}) = 0 \]  

(1)

where \( a, b \) and \( c \) are all constants, while \( u(x, y, t) \) represents the electrostatic wave potential in plasmas. Eq.(1) supports soliton solutions that are the outcome of a delicate balance between dispersion and nonlinearity.

The outline of this paper is as follows. In Sections 2, 3 and 4, a brief description of the simplest equation methods for extracting traveling wave solutions of nonlinear partial differential equations (NPDEs) is presented. Thus, we construct the conservation laws using a new conservation theorem in Sec.5 for the underlying equation. Some conclusions are given in Sec.6.

2. Description of the simplest equation method [22-24]

Consider a general nonlinear PDE in the form

\[ P(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, ...) = 0. \]  

(2)

where \( P \) is a polynomial in its arguments.

Using the traveling wave transformation \( u(x,t) = u(\xi) \), \( \xi = x - ct + \xi_0 \), where \( c \) is the wave speed, and \( \xi_0 \) is an arbitrary constant, to transform Eq.(2) to a nonlinear ordinary differential equation (ODE) as

\[ Q(u, u', u'', u''', ..., ) = 0, \]  

(3)

where the prime denotes the derivation with respect to \( \xi \).

Suppose the solution of Eq.(3) can be expressed as in the form

\[ u(\xi) = \sum_{i=0}^{N} \alpha_i (\phi(\xi))^i, \]  

(4)
where \( \varphi(\xi) \) satisfies either the Bernoulli or Riccati equation, \( N \) is a positive integer that can be determined by balancing procedure and \( \alpha_i \) are parameters to be determined, \( i = 0,1,2,\ldots,N \).

Let us consider the Bernoulli equation in the following form
\[
\varphi'(\xi) = p \varphi(\xi) + q \varphi^2(\xi),
\]
where \( p \) and \( q \) are arbitrary constants. The solutions of Eq.(5) can be written as
\[
\varphi(\xi) = \frac{-p A_1}{q (A_1 + \cosh(p(\xi + \xi_0)) - \sinh(p(\xi + \xi_0)))},
\]
\[
\varphi(\xi) = \frac{-p (\cosh(p(\xi + \xi_0)) - \sinh(p(\xi + \xi_0)))}{q (A_2 + \cosh(p(\xi + \xi_0)) + \sinh(p(\xi + \xi_0)))},
\]
where \( A_1, A_2 \) and \( \xi_0 \) are arbitrary constants.

Also let us consider the Riccati equation as
\[
\varphi'(\xi) = p \varphi^2(\xi) + q \varphi(\xi) + s,
\]
where \( p, q \) and \( s \) are constants. It should be mentioned that Eq.(8) has the following solutions [25]
\[
\varphi(\xi) = \left(-q/2p\right) - \left(\theta/2p\right) \tanh[(\theta/2)(\xi + \xi_0)],
\]
\[
\varphi(\xi) = \left(-q/2p\right) - \left(\theta/2p\right) \tanh[(\theta/2)\xi]
+ \frac{\sec h((\theta/2)\xi)}{C \cosh((\theta/2)\xi) - (2p/\theta) \sinh((\theta/2)\xi)},
\]
where \( \theta^2 = q^2 - 4ps \).

Substituting (4) into (3) with (5) (or (8)), then the left-hand side of (3) is reduced to a polynomial in \( \varphi(\xi) \). Setting each coefficient of this polynomial to zero, yields a system of algebraic equations which can be solved for \( \alpha_i, p, q \). Hence, substituting these values into (4), the exact traveling wave solutions are reported then for Eq.(2).

Note. If we put \( p = A \) and \( q = -1 \) in Bernoulli equation (5), we obtain
\[
\varphi'(\xi) = A \varphi(\xi) - \varphi^2(\xi),
\]
which has the exact solution
\[ \varphi(\xi) = (A/2)[1 + \tanh((A/2)(\xi + \xi_0))], \]  

(12)

provided \( A > 0 \), and the other exact solution
\[ \varphi(\xi) = (A/2)[1 - \tanh((A/2)(\xi + \xi_0))], \]  

(13)

if \( A < 0 \).

**Application of the simplest equation method using Bernoulli equation**

Introducing a wave variable
\[ \xi = kx + ry - vt + \xi_0, \]

and the other exact solution
\[ \varphi(\xi) = (A/2)[1 - \tanh((A/2)(\xi + \xi_0))], \]  

(13)

if \( A < 0 \).

\[ \frac{d}{d \xi} + (b(k^3 + r^3) + c k r(k + r))u'' = 0 \]  

(14)

Integrating (2) with respect to \( \xi \) and setting the integration constant to zero, yields
\[ -v u + \left(\frac{1}{2} ak\right)u^2 + (b(k^3 + r^3) + c k r(k + r))u^* = 0. \]  

(15)

Thus, we are concerned to solve the equation (15).

Considering the homogeneous balance between \( u'' \) and \( u^2 \), we get \( N = 2 \), therefore, the solution of (24) can be written as
\[ u(\xi) = \alpha_0 + \alpha_1 \varphi(\xi) + \alpha_2 \varphi^2(\xi) \]  

(16)

Substituting (16) into (15) and making use of the Bernoulli equation (5) and then equating the coefficients of the functions \( \varphi^i(\xi) \) to zero, we obtain the following system of algebraic equations
\[ \varphi^4(\xi) : \frac{1}{2} ak \alpha_2^2 + 6q^2 \alpha_2 M = 0 \]  

(17)

\[ \varphi^3(\xi) : ak \alpha_1 \alpha_2 + (2q^2 \alpha_1 + 10pq \alpha_2) M = 0 \]  

(18)

\[ \varphi^2(\xi) : \frac{1}{2} ak(\alpha_1^2 + 2\alpha_0 \alpha_2) - v \alpha_2 + (4p^2 \alpha_2 + 3pq \alpha_1) M = 0 \]  

(19)

\[ \varphi^1(\xi) : -v \alpha_1 + ak \alpha_0 \alpha_1 + p^2 \alpha_1 M = 0 \]  

(20)

\[ \varphi^0(\xi) : -v \alpha_0 + \frac{1}{2} ak \alpha_0^2 = 0 \]  

(21)

\[ M = (b(k^3 + r^3) + c k r(k + r)). \]  

(22)
On solving the above system of over-determined algebraic equations, we obtain two sets of solution,
\[ \alpha_0 = -(2 p^2 / a k) M, \alpha_1 = -(12 p q / a k) M, \alpha_2 = -(12 q^2 / a k) M, v = - p^2 M, \]  
(23)
\[ \alpha_0 = 0, \alpha_1 = -(12 p q / a k) M, \alpha_2 = -(12 q^2 / a k) M, v = p^2 M. \]  
(24)
Therefore, using solutions (6) and (7) of (5) and the values of \( \alpha_0, \alpha_1, \alpha_2 \) from (23) into solution ansatz (16), we obtain the exact solutions of (1) in the form:
\[ u_1(x, y, t) = -(2 p^2 / a k) M - \]
\[ (12 p q / a k) M \left( \frac{-p A_1}{q \left( A_1 + \cosh(p(\xi + \xi_0)) + \sinh(p(\xi + \xi_0)) \right)} \right) \]  
(25)
\[ -(12 q^2 / a k) M \left( \frac{-p A_1}{q \left( A_1 + \cosh(p(\xi + \xi_0)) + \sinh(p(\xi + \xi_0)) \right)} \right)^2, \]
\[ u_2(x, y, t) = -(2 p^2 / a k) M - \]
\[ (12 p q / a k) M \left( \frac{-p (\cosh(p(\xi + \xi_0)) + \sinh(p(\xi + \xi_0)))}{q \left( A_2 + \cosh(p(\xi + \xi_0)) + \sinh(p(\xi + \xi_0)) \right)} \right) \]  
(26)
\[ -(12 q^2 / a k) M \left( \frac{-p (\cosh(p(\xi + \xi_0)) + \sinh(p(\xi + \xi_0)))}{q \left( A_2 + \cosh(p(\xi + \xi_0)) + \sinh(p(\xi + \xi_0)) \right)} \right)^2, \]
where \( \xi = k x + r y + (p^2 M) t + \xi_0, M = (b(k^3 + r^3) + c k r(k + r)). \)

In the same way, using solutions (6) and (7) of (5) and the values of \( \alpha_0, \alpha_1, \alpha_2 \) from (24) into solution ansatz (16), we obtain the following exact solutions of (1):
\[ u_3(x, y, t) = -(12 p q / a k) M \left( \frac{-p A_1}{q \left( A_1 + \cosh(p(\xi + \xi_0)) + \sinh(p(\xi + \xi_0)) \right)} \right) \]  
(27)
\[ -(12 q^2 / a k) M \left( \frac{-p A_1}{q \left( A_1 + \cosh(p(\xi + \xi_0)) + \sinh(p(\xi + \xi_0)) \right)} \right)^2, \]
\[
\begin{aligned}
&u_4(x,y,t) = -(12 p q / a k) M \left( -p \left( \cosh(p(\xi + \xi_0)) + \sinh(p(\xi + \xi_0)) \right) \right) \\
&\quad \quad \quad \quad \quad + \left( q \left( A_2 + \cosh(p(\xi + \xi_0)) + \sinh(p(\xi + \xi_0)) \right) \right)^2 \\
&\quad -(12 q^2 / a k) M \left( -p \left( \cosh(p(\xi + \xi_0)) - \sinh(p(\xi + \xi_0)) \right) \right) \\
&\quad \quad \quad \quad \quad + \left( q \left( A_2 + \cosh(p(\xi + \xi_0)) + \sinh(p(\xi + \xi_0)) \right) \right)^2 \\
\end{aligned}
\]

where \( \xi = k x + r y - (p^2 M) t, M = (b(k^3 + r^3) + c k r (k + r)) \).

and \( a, b, c, p, q, r, k, A_1, \) and \( A_2 \) are arbitrary constants.

Application of the simplest equation method using Riccati equation

In the case of Riccati equation, the solution ansatz can be written as,

\[
u(\xi) = \beta_0 + \beta_1 \varphi(\xi) + \beta_2 \varphi^2(\xi)
\]

Substituting (29) into (15) and making use of the Riccati equation (8) and then equating the coefficients of the functions \( \varphi^i(\xi) \) to zero, we thus obtain a system of algebraic equations and by solving it, we obtain

\[
\begin{aligned}
\beta_0 &= 0, \beta_1 = \beta_1, \quad \beta_2 = (p/q) \beta_1, v = -3 q^2 M, c = -\frac{12 b p^2(k^3 + r^3) + a k \beta_2}{12 k r p^2 (k + r)}, \\
p &= -q^2 / 2s, \\
M &= (b(k^3 + r^3) + c k r (k + r)).
\end{aligned}
\]

Thus, using solutions (9) and (10) of (8), (30) and (16), yields following exact solutions of Eq. (1):

\[
\begin{aligned}
&u_5(x,y,t) = \beta_1 \left\{ (-q/2 p) - (\theta/2 p) \tanh([\theta/2](\xi + \xi_0)) \right\} \\
&\quad + \left( (p/q)((-q/2 p) - (\theta/2 p) \tanh([\theta/2](\xi + \xi_0))) \right)^2
\end{aligned}
\]

\[
\begin{aligned}
&u_6(x,y,t) = \beta_1 \left\{ \left( -q/2 p - (\theta/2 p) \tanh([\theta/2][\xi]) \right) \right. \\
&\quad \quad + \left. \frac{\text{sech}([\theta/2][\xi])}{C \cosh([\theta/2][\xi]) - (2 p/\theta) \sinh([\theta/2][\xi])} \right\} \\
&\quad + \left( \frac{\text{sech}([\theta/2][\xi])}{C \cosh([\theta/2][\xi]) - (2 p/\theta) \sinh([\theta/2][\xi])} \right)^2
\end{aligned}
\]
where

\[\xi = kx + ry + (3q^2 M)t + \xi_0, M = (b(k^3 + r^3) + ck r(k + r)), \theta = \sqrt{q^2 - 4ps}, c\]

and p are determined in (39), \(a, b, p, q, r, s, k, \beta_1, C\) are arbitrary constants.

3 The modified simplest equation method [26,27]

**Step 1.** We suppose that (15) admits the formal solution

\[\phi(\xi) = \sum_{i=0}^{N} a_i \left( \frac{\phi'}{\phi} \right)^i, \tag{33}\]

where \(a_i\) are constants to be determined such that \(a_N \neq 0, i = 0, 1, ..., N\).

The function \(\phi(\xi)\) is an unknown function to be determined later provided that \(\phi'(\xi) \neq 0\).

**Step 2.** We substitute (33) into (15), we calculate all the necessary derivatives \(\phi', \phi'', \phi''', ...,\) and then we determine the function \(\phi(\xi)\).

As a result, we get a polynomial of \(\phi^{-j}(\xi) (j = 0, 1, ..., )\), with the derivatives of \(\phi(\xi)\).

Setting all the coefficients of \(\phi^{-j}(\xi) (j = 0, 1, ..., )\), to zero yields a system of algebraic equations which can be solved to obtain \(a_i\) and \(\phi(\xi)\).

**Step 3.** Substituting the values of \(a_i\) and \(\phi(\xi)\) and its derivative \(\phi'(\xi)\) into (33) leads to exact traveling wave solutions of (1).

**Application of the modified simplest equation method**

Suppose that (15) admits the formal solution

\[u(\xi) = a_0 + a_1 \left( \frac{\phi'(\xi)}{\phi(\xi)} \right) + a_2 \left( \frac{\phi'(\xi)}{\phi(\xi)} \right)^2, \tag{34}\]

where \(a_0, a_1\) and \(a_2\) are constants to be determined such that \(a_2 \neq 0\).

Substituting (34) into (15) and equating all the coefficients of \(\phi^0, \phi^{-1}, \phi^{-2}, \phi^{-3}, \phi^{-4}\), to zero, we respectively obtain

\[\phi^0 : -v a_0 + \frac{1}{2} a k \ a_0^2 = 0, \tag{35}\]
\[
\varphi^{-1} = -v a_1 \varphi' + a k a_0 a_1 \varphi' + M a_1 \varphi'' = 0,
\]  
(36)

\[
\varphi^{-2} = -v a_2 \varphi'^2 + a k a_0 a_2 \varphi'^2 + \frac{1}{2} a k a_1 2 \varphi'^2 - 3 M a_1 \varphi' \varphi'' + 2 Ma_2 \varphi' \varphi'' + 2 Ma_2 \varphi'^2 = 0,
\]  
(37)

\[
\varphi^{-3} = a k a_1 a_2 \varphi'^3 + 2 Ma_1 \varphi'^3 - 10 M a_2 \varphi'^2 \varphi' = 0
\]  
(38)

\[
\varphi^{-4} = \frac{1}{2} a k a_2 \varphi'^4 + 6 Ma_2 \varphi'^4 = 0
\]  
(39)

where \( M = (b(k^3 + r^3) + c k r (k + r)) \).

From (35) and (39), we deduce that

| \( a_0 = 0, a_0 = 2v / a k \), \( a_2 = -12 M / a k \),

provided \( a k \neq 0 \). Now we consider the following cases:

**Case 1.** When \( a_0 = 0, a_1 \neq 0 \), and \( \varphi' \neq 0 \), then we deduce from Eqs.(36-38) that

\[
\varphi' = -v + a k a_0 + M \varphi'' = 0,
\]  
(41)

\[
(12v M / a k + \frac{1}{2} a k a_1 2 \varphi'^2 - 3 a_1 M a k \varphi'' - (24 M^2 / a k) (\varphi' \varphi'' + \varphi'^2)) = 0
\]  
(42)

\[
-a k a_1 \varphi' + 12 M \varphi'' = 0
\]  
(43)

From (41) and (43) we have

\[
\varphi' = \frac{-M \varphi''}{-v + a k a_0} = \frac{12 M \varphi''}{a k a_1}
\]  
(44)

and consequently we get

\[
\varphi'' / \varphi' = 12v / a k a_1
\]  
(45)

Integrating (45), we get

\[
\varphi'' = C_1 e^{\frac{12v}{a k a_1}}
\]  
(46)

and substituting from (46) into (44), we get
Conservation laws and multiple simplest equation methods

\[ \phi'(\xi) = \frac{12M}{ak a_1} C_1 e^{\left(\frac{12v}{ak a_1}\right) \xi} \]

Integrating (47), we obtain

\[ \phi(\xi) = C_2 + \frac{M}{v} C_1 e^{\left(\frac{12v}{ak a_1}\right) \xi}, \]

where \( C_1 \) and \( C_2 \) are arbitrary constants of integration.

Substituting (44) into (42), we have

\[ a_i = (\pm 12/ak)\sqrt{vM}, \]

provided \( ak \neq 0 \).

Thus the exact solution of (1) in this case has the form

\[ u(\xi) = (12C_1 M/ak) \left( \frac{12v}{ak a_1} \right) \frac{(k x + r y - vt + \xi_0)}{e} \left( \frac{12v}{ak a_1} \right) (k x + r y - vt + \xi_0) \left( C_2 + (C_1 M/v)e\right) \]

\[ -(1728C_1^2 M^3 / a^3 k^3 a_1^2) \left( \frac{12v}{ak a_1} \right) (k x + r y - vt + \xi_0) \left( \frac{12v}{ak a_1} \right) (k x + r y - vt + \xi_0) \left( C_2 + (C_1 M/v)e\right)^2 \]

where the value of \( a_i \) is provided in (49).

Since \( C_1 \) and \( C_2 \) are constants of integration, we might arbitrarily choose their values. Therefore, if we choose \( C_2 = \pm 1, C_1 = v/M \), we have respectively the solitary wave solution

\[ u_1(\xi) = (6v/ak) \left( 1 + \tanh\left( \frac{6v}{ak a_1} (k x + r y - vt + \xi_0) \right) \right) \]

\[ - (432v^2 M / a^3 k^3 a_1^2) \left( 1 + \tanh\left( \frac{6v}{ak a_1} (k x + r y - vt + \xi_0) \right) \right)^2 \]

(51)
\[ u_2(\xi) = (6v/ak) \left( 1 + \coth\left( \frac{6v}{ak a_1} (k x + r y - vt + \xi_0) \right) \right) \]

\[- (432v^2 M/a^3 k^3 a_1^2) \left( 1 + \coth\left( \frac{6v}{ak a_1} (k x + r y - vt + \xi_0) \right) \right)^2 \]

Case 2. When \( a_0 \neq 0 \), \( a_1 \neq 0 \), and \( \phi' \neq 0 \), then we deduce from Eqs. (36) to (38) that

\[ v \phi' + M \phi'' = 0 \]  

(53)

\[ (-12vM + \frac{1}{2} a^2 k^2 a_1^2) \phi'^2 - 3a_1 M a k \phi' \phi'' - 24M^2 (\phi' \phi''' + \phi''^2) = 0 \]  

(54)

\[ a k a_1 \phi' - 12 M \phi'' = 0 \]  

(55)

Consequently, we deduce from Eqs. (53) and (55) that

\[ \phi' = \frac{-M \phi''}{v} = \frac{12M \phi''}{ak a_1} \]  

(56)

and thus, we get

\[ \phi''/\phi'' = -12v/ak a_1 \]  

(57)

Integrating (57), we then have

\[ (- \frac{12v}{ak a_1}) ^{\frac{\xi}{\phi''}} = C_1 e \]  

(58)

From (58) into (56), we get

\[ \phi'(\xi) = \frac{12M}{ak a_1} C_1 e^{- \frac{12v}{ak a_1} \xi} \]  

(59)

Integrating (59), we obtain

\[ \phi(\xi) = C_2 - \frac{M}{v} \frac{1}{C_1} e^{- \frac{12v}{ak a_1} \xi} \]  

(60)

where \( C_1 \) and \( C_2 \) are arbitrary constants of integration.

Consequently, we conclude from (54) that
\[ a_1 = (\pm 12 i / a k) \sqrt{3 v M}, \quad i = \sqrt{-1} \]  

(61)

provided \( a k \neq 0 \).

Thus, the exact solution of (1) in this case has the form

\[
\begin{align*}
\left. \begin{array}{c}
\left( \frac{-12 c}{a k a_1} (k x + r y - vt + \xi_0) \\
\left( \frac{12 c}{a k a_1} (k x + r y - vt + \xi_0) \right)
\end{array} \right) \\
\left( \frac{12 c}{a k a_1} (k x + r y - vt + \xi_0) \right)
\end{align*}
\]

\[
\begin{align*}
\left( \frac{-12 c}{a k a_1} (k x + r y - vt + \xi_0) \right)^2
\end{align*}
\]

(62)

where \( a_1 \) is as given in (61).

The value of \( C_1 \) and \( C_2 \) can be selected randomly as they are integration constant. Therefore, if we select \( C_2 = \pm 1, C_1 = v / M \), we have the solitary wave solution

\[
\begin{align*}
u_1(\xi) = (2v / a k) & \left[ 4 - 3 \coth\left( \frac{-6 c}{a k a_1} (k x + r y - vt + \xi_0) \right) \\
& - (216 v / a^2 k^2 a_1^2) \left( 1 - \coth\left( \frac{-6 c}{a k a_1} (k x + r y - vt + \xi_0) \right) \right)^2 \right]
\end{align*}
\]

\[
\begin{align*}
u_2(\xi) = (2v / a k) & \left[ 4 - 3 \tanh\left( \frac{-6 c}{a k a_1} (k x + r y - vt + \xi_0) \right) \\
& - (216 v / a^2 k^2 a_1^2) \left( 1 - \tanh\left( \frac{-6 c}{a k a_1} (k x + r y - vt + \xi_0) \right) \right)^2 \right]
\end{align*}
\]

(63)

(64)

4. The extended simplest equation method [28, 29]

Suppose the solution of Eq.(5) can be expressed as another ansatz in the following form
\[ u(\xi) = \sum_{i=0}^{N} \alpha_i \left( \frac{\varphi'}{\varphi} \right)^i + \sum_{j=0}^{N-1} \beta_j \left( \frac{\varphi'}{\varphi} \right)^j \left( \frac{1}{\varphi} \right), \]  

(65)

where \( \alpha_i, \beta_j \) are constants to be determined later, and \( \alpha_N \beta_{N-1} \neq 0, i = 0, 1, 2, ..., N, j = 0, 1, 2, ..., N-1 \).

The function \( \varphi = \varphi(\xi) \) satisfies the second order linear ODE

\[ \varphi'' + \delta \varphi = \mu, \]  

(66)

where \( \delta \) and \( \mu \) are constants.

Eq.(66) has the following three types of general solutions with double arbitrary parameters, namely,

\[ \varphi(\xi) = \begin{cases} 
A_1 \cosh(\sqrt{-\delta} \xi) + A_2 \sinh(\sqrt{-\delta} \xi) + \mu \delta, & \delta < 0 \\
A_1 \cos(\sqrt{-\delta} \xi) + A_2 \sin(\sqrt{-\delta} \xi) + \mu \delta, & \delta > 0 \\
(\mu/2)\xi^2 + A_1 \xi + A_2, & \delta = 0
\end{cases} \] 

(67)

(\delta A_1^2 - \delta A_2^2 - \mu^2 / \delta) \left( \frac{1}{\varphi} \right)^2 - \delta + 2\mu / \varphi, \delta < 0

\[ \left( \frac{\varphi'}{\varphi} \right)^2 = \begin{cases} 
(\delta A_1^2 + \delta A_2^2 - \mu^2 / \delta) \left( \frac{1}{\varphi} \right)^2 - \delta + 2\mu / \varphi, & \delta > 0 \\
(A_1^2 - 2\mu A_2^2) \left( \frac{1}{\varphi} \right)^2 + 2\mu / \varphi, & \delta = 0
\end{cases} \] 

(68)

where \( A_1 \) and \( A_2 \) are arbitrary constants.

By substituting (65) into (3) and using the second order linear ODE (66) and Eq.(68), the left-hand side of (3) is transformed to a polynomial in \( 1/\varphi^i \) and \( 1/\varphi \)(\varphi'/\varphi). Setting each coefficient of this polynomial to zero, yields a set of algebraic equations which can be solved for the constants \( \alpha_i, \beta_j, i = 0, 1, 2, ..., N, j = 0, 1, 2, ..., N - 1 \), \( v_0 \), \( \delta \) and \( \mu \). Therefore, substituting the resultant values of these constants and the general solutions (67) of (66) into (5), we can obtain the exact traveling wave solutions of Eq.(4) in a concise manner.
Application of the extended simplest equation method

Suppose the solution of (3) is of the form

$$u(\xi) = a_0 + a_1 \frac{\varphi'}{\varphi} + a_2 \left( \frac{\varphi'}{\varphi} \right)^2 + a_3 \left( \frac{1}{\varphi} \right) + a_4 \left( \frac{\varphi'}{\varphi} \right) \left( \frac{1}{\varphi} \right),$$  \hspace{1cm} (69)

where $a_0$, $a_1$, $a_2$, $a_3$, and $a_4$ are constants to be determined later and the function $\varphi = \varphi(\xi)$ satisfies the linear ODE (66).

By substituting (69) into (3) and using (66), the left-hand side of (3) becomes a polynomial in $1/\varphi^i$ and $(1/\varphi^i)(\varphi'/\varphi)$. Putting every coefficient of this polynomial as zero, yields a system of algebraic equations where by solving it via Mathematica 9, we therefore have the following cases:

If $\delta < 0$, we obtain

$$a_0 = 0, a_3 = 0, a_4 = 0, a_2 = \frac{i}{\sqrt{\delta}}, \quad A_2 = \pm A_1, \quad v = -ak\alpha_2 \delta, \quad \mu = 0,$$  \hspace{1cm} (70)

$$a_0 = 0, a_1 = 0, a_4 = 0, a_3 = \mp 3i\sqrt{2\delta}, \quad A_2 = \mp \sqrt{\frac{9 + 2\delta A_1^2}{2\delta}},$$  \hspace{1cm} (71)

Substituting (70) and (71) into (69) and making use of solutions (67) of (66), then the exact traveling wave solutions which expressed by hyperbolic functions are obtained and can be written as

$$u_1(\xi) = a_1 \sqrt{-\delta} \left( \frac{\sinh(\sqrt{-\delta} \xi) + \cosh(\sqrt{-\delta} \xi)}{\cosh(\sqrt{-\delta} \xi) + \sinh(\sqrt{-\delta} \xi)} \right) \left( \frac{1}{\cosh(\sqrt{-\delta} \xi) + \sinh(\sqrt{-\delta} \xi)} \right),$$  \hspace{1cm} (72)

$$\xi = kx + ry \mp (iak\alpha_1 \sqrt{\delta})t + \xi_0,$$

$$u_2(\xi) = -a_2 \sqrt{-\delta} \left[ \frac{A_1 \sinh(\sqrt{-\delta} \xi) \mp \sqrt{\frac{9 + 2\delta A_1^2}{2\delta} \cosh(\sqrt{-\delta} \xi)}}{A_1 \cosh(\sqrt{-\delta} \xi) \mp \sqrt{\frac{9 + 2\delta A_1^2}{2\delta} \cosh(\sqrt{-\delta} \xi) \pm 3i/\sqrt{2\delta}}} \right]^2,$$  \hspace{1cm} (73)

$$\xi = kx + ry + \left( \frac{1}{2ak\alpha_2\delta} \right) t + \xi_0.$$
where \( \alpha_1, \alpha_2, A_1, A_2, \delta \) and \( \xi_0 \) are arbitrary constants.

If \( \delta > 0 \), we obtain
\[
\alpha_0 = 0, \alpha_1 = 0, \alpha_4 = 0, \alpha_2 = \mp \frac{i \alpha_1}{\sqrt{\delta}}, A_2 = \mp A_1, \quad v = -ak \alpha_2 \delta, \mu = 0 \tag{74}
\]
\[
\alpha_0 = 0, \alpha_1 = 0, \alpha_4 = 0, \alpha_3 = \mp 3i \sqrt{2 \delta} \alpha_2, A_2 = \mp \sqrt{-9 - 2 \delta A_1^2 / 2 \delta},
\]
\[
c = -\frac{b(k^2 + r^2 - k r)}{k r}, \quad v = -\frac{1}{2} a k \alpha_2 \delta, \quad \mu = \pm 3i \frac{\sqrt{\delta}}{\sqrt{2}} \tag{75}
\]
Substituting (74) and (75) into (69) and making use of solutions (67) of (66), the exact traveling wave solutions expressed by trigonometric functions are obtained as
\[
u_1(\xi) = \alpha_1 \sqrt{\delta} \left( \frac{\sin(\sqrt{\delta} \xi) \mp \sqrt{-9 - 2 \delta A_1^2 / 2 \delta} \cos(\sqrt{\delta} \xi)}{\cos(-\sqrt{\delta} \xi) \mp \sin(\sqrt{\delta} \xi)} \right) \left[ \cos(\sqrt{\delta} \xi) \mp i \sin(\sqrt{\delta} \xi) \right]
\]
\[\xi = k x + r y \mp (i a k \alpha_1 \sqrt{\delta}) t + \xi_0 \tag{76}\]
\[
u_2(\xi) = \alpha_2 \sqrt{\delta} \left( \frac{A_1 \sin(\sqrt{\delta} \xi) \mp \sqrt{-9 - 2 \delta A_1^2 / 2 \delta} \cos(\sqrt{\delta} \xi)}{A_1 \cos(\sqrt{\delta} \xi) \mp \sqrt{-9 - 2 \delta A_1^2 / 2 \delta} \sin(\sqrt{\delta} \xi) \pm 3i / \sqrt{2 \delta}} \right)^2
\]
\[\xi = k x + r y + \left( \frac{1}{2} a k \alpha_2 \delta \right) t + \xi_0 \tag{77}\]
where \( \alpha_1, \alpha_2, A_1, A_2, \delta \) and \( \xi_0 \) are arbitrary constants.

If \( \delta = 0 \), we obtain
\[
\alpha_0 = 0, \alpha_1 = 0, \alpha_4 = 0, \quad A_2 = \mp i \sqrt{\frac{\alpha_2}{\alpha_3}} A_1, \quad \mu = A_1^2 / 2 A_2^2 \tag{78}
\]
Substituting (78) into (69) and making use of solutions (67) of (66), we obtain the rational function solutions of Eq.(1) and can be written as
Conservation laws and multiple simplest equation methods

\[ u(\xi) = 4\alpha_2 \left( \frac{A_1 \xi + 2A_2}{A_1 \xi^2 + 4A_2 \xi + 4i\sqrt{\alpha_2 / \alpha_3 A_2^2}} \right)^2 + \frac{4\alpha_3 (A_2^2 / A_1)}{A_1 \xi^2 + 4A_2 \xi + 4i\sqrt{\alpha_2 / \alpha_3 A_2^2}}. \]

\[ \xi = kx + ry - vt + \xi_0. \] (79)

where \( \alpha_2, \alpha_3, A_1, A_2 \), and \( \xi_0 \) are arbitrary constants.

5. Conservation laws for the extended quantum Zakharov-Kuznetsov (qZK) equation

Preliminaries[30]

Consider an \( r^{th} \)-order partial differential equation

\[ F(x, u, u_1, \ldots, u_r) = 0 \] (80)

with independent variables \( x = (x^1, \ldots, x^n) \) and one dependent variable \( u \), where \( u_1 = \{u_i\}, u_2 = \{u_{ij}\}, \ldots \) denote the set of partial derivatives of the first order, second order, etc., and \( u_i = \partial u / \partial x^i, u_{ij} = \partial^2 u / \partial x^i \partial x^j \).

The adjoint equation to (80) is

\[ F^*(x, u, v, u_1, \ldots, u_r, v_1, \ldots, v_r) = 0 \] (81)

where

\[ F^*(x, u, v, u_1, \ldots, u_r, u_r) = \delta(vF) / \delta u \] (82)

and

\[ \frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{r=1}^{\infty} (-1)^r D_{i_1 \ldots i_r} \frac{\partial}{\partial u_{i_1 \ldots i_r}} \] (83)

denote the Euler-Lagrange operator, and \( v \) is a new dependent variable. Here

\[ D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \ldots \] (84)

are the total differentiations with respect to \( x^i, i = 1, \ldots, n \).

Eq.(80) is called self-adjoint if the equation

\[ F^*(x, u, u_1, \ldots, u_r, u_r) = 0 \] (85)
obtained from the adjoint equation (82) by the substitution $v=u$ , is identical to the original equation (81). This concept can be written in an equivalent form as

$$F^*(x,u,u,(1),u,...,u,(r),u,(1),...,u,(r)) = \lambda(x,u,u,(1),...,u,(r)) F(x,u,u,(1),...,u,(r))$$

(86)

**Theorem of Ibragimov on conservation laws [31]:**

Every Lie point, Lie –Backlund or non-local symmetry

$$X = \xi^i(x,u,u,(1),...,\frac{\partial}{\partial x^i}) + \eta(x,u,u,(1),...,\frac{\partial}{\partial u})$$

(87)

of Eq.(80) provides a conservation law $D_i(C^i) = 0$, for the system of equations (81) and (82).

The conserved vector is given by

$$T^i = \xi^i L + W \left[ \frac{\partial L}{\partial u_j} - D_j \left( \frac{\partial L}{\partial u_{ij}} \right) + D_j D_k \left( \frac{\partial L}{\partial u_{ijk}} \right) - ... \right] + D_j D_k (W) \left[ \frac{\partial L}{\partial u_{ijk}} - ... \right] + ...$$

(88)

where $W$ (the characteristic function) and $L$ (the Lagrangian) are defined as

$$W = \eta - \xi^j u_j, \quad L = v F(x,u,u,(1),...,u,(r)), j = 1,...,n.$$  

(89)

Let us recall the symmetry algebra of Eq.(1) as given in [29] which is spanned by the five vector fields

$$X_1 = \partial/\partial t, X_1 = \partial/\partial x, X_3 = \partial/\partial y, X_4 = 3(t(\partial/\partial t) + x(\partial/\partial x) + y(\partial/\partial y) - 2u(\partial/\partial u),$$

$$X_5 = t(\partial/\partial x) + (1/2)(\partial/\partial u).$$

(90)

Now, the extended quantum Zakharov-Kuznetsov equation (1) together with its adjoint equation are given by

$$F = u_t + a u u_x + b(u_{xxx} + u_{yyy} + c(u_{xxy} + u_{xyy})) = 0$$

(91)

$$F^* = v_t + a u v_x + b(v_{xxx} + v_{yyy} + c(v_{xxy} + v_{xyy})) = 0$$

(92)

The third-order Lagrangian $L$ for the system of equations (81) and (82) is given by

$$L = v F = v(u_t + a u u_x + b(u_{xxx} + u_{yyy} + c(u_{xxy} + u_{xyy}))$$

(93)
Therefore, we must consider the following five cases:

(i) Considering the vector field \( X_1 = \partial/\partial t \), for Eq.(1). By using (88) and taking \( v = u \), due to self-adjointness of Eq.(1), the resultant conserved vector is given by

\[
C_1^t = au^2 u_x + buu_{xxx} + buu_yyy + cuu_{xyy} + cuu_{xyx},
\]

\[
C_1^x = -au^2 u_t - buu_{xx} - 2cuu_{uy} - cuu_{uyy} - buu_{yy} + buu_{xu} + cuu_{uy},
\]

\[
C_1^y = -cuu_{uy} - buu_{xy} - 2cuu_{xx} - cuu_{xy} - buu_{yy} + cuu_{uy} + buu_{yx}.
\]

(ii) Using the Lie point symmetry generator \( X_2 = \partial/\partial x \), for Eq.(1) and (88), the components of the conserved vector are given by

\[
C_2^t = -uu_x,
\]

\[
C_2^x = uu_t + buu_{yy} - cuu_{xyy} - cuu_{xyx} - cuu_{xy} - cuu_{uy} + cuu_{xx} + cuu_{xy} + cuu_{uy},
\]

\[
C_2^y = cuu_{xx} - cuu_{xx} - cuu_{xy} - cuu_{yy} + buu_{yy} - 2cuu_{xx} - buu_{xy}.
\]

(iii) Using \( X_3 = \partial/\partial y \), for Eq.(1) and (88), the component of the conserved vector are obtained and can be written as

\[
C_3^t = -uu_y,
\]

\[
C_3^x = -au^2 u_y - buu_{yy} - cuu_{xx} - cuu_{xy} + buu_{xy} + cuu_{yy} - buu_{yy} - 2cuu_{xx} - cuu_{xy} - cuu_{yy},
\]

\[
C_3^y = uu_t + au^2 u_x + buu_{xx} - cuu_{xxy} - cuu_{xyx} - cuu_{xy} - cuu_{yy} - cuu_{yy} + cuu_{xy} + cuu_{yy}.
\]

(iv) The Lie point symmetry generator

\[
X_4 = 3u(\partial/\partial t) + x(\partial/\partial x) + y(\partial/\partial y) - 2u(\partial/\partial u)
\]

For Eq.(1), leads to a conserved vector with component given by
\[ C_4^t = 3atu^2 u_x + 3buu_{xxx} + 3bvu_{yyy} + 3ctuu_{xyy} + 3ctuu_{xxy} - 2u^2 - uuu_x - uuu_y \]
\[ C_4^x = xuu_t + 3bu^2 u_x + 3cu^2 u_y - 2au^3 - cuu_{xy} - cuu_{yxy} - 6ctu_t u_x \]
\[ + cuu_{uxy} + buu_{uxy} - 12cuu_{xyy} - 6buu_{xxx} - 3btu_{x} u_x - ybu_{uu} u_x + cuu_{uxx} \]
\[- 6ctu_{u} u_y - 3ctu_{u} u_y - 3atuu_{u} u_t + 6cu_{u} y - ayu^2 u_y + \]
\[ 3btu_{u} u_x + 3ctu_{u} u_x + 3ctu_{u} u_y + xbu_{u} u_y - 3btuu_{xy} \]
\[- ybu_{u} u_y - 6ctuu_{xy} - xxu_{u} u_y - 2cyuu_{u} u_y - 3ctuu_{u} u_y - cyuu_{u} uy_y. \]
\[ C_4^y = yuu_t + ayu^2 u_x + 3cu^2 u_x + 3bu^2 u_y - 6cuu_{xx} - 3ctu_{u} u_t - cuu_{u} y_x + cuu_{u} u_x \]
\[- 12cuu_{xy} - cuu_{u} x_x - cuu_{u} y_y - 6ctu_{u} u_x + cuu_{u} u_y + xbu_{u} u_y - 6buu_{yy} - 3btu_{u} u_y \]
\[- xbu_{u} u_y + cuu_{u} x_x + 3ctu_{u} u_x + 6cu_{u} u_y + 3ctu_{u} u_y + 3ctu_{u} u_y + 3btu_{u} u_y \]
\[ + ybu_{uu} x_x - cyuu_{u} u_y - 3ctuu_{xy} - cxxu_{u} u_y - 6ctuu_{xy} - 2cyuu_{u} u_y \]
\[- 3btuu_{xy} + xbu_{u} u_y. \] (97)

(v) The last symmetry generator \( X_5 = t(\partial / \partial x) + (1/a)(\partial / \partial u) \), give the following conserved quantities
\[ C_5^t = (1/a)u - tu_x, \]
\[ C_5^x = u^2 + tuu_t + (b/a)u_{xx} + (2c/a)u_{xy} + (c/a)u_{yy} - ctu_{x} u_x - ctu_{x} u_y + ctu_{u} u_x \]
\[ + ctu_{y} u_y + btuu_{yy} - ctuu_{xxy}, \]
\[ C_5^y = (c/a)u_{xx} + (2c/a)u_{xy} + (b/a)u_{yy} - ctu_{x} u_y - btu_{u} u_x - 2ctuu_{xy} - btuu_{xyy}. \] (98)

6. Conclusion

In this article, we have obtained wider classes of exact traveling wave solutions of the extended qZK equation by using distinct types of the simplest equation methods. In addition, we have constructed conservation laws for the extended qZK equation via the new conservation theorem. The conservation laws can be associated with Lie symmetry generators of NPDEs to yield another conservation laws and more exact solutions of these nonlinear equations with the aid of double reduction theory and Kara and Mahomed results. These useful ideas are in urgent need to be conducted in a separate study in a future work.
**Competing interests**
Authors have declared that no competing interests exist.

**Authors’ contributions**
This research work was carried out in collaboration between the authors. Both authors have a good contribution to design the study and to perform the analysis of this research work. Both the authors read and approved the final manuscript.

**References**


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