Influences of Material and Structure Parameter on Everted Deformation for a Spherical Shell Composed of Incompressible Hyperelastic Materials

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Abstract
In this paper, the problem of finite deformation is examined for a thin-walled everted spherical shell composed of a class of neo-Hookean materials, and then it is described as a class of boundary value problems (BVPs) of a certain second-order nonlinear ordinary differential equation (ODE). The implicit solutions are solved. By using numerical example, the results reveal the thickness of the everted cylindrical tube increase with the increasing initial thickness, the influences of the dimensionless radial perturbation parameters on the inner radius is significant.

Keywords: Incompressible hyperelastic material, Spherical shell, Eversion, material and structure parameter

1. Introduction
The everted cylindrical tubes, which can be regarded as an ideal anti-collision energy absorbing component, have great applications in engineering design, aero-
space and many other fields of real life. While there are similarities for the corresponding spherical shell we think that there are sufficient differences to make an investigation of the spherical case worthwhile. The problem of inverting a spherical shell composed of isotropic hyperelastic material was originally considered by Armanni [1] for a particular compressible material model. A number of exact axially symmetric and fully three-dimensional deformations are given in Hill [2, 3] for a neo-Hookean elastic material. Chen and Haughton [4] proved that, if the material satisfied the Baker-Ericksen inequalities, no matter how the thickness was, there was a unique spherically symmetric inverted solution, and they gave a sufficient condition of cavity formation for the inverted spherical shell. Otherwise, they also found that thicker spherical shells could undergo a bifurcation on inversion. Haughton and Chen [5] applied the WKB method to the bifurcation analysis of inverted cylindrical and spherical shells composed of Varga material. The method is degenerate but they obtain explicit bifurcation criteria and compare with previous numerical approximations.

In this paper, the inversion problem of a spherical shell composed of a class of isotropic incompressible neo-Hookean materials is considered. The corresponding mathematical model can be treated as a BVP. The implicit solution is obtained by using the incompressible condition and the semi-inverse method. The effects of structure parameter and material parameters on the inverted spherical shells are discussed by numerical examples.

2. Mathematical model and Solutions

Here we consider the solution of the inversion problem for an incompressible hyperelastic thin-walled spherical shell.

Let \((R, \Theta, \Phi)\) and \((r, \theta, \phi)\) be the initial and deformed spherical polar coordinates of incompressible hyperelastic spherical shell, respectively. The undeformed spherical shell occupies the region

\[
A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq \Phi \leq 2\pi.
\]  

(1)

The deformed spherical shell occupies the region

\[
a \leq r \leq b, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq 2\pi,
\]  

(2)

where \(A\) and \(B\) are the inner and outer radii of the undeformed shell, \(a\) and \(b\) are the inner and outer radii of the deformed shell. Note that \(r(A) = b, r(B) = a\).

Since the deformation is spherical symmetry, the deformation configuration is given by

\[
r = r(R), \quad \theta = \pi - \Theta, \quad \phi = \Phi.
\]  

(3)

The principal stretches are given by

\[
\lambda_r = -r' = -\frac{dr}{dR}, \quad \lambda_\theta = \lambda_\phi = \frac{r}{R} = \hat{\lambda},
\]  

(4)

where \(\hat{\lambda} > 0\) is the axial stretch rate. Since the material is incompressible, we get
\[
\det F = \lambda_r \lambda_\theta \lambda_\phi = -\frac{r^2}{R^2} \frac{dr}{dR} = 1, \quad (5)
\]
then \( r^3 - a^3 = B^3 - R^3 \). Moreover, \( b^3 - a^3 = B^3 - A^3 \).

In this paper, we consider a class of incompressible materials obtained by introducing two nonlinear perturbation terms into the strain energy function as the homogeneous isotropic neo-Hookean materials have some radial imperfections, namely

\[
W = \frac{\mu}{2} [\lambda_r^2 + \lambda_\theta^2 + \lambda_\phi^2 - 3 + \alpha (\lambda_r^2 - 1)^2 + \beta (\lambda_r^2 - 1)^3], \quad (6)
\]
where \( \mu > 0 \) is the shear modulus for infinitesimal deformations, and \( \alpha, \beta \geq 0 \) are the dimensionless radial perturbation parameters which serve as measures of the degree of anisotropy of the materials. By setting \( \alpha = \beta = 0 \), Eq. (6) reduces to the well-known isotropic neo-Hookean materials model. Thus Eq. (6) may be viewed as representing a class of generalized incompressible neo-Hookean materials.

The principle components of the Cauchy stress tensor are given by

\[
\sigma_{rr} = \lambda_r \frac{\partial W}{\partial \lambda_r} - P = \mu \lambda_r^2 [1 + 2 \alpha (\lambda_r^2 - 1)^2 + 3 \beta (\lambda_r^2 - 1)^3] - P, \quad (7)
\]

\[
\sigma_{\theta\theta} = \lambda_\theta \frac{\partial W}{\partial \lambda_\theta} - P = \mu \lambda_\theta^2 - P, \quad (8)
\]

\[
\sigma_{\phi\phi} = \lambda_\phi \frac{\partial W}{\partial \lambda_\phi} - P = \mu \lambda_\phi^2 - P, \quad (9)
\]
where \( P \) is the hydrostatic pressure related to the incompressibility constraint. In the absence of body force, the equilibrium differential equation describing the spherically symmetric deformation can be reduced to

\[
\frac{d\sigma_{rr}}{dr} - 2 \frac{\sigma_{\theta\theta} - \sigma_{rr}}{r} = 0. \quad (10)
\]

Integrating Eq. (10) with respect to \( r \), we get

\[
\sigma_{rr}(r) = \sigma_{rr}(a) + 2 \int_a^r \frac{\sigma_{\theta\theta} - \sigma_{rr}}{r} dr. \quad (11)
\]
Assuming that the inner and outer surfaces are traction-free, the boundary conditions are obtained

\[
\sigma_{rr}(b) = \sigma_{rr}(a) = 0. \quad (12)
\]
Substituting Eq. (11) into Eq. (12),
\[
\int_{a}^{b} \frac{\sigma_{\theta \theta} - \sigma_{r r}}{r} dr = 0 ,
\]

Substituting Eqs. (7) and (8) into Eq. (12),

\[
\int_{a}^{b} (\lambda_{r}^{2} [1 + 2\alpha(\lambda_{r}^{2} - 1) + 3\beta(\lambda_{r}^{2} - 1)^{2}] - \lambda_{r}^{2} \lambda_{o}^{2}) \frac{dr}{r} = 0 ,
\]

then,

\[
\int_{a}^{b} (r^{2} [1 + 2\alpha(r^{2} - 1) + 3\beta(r^{2} - 1)^{2}] - \lambda_{r}^{2}) \frac{dr}{r} = 0 .
\]

For convenience, we introduce the following notations:

\[
\delta = \frac{A}{B} , \quad m = \frac{a}{B} , \quad M = \frac{b}{A} = \left(\frac{1 - \delta^{3} + m^{3}}{\delta^{3}}\right) ,
\]

then we have

\[
\lambda = \frac{r}{R} \left(\frac{B^{3} + a^{3}}{r^{3}} - 1\right) , \quad \frac{dr}{r} = \frac{r^{3} d\lambda}{\lambda^{3}} = \frac{B^{3} + a^{3}}{1 + \lambda^{3}} \frac{d\lambda}{\lambda} , \quad r' = -\frac{R^{2}}{r^{3}} = -\lambda^{-2} .
\]

Using the above notations, we rewrite Eq. (15) as

\[
\int_{0}^{M} \frac{3\beta}{(1 + \lambda^{3}) \lambda^{3}} d\lambda + \int_{0}^{M} \frac{2\alpha - 6\beta}{(1 + \lambda^{3}) \lambda^{3}} d\lambda - \int_{0}^{M} \frac{\lambda}{1 + \lambda^{3}} d\lambda + \int_{0}^{M} \frac{3\beta - 2\alpha + 1}{(1 + \lambda^{3}) \lambda^{3}} d\lambda = 0 ,
\]

Decomposing rational function into partial fraction, then we have the following equation

\[
(3\beta - 2\alpha + 1) \lambda^{-1} - (\alpha - 3\beta) \lambda^{-2} + \beta \lambda^{-3} - \frac{3\beta - 2\alpha + 1}{4} \lambda^{-4} + \frac{2\alpha - 6\beta}{5} \lambda^{-5} - \frac{\beta}{2} \lambda^{-6} - \frac{\alpha - 3\beta}{4} \lambda^{-8} + \frac{\beta}{3} \lambda^{-9} - \frac{\beta}{4} \lambda^{-12} + 3\beta \ln \lambda - \frac{3}{2} \beta \ln(1 + \lambda^{2} - \lambda) - \frac{\sqrt{3}}{3} (4\alpha - 9\beta) \arctan \frac{2\lambda - 1}{\sqrt{3}} |_{m} = 0 .
\]

Obviously, Eq. (19) is a nonlinear equation with respect to \(\delta\) and \(m\), describing the finite deformation of the everted spherical shell. From Eq. (19), we can get the relations among structure parameter, material parameters and inner radius of the everted shell.
3. Numerical Simulations

Figure 1. The relations among $\delta$, $a/B$, $b/A$ and $\eta$

Figure 2. The relations between $\delta$ and $a/B$ for different values of $\alpha$

Figure 3. The relations between $\delta$ and $a/B$ for different values of $\beta$
For the given values of $\alpha$ and $\beta$, as shown in Figure 1, the inner radius of the everted spherical shell increases with the increasing initial thickness $\delta$, and the thickness of the everted spherical shell increases with the increasing initial thickness $\delta$. The outer radius of the everted spherical shell decreases with the increasing initial thickness $\delta$. In Figure 2, for the given value of $\beta$, it is shown that the influences of the material constants $\alpha$ on the inner radius is significant, the increasing relation is linear approximately, if $\alpha = 0$. In Figure 3, for the given value of $\alpha$, the increasing relation increases with the increasing initial thickness $\beta$.

4. Conclusions

In this work, the finite deformation problem of the everted thin-walled spherical shell composed of a class of isotropic incompressible neo-Hookean materials is investigated. The relations among the structure parameter, the material parameters and the inner radius of the everted shell are obtained. Numerical simulations show that the inner radius of the everted spherical shell increases with the increasing initial thickness, the influences of the material constants $\alpha$ and $\beta$ on the inner radius are significant.

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References


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