Existence and Uniqueness Results for Three-Point Boundary Value Problems for Caputo-Hadamard-Type Fractional Differential Equations

Wafa Shammakh

Faculty of Science-AL Faisaliah Campus, King Abdulaziz University
Jeddah, Saudi Arabia

Abstract

In this paper, we study the existence and uniqueness results of solutions for three-point boundary value problems for Caputo-Hadamard fractional differential equations by using some fixed point theorems. Example illustrating the results are obtained.

Keywords: Existence, boundary value problems, Caputo-Hadamard derivatives, fractional differential equations

1. Introduction

The Hadamard fractional derivative is a kind of fractional derivatives due to Hadamard in 1892[5], this fractional derivative differs from the Riemann-Liouville and Caputo fractional derivatives [9] in the sense that the kernel of the integral contains a logarithmic function of arbitrary exponent. The Riemann-Liouville and Hadamard derivative has its own disadvantages as well, one of which is the fact that the derivative of a constant is not equal to zero in general.

Jarad et al. [6], modified the Hadamard fractional derivative into a more suitable one having physically interpretable initial conditions similar to the ones in the Caputo setting. For some recent work on the Hadamard fractional derivative and integral, see ([1, 2, 3, 8, 10, 11]). In particular, B. Ahmad et al. [4] studied fractional boundary value problem involving Hadamard-type fractional differential
inclusions and integral boundary conditions. In this paper, we study the existence and uniqueness of solutions for the three-point boundary value problems for Caputo-Hadamard fractional differential equations of the form

\[
\mathcal{C} \mathcal{D}^{\alpha} x(t) + f(t, x(t)) = 0, \quad 1 \leq t \leq e, \quad 1 < \alpha \leq 2, \\
x(1) = 0, \quad \mathcal{C} \mathcal{D}^{\alpha} x(e) = \gamma \mathcal{H} \mathcal{D} x(\xi),
\]

where \( \mathcal{C} \mathcal{D}^{\alpha} \) is the Caputo-Hadamard fractional derivative of order \( 1 < \alpha \leq 2 \), \( 0 \leq \gamma < 1 \), \( \xi \in (1, e) \), \( \mathcal{H} \mathcal{D} = t \frac{d}{dt} \) and \( f : [1, e] \to [0, \infty) \).

2. Preliminaries

In this section, we introduce some notations and definitions of Hadamard-type fractional calculus.

**Definition 2.1.** [7]
The Hadamard derivative of fractional order \( \alpha \) for a function \( g : [1, \infty) \to \mathbb{R} \) is defined as

\[
D^{\alpha} g(t) = \left( \frac{d}{dt} \right)^n \int_{1}^{t} \frac{(\log s)^{n-1}}{\Gamma(n-\alpha)} g(s) \frac{ds}{s}, \quad n - 1 < \alpha < n, \quad n = [\alpha] + 1,
\]

where \( [\alpha] \) denotes the integer part of the real number \( \alpha \) and \( \log(.) = \log_e(.) \).

**Definition 2.2.** [7]
The Hadamard fractional integral of order \( \alpha \in \mathbb{R^+} \) of a function \( g(x), \forall x > 0 \), is defined as

\[
I^{\alpha}_{a+} g(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \left( \log \frac{x}{t} \right)^{\alpha-1} \frac{g(t)}{t} dt, \\
I^{\alpha}_{b-} g(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} \left( \log \frac{t}{x} \right)^{\alpha-1} \frac{g(t)}{t} dt.
\]

**Definition 2.3.** [5, 6]
Let \( \Re(\alpha) \geq 0, n = [\Re(\alpha) + 1] \) and \( g \in AC^n_\Re[a, b], 0 < a < b < \infty \). Then

\[
\mathcal{C} D^{\alpha}_{a+} g(x) = D^{\alpha}_{a+} \left[ g(x) - \sum_{k=0}^{n-1} \frac{\delta^{k} g(a)}{k!} (\log \frac{t}{a})^k \right](x), \\
\mathcal{C} D^{\alpha}_{b-} g(x) = D^{\alpha}_{b-} \left[ g(x) - \sum_{k=0}^{n-1} \frac{(-1)^{k} \delta^{k} g(b)}{k!} (\log \frac{b}{t})^k \right](x).
\]

Here \( \Re(\alpha) \geq 0, n = [\Re(\alpha) + 1], 0 < a < b < \infty \) and
\[ g \in AC^\alpha_b[a,b] = \{ g : [a,b] \rightarrow \mathbb{C} : \delta^{(n-1)} g(x) \in AC[a,b], \delta = x \frac{d}{dx} \}. \]

In particular, if \( 0 < \Re(\alpha) < 1 \), then
\[ ^cD^\alpha_{a+} g(x) = D^\alpha_{a+} [g(x) - g(a)](x), \]
\[ ^cD^\alpha_{b-} g(x) = D^\alpha_{b-} [g(x) - g(b)](x). \]

**Theorem 2.4. [5, 6]**
Let \( \Re(\alpha) \geq 0, n = [\Re(\alpha) + 1] \text{and} \ g \in AC^\alpha_b[a,b], 0 < a < b < \infty. \) Then \(^cD^\alpha_{a+} g(x) \) and \(^cD^\alpha_{b-} g(x) \) exist everywhere on \([a,b] \) and

(a) if \( \alpha \notin \mathbb{N}_0, \)
\[ ^cD^\alpha_{a+} g(x) = \frac{1}{\Gamma(n-\alpha)} t^x \left( \log \frac{t}{a} \right)^{n-\alpha-1} \delta^n g(t) \frac{dt}{t} = I^\alpha_{a+} \delta^n g(x), \]
\[ ^cD^\alpha_{b-} g(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} b^b \left( \log \frac{b}{t} \right)^{n-\alpha-1} \delta^n g(t) \frac{dt}{t} = (-1)^n I^\alpha_{a+} \delta^n g(x), \]

(b) if \( \alpha = n \in \mathbb{N}_0, \)
\[ ^cD^\alpha_{a+} g(x) = \delta^n g(x), \quad ^cD^\alpha_{b-} g(x) = (-1)^n \delta^n g(x), \]

In particular,
\[ ^cD^\alpha_{a+} g(x) = ^cD^\alpha_{b-} g(x) = g(x). \]

**Lemma 2.5. [5, 6]**
Let \( \Re(\alpha) \geq 0, n = [\Re(\alpha) + 1] \text{and} \ g \in C[a,b]. \)
If \( \Re(\alpha) \neq 0 \text{ or} \alpha \in \mathbb{N}, \) then
\[ ^cD^\alpha_{a+} (I^\alpha_{a+} g)(x) = g(x), \quad ^cD^\alpha_{b-} (I^\alpha_{b-} g)(x) = g(x). \]

**Lemma 2.6. [5, 6]**
Let \( g \in AC^\alpha_b[a,b] \) or \( C^\alpha_b[a,b] \text{and} \alpha \in C, \) then
\[ I^\alpha_{a+} (^cD^\alpha_{a+} g)(x) = g(x) - \frac{1}{\Gamma(n-\alpha)} \delta^n g(a) \frac{dt}{t}, \]
\[ I^\alpha_{b-} (^cD^\alpha_{b-} g)(x) = g(x) - \frac{1}{\Gamma(n-\alpha)} \delta^n g(b) \frac{dt}{t}. \]

**Lemma 2.7.**
For \( 1 < \alpha \leq 2 \) and \( h \in C([1, e], \mathbb{R}), \) the unique solution of the problem
\[ ^cD^\alpha h x(t) + h(t) = 0, \quad 1 \leq t \leq e, \quad 1 < \alpha \leq 2, \]
\[ x(1) = 0, \quad ^cD^\alpha h x(e) = \gamma \int_1^e h(x) \ dx \]
\[ x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} h(s) \frac{ds}{s} + \frac{\log t}{t(1-\gamma)\Gamma(\alpha-1)}, \]

is given by
\[
\gamma \int_{1}^{e} \left( \log \frac{e}{s} \right)^{\alpha-2} h(s) \frac{ds}{s} - \int_{1}^{e} \left( \log \frac{e}{s} \right)^{\alpha-2} h(s) \frac{ds}{s}.
\]  

(2.2)

**Proof.**

In view of Lemma 2.6, the solution of the Hadamard differential equation (2.1) can be written as

\[
x(t) = -H^{\alpha} h(t) + c_0 + c_1 \log t,
\]

and

\[
\frac{d}{dt} D^{-1}_H x(t) = -H^{1-\alpha} h(t) + c_1.
\]

The boundary condition \(x(1) = 0\) implies that \(c_0 = 0\).

Thus

\[
x(t) = -H^{\alpha} h(t) + c_1 \log t.
\]

In view of the boundary condition \(\frac{d}{dt} D^{-1}_H x(e) = \gamma \frac{d}{dt} D^{-1}_H (\xi)\), we conclude that

\[
c_1 = \frac{1}{1-\gamma} \left[ H^{1-\alpha} h(e) - \gamma H^{1-\alpha} h(\xi) \right].
\]

Substituting the values of \(c_0\) and \(c_1\) in (2.3), we obtain (2.2).

### 3. Existence results

Let us denote by \(E = C([1, e], \mathbb{R})\) be the Banach space of all continuous functions from \([1, e]\) to \(\mathbb{R}\) endowed with the norm \(\|x\| = \sup_{t \in [1, e]} |x(t)|\).

By lemma 2.6, we obtain an operator \(\mathcal{F}: E \to E\) as

\[
(\mathcal{F}x)(t) = -\frac{1}{\Gamma(\alpha)} \int_{1}^{t} \left( \log \frac{e}{s} \right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} + \frac{\log t}{(1-\gamma)\Gamma(\alpha-1)} + \int_{1}^{e} \left( \log \frac{e}{s} \right)^{\alpha-2} f(s, x(s)) \frac{ds}{s} - \gamma \int_{1}^{\xi} \left( \log \frac{\xi}{s} \right)^{\alpha-2} f(s, x(s)) \frac{ds}{s}, t \in [1, e],
\]

(3.1)

It should be noticed that problem (1.1) has solutions if and only if the operator \(\mathcal{F}\) has fixed points. The first existence and uniqueness result is based on the Banach contraction principle.

**Theorem 3.1** Assume that there exists a constant \(L > 0\) such that

\[
|f(t, x(t) - f(t, y(t))| \leq L|x-y|, \forall t \in [1, e], L > 0, x, y \in \mathbb{R}.
\]

Then the problem (1.1) has a unique solution provided \(L\Psi < 1\), where

\[
\Psi = \frac{1}{\Gamma(\alpha+1)} + \frac{1}{(1-\gamma)\Gamma(\alpha)} \left[ 1 + \gamma (\log \xi)^{\alpha-1} \right].
\]

(3.3)

**Proof.** We set \(\sup_{t \in [1, e]} |f(s, 0)| = m < \infty\) and choose \(r \geq \frac{\Psi M}{1-\Psi L}\).
Now, we show that \( \mathcal{F}B_r \subseteq B_r \), where \( B_r = \{ x \in E : \| x \| \leq r \} \). For any \( x \in B_r \), we have

\[
\| (Fx)(t) \| = \sup_{t \in [1, e]} \left\{ -\frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} + \frac{\log t}{(1 - \gamma)\Gamma(\alpha - 1)} \int_1^e \left( \log \frac{e}{s} \right)^{\alpha-2} f(s, x(s)) \frac{ds}{s} \right. \\
- \gamma \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-2} f(s, x(s)) \frac{ds}{s} \left. \right\}
\]

\[
\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) \frac{ds}{s} + \frac{\log t}{(1 - \gamma)\Gamma(\alpha - 1)} \int_1^e \left( \log \frac{e}{s} \right)^{\alpha-2} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) \frac{ds}{s} \\
+ \gamma \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-2} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) \frac{ds}{s} \leq (Lr + M) \left[ \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} \frac{ds}{s} \right. \\
+ \frac{\log t}{(1 - \gamma)\Gamma(\alpha - 1)} \left( \int_1^e \left( \log \frac{e}{s} \right)^{\alpha-2} \frac{ds}{s} \right) \left. \right\] \\
\leq (Lr + M) \left[ \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{(1 - \gamma)\Gamma(\alpha)} (1 + \gamma (\log \xi)^{\alpha-1}) \right] \leq \Psi(Lr + M) \leq r.
\]

(3.4)

It follows that \( \mathcal{F}B_r \subseteq B_r \). For \( x, y \in E \) and for each \( t \in [1, e] \), we have

\[
| (Fx)(t) - (Fy)(t) | = \left| -\frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} (f(s, x(s)) - f(s, y(s))) \frac{ds}{s} + \frac{\log t}{(1 - \gamma)\Gamma(\alpha - 1)} \int_1^e \left( \log \frac{e}{s} \right)^{\alpha-2} (f(s, x(s)) - f(s, y(s))) \frac{ds}{s} \right. \\
- \gamma \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-2} (f(s, x(s)) - f(s, y(s))) \frac{ds}{s} \left. \right| \\
\leq (Lr + M) \left[ \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{(1 - \gamma)\Gamma(\alpha)} (1 + \gamma (\log \xi)^{\alpha-1}) \right] \leq \Psi(Lr + M) \leq r.
\]
\[
\begin{align*}
&\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - 1} \left| f(s, x(s)) - f(s, y(s)) \right| \frac{ds}{s} \\
&\quad + \frac{\log t}{(1 - \gamma) \Gamma(\alpha - 1)} \left[ \int_1^e \left( \log \frac{s}{e} \right)^{\alpha - 2} \left| f(s, x(s)) - f(s, y(s)) \right| \frac{ds}{s} \right] \\
&\quad + \gamma \int_1^\xi \left( \log \frac{s}{\xi} \right)^{\alpha - 2} \left| f(s, x(s)) - f(s, y(s)) \right| \frac{ds}{s} \\
&\leq L \left[ \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - 1} \left| x(s) - y(s) \right| \frac{ds}{s} \\
&\quad + \frac{\log t}{(1 - \gamma) \Gamma(\alpha - 1)} \left[ \int_1^e \left( \log \frac{s}{e} \right)^{\alpha - 2} \left| x(s) - y(s) \right| \frac{ds}{s} \right] \\
&\quad + \gamma \int_1^\xi \left( \log \frac{s}{\xi} \right)^{\alpha - 2} \left| x(s) - y(s) \right| \frac{ds}{s} \right] \\
&\leq L \Psi \| x - y \|.
\end{align*}
\]

Hence it follows that \( \|(Fx)(t) - (Fy)(t)\| \leq L \Psi \| x - y \| \), where \( L \Psi < 1 \). Therefore \( F \) is a contraction. Hence by the contraction mapping principle the problem (1.1) has a uniqueness solution.

**Theorem 3.2** (Leray- Schauder alternative)
Let \( X \) be a Banach space. Assume that \( T: X \to X \) is completely continuous operator and the set \( V = \{ \mu \omega X \mid \mu X = \mu T u, 0 < \mu < 1 \} \) is bounded. Then \( T \) has a fixed point in \( X \).

**Theorem 3.3** Assume that there exists a positive constant \( L_1 \) such that \( |f(t, x)| \leq L_1 \) for \( t \in [1, e], x \in \mathbb{R} \). Then the problem (1.1) has at least one solution.

**Proof.** First, we show that the operator \( F \) is completely continuous. Note that the operator \( F \) is continuous in view of the continuity of \( f \).
Let \( F \subset F \) be a bounded set. By the assumption that \( |f(t, x)| \leq L_1 \), for \( x \in F \), we have
\[ |(Fx)(t)| = \left| -\frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} \right. \]
\[ \quad + \frac{1}{1 - \gamma} \int_1^e \left( \log \frac{e}{s} \right)^{\alpha-2} f(s, x(s)) \frac{ds}{s} \left. - \gamma \int_1^\xi \left( \log \frac{\xi}{s} \right)^{\alpha-2} f(s, x(s)) \frac{ds}{s} \right| \]
\[ \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} \left| f(s, x(s)) \right| \frac{ds}{s} \]
\[ \quad + \frac{1}{(1 - \gamma)\Gamma(\alpha - 1)} \left[ \int_1^e \left( \log \frac{e}{s} \right)^{\alpha-2} \frac{ds}{s} + \gamma \int_1^\xi \left( \log \frac{\xi}{s} \right)^{\alpha-2} \frac{ds}{s} \right] \]
\[ \leq L_1 \left[ \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{(1 - \gamma)\Gamma(\alpha)} \left( 1 + \gamma(\log \xi)^{\alpha-1} \right) \right] = L_2, \quad (3.6) \]

which implies that \( \| (Fx)(t) \| \leq L_2 \). Further, we find that

\[ |D_n^\alpha (Fx)(t)| = \left| -\frac{1}{\Gamma(\alpha - 1)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-2} f(s, x(s)) \frac{ds}{s} \right. \]
\[ \quad + \frac{1}{(1 - \gamma)\Gamma(\alpha - 1)} \left[ \int_1^e \left( \log \frac{e}{s} \right)^{\alpha-2} f(s, x(s)) \frac{ds}{s} \right. \left. - \gamma \int_1^\xi \left( \log \frac{\xi}{s} \right)^{\alpha-2} f(s, x(s)) \frac{ds}{s} \right| \]
\[ \leq \frac{1}{\Gamma(\alpha - 1)} \int_{1}^{t} \left( \log \frac{t}{s} \right)^{\alpha-2} \left| f(s, x(s)) \right| \frac{ds}{s} \]

\[ + \frac{1}{(1 - \gamma)\Gamma(\alpha - 1)} \left[ \int_{1}^{e} \left( \log \frac{e}{s} \right)^{\alpha-2} \left| f(s, x(s)) \right| \frac{ds}{s} \right] \]

\[ + \gamma \int_{1}^{e} \left( \log \frac{e}{s} \right)^{\alpha-2} \left| f(s, x(s)) \right| \frac{ds}{s} \]

\[ \leq L_1 \left[ \frac{1}{\Gamma(\alpha)} + \frac{1}{(1 - \gamma)\Gamma(\alpha)} \right] = L_3. \quad (3.7) \]

Hence, for \( t_1, t_2 \in [1, e] \), we have

\[ \| (Fx)(t_2) - (Fx)(t_1) \| \leq \int_{t_1}^{t_2} \mu \mathcal{D}(Fx)(s) \frac{ds}{s} \leq L_3(t_2 - t_1). \]

This implies that \( F \) is continuous on \([1, e]\). Thus, by the Arzela-Ascoli theorem, the operator \( F: E \to E \) is completely continuous.

Next, we consider the set \( V = \{ u \in E \mid x = \mu Fx, 0 < \mu < 1 \} \), and show that the set \( V \) is bounded. Let \( x \in V \), then \( x = \mu Fx, 0 < \mu < 1 \). For any \( t \in [1, e] \), we have

\[ |x(t)| = \mu |(Fx)(t)| \]

\[ \leq \frac{1}{\Gamma(\alpha)} \int_{1}^{t} \left( \log \frac{t}{s} \right)^{\alpha-1} \left| f(s, x(s)) \right| \frac{ds}{s} \]

\[ + \frac{1}{(1 - \gamma)\Gamma(\alpha - 1)} \left[ \int_{1}^{e} \left( \log \frac{e}{s} \right)^{\alpha-2} \left| f(s, x(s)) \right| \frac{ds}{s} \right] \]

\[ + \gamma \int_{1}^{e} \left( \log \frac{e}{s} \right)^{\alpha-2} \left| f(s, x(s)) \right| \frac{ds}{s} \]

\[ \leq L_1 \left[ \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{(1 - \gamma)\Gamma(\alpha)} \right] = M_1. \quad (3.8) \]

Thus, \( \| x(t) \| \leq M_1 \) for any \( t \in [1, e] \). So, that set \( V \) is bounded. Thus, by the conclusion of Theorem 3.2, the operator \( F \) has at least one fixed point, which implies that the problem (1.1) has at least one solution.

In the next we give an existence result for boundary value problem (1.1) by using the following fixed point theorem.
**Theorem 3.4.** Let $X$ be a Banach space. Assume that $Ω$ is an open bounded subset of $X$ with $θ ∈ Ω$ and let $T: \bar{Ω} → X$ be a completely continuous operator such that
\[
||Tu|| ≤ ||u||, \quad \forall u ∈ ∂U.
\] (3.9)
Then $T$ has a fixed point in $Ω$.

**Theorem 3.5.** Let there exist a small positive number $\bar{r}$ such that $|f(t,x)| ≤ p|x|$ for $0 < |x| < \bar{r}$, with $0 < p < \frac{1}{\Psi}$, where $Ψ$ is defined by (3.3). Then the problem (1.1) has at least one solution.

**Proof.** We define $ℬ_{\bar{r}} = \{x ∈ E: ||x|| ≤ \bar{r}\}$ and take $x ∈ E$ such that $||x|| = \bar{r}$, that is, $x ∈ ∂ℬ_{\bar{r}}$. As before, it can be shown that $F$ is completely continuous and
\[
||\mathcal{F}(x)(t)|| = \text{Sup}_{t∈[1,e]} \left\{ -\frac{1}{\Gamma(\alpha)} \int_{1}^{t} \left( \log \frac{t}{s} \right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} \right. \right.
+ \frac{\log t}{(1-γ)\Gamma(\alpha-1)} \left[ \int_{1}^{e} \left( \log \frac{s}{\xi} \right)^{\alpha-2} f(s, x(s)) \frac{ds}{s} \right. \right.
- \left. \gamma \int_{1}^{\xi} \left( \log \frac{s}{\xi} \right)^{\alpha-2} f(s, x(s)) \frac{ds}{s} \right\} \leq Ψp||x||.
\] (3.10)

The above result implies that $||\mathcal{F}(x)(t)|| ≤ ||x||, x ∈ ∂ℬ_{\bar{r}}$. Therefore, by Theorem (3.4), $F$ has at least one fixed point. So that the problem (1.1) has at least one solution on $[1, e]$.

**Example.**
Consider the following BVP for Hadamard fractional differential equation
\[
\frac{c}{\bar{H}}D^{3/2}x(t) + \frac{1}{64}(\sqrt{t} + \log t)(\frac{x^2}{|x| + 1} + \frac{\sqrt{|x|}}{2(1 + \sqrt{|x|})} + \frac{1}{2}) = 0, \quad 1 ≤ t ≤ e,
\]
\[
x(1) = 0, \quad \frac{c}{\bar{H}}Dx(e) = γ\frac{c}{\bar{H}}Dx(\xi),
\]
Here $α = 3/2$, $ξ = 1.7$, $γ = 0.25$. Clearly $|f(t,x)| ≤ \frac{1}{64}(\sqrt{t} + 1)(|x| + 1)$ and $|f(t,x) - f(t,y)| ≤ \frac{1}{64}(\sqrt{t} + 1)|x - y| ≤ \frac{1}{32}|x - y|$.

Hence, by Theorem 3.1, the BVP (1.1) has a unique solution on $[1, e]$ with $L = \frac{1}{32} = 0.0313$.

We can show that $Ψ = 2.513$, $LΨ = 0.079 < 1$. 
References


**Received: February 28, 2016; Published: April 9, 2016**