Weak Solutions of a Functional Integral Equation in Reflexive Banach Space

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Abstract

In this paper we study the existence of weak continuous solutions of the nonlinear integral equation

\[ x(t) = a(t) + \int_0^t f(t, g(s, x(m(s))))ds, \quad t \in [0, T] \]

under two different groups of assumptions imposed on the two functions \( f \) and \( g \).

Keywords: Weak solutions, Integral equations, Coupled system, Fixed point theorem

1 Introduction and Preliminaries

Let \( I = [0, T] \), and let \( L^1(I) \) be the class of all Lebesgue integrable functions defined on the interval \( I \). Let \( E \) be a reflexive Banach space with norm \( \| \| \) and dual \( E^* \).
Denote \( C[I, E] \) the Banach space of strongly continuous functions \( x : I \to E \) with sup-norm.
The existence of weak solutions of the integral and differential equations have
been extensively investigated by a number of authors and there are many interesting results concerning this problems (see\cite{1}-\cite{3}, \cite{5}-\cite{6} and \cite{8}-\cite{11}).

Let \( m : [0, T] \rightarrow [0, T] \), \( m(t) \leq t \) is continuous. In this paper, we study the existence of at least one weak continuous solution \( x \in C[I, E] \) of the nonlinear integral equation

\[
 x(t) = a(t) + \int_0^t f(t, g(s, x(m(s)))) ds, \quad t \in [0, T] 
\]

under a set of several suitable assumptions on the two function \( f \) and \( g \). We can reformulate the integral equation (1) according to these several assumptions and studying two cases of such problem.

Now we present some auxiliary results that will be needed in this work.

Let \( E \) be a Banach space and let \( x : I \rightarrow E \), then

(1) \( x(.) \) is said to be weakly continuous (measurable) at \( t_0 \in I \) if for every \( \phi \in E^* \), \( \phi(x(.)) \) is continuous (measurable) at \( t_0 \).

(2) A function \( h : E \rightarrow E \) is said to be sequentially continuous if \( h \) maps weakly convergent sequence in \( E \) to weakly convergent sequence in \( E \).

If \( x \) is weakly continuous on \( I \), then \( x \) is strongly measurable and hence weakly measurable (see\cite{4} and\cite{7}). Note that in reflexive Banach spaces weakly measurable functions are pettis integrable (see\cite{7} and \cite{11} for the definition) if and only if \( \phi(x(.)) \) is Lebesgue integrable on \( I \) for every \( \phi \in E^* \).

Now we state a fixed point theorem and some propositions which will be used in the sequel (see\cite{9}).

**Theorem 1.1 "O’Regan fixed point theorem"**

Let \( E \) be a Banach space and let \( Q \) be a nonempty, bounded, closed and convex subset of the space \( (C[0, T], E) \) and let \( A : Q \rightarrow Q \) be a weakly sequentially continuous and assume that \( AQ(t) \) is relatively weakly compact in \( E \) for each \( t \in [0, T] \). Then \( A \) has a fixed point in the set \( Q \).

**proposition 1** A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.

**proposition 2** Let \( E \) be a normed space with \( y \neq 0 \). Then there exists a \( \phi \in E^* \) with \( \|\phi\| = 1 \) and \( \|y\| = \phi(y) \).

## 2 Existence of weak continuous solutions

### 2.1 Coupled system approach

Let \( f, g : I \times E \rightarrow E \) satisfy the following assumptions:

(i) \( f(t, .) \) is weakly sequentially continuous for each \( t \in I \).
(ii) \( f(., x) \) is continuous on \( I \) for every \( x \in E \).
(iii) There exists a constant $M_1 > 0$ such that $\|f(t,x)\| \leq M_1$.
(iv) $g(.,x)$ is continuous on $I$ for every $x \in E$.
(v) $g$ satisfy Lipschitz condition
\[
\|g(t,u) - g(t,v)\| \leq L\|u - v\| \quad \forall (t,u), (t,v) \in I \times E.
\]
(vi) There exists a constant $M_2 > 0$ such that $M_2 = \sup |g(t,0)|$
(vii) $a \in C[I, E]$.

Now, let $y(t) = g(t,x(m(t)))$, $t \in [0, T]$, then the nonlinear integral equation (1) can written in the form of the coupled system of integral and functional equations

\[
\begin{align*}
x(t) &= a(t) + \int_0^t f(t,y(s))ds, \quad t \in [0, T] \quad (2) \\
y(t) &= g(t,x(m(t))), \quad t \in [0, T] \quad (3)
\end{align*}
\]

**Definition 2.1** Let $X$ be the class of all ordered pairs $(u,v)$, $u,v \in C[I, E]$, with norm $\|(u,v)\| = \|u\| + \|v\|$.

**Definition 2.2** By a weak solution of the coupled system (2)-(3) we mean that a pair of functions $(x,y) \in X$, $x,y \in C[I, E]$ such that
\[
\phi(x(t)) = \phi(a(t)) + \int_0^t \phi(f(t,y(s)))ds, \quad t \in [0, T]
\]
\[
\phi(y(t)) = \phi(g(t,x(m(t)))), \quad t \in [0, T]
\]
for all $\phi \in E^*$.

Now for the existence of a weak continuous solution of (2)-(3) we have the following theorem

**Theorem 2.2** Let the assumptions (i)-(vii) be satisfied. Then the coupled system of the nonlinear integral and functional equations (2)-(3) has at least one weak continuous solution $U = (x,y) \in X$, $x,y \in C[I, E]$.

**Proof.** Let
\[
U(t) = (x(t), y(t)) = (a(t) + \int_0^t f(t,y(s))ds, g(t,x(m(t))), \quad t \in [0, T] \quad (4)
\]

Let $A$ be any operator defined by
\[
AU(t) = A(x(t), y(t)) = (A_1y(t), A_2x(t))
\]
where
\[
A_1y(t) = a(t) + \int_0^t f(t,y(s))ds, \quad t \in [0, T]
\]
\[
A_2x(t) = g(t,x(m(t))), \quad t \in [0, T].
\]
Let the set \( Q_r \) defined by
\[
Q_r = \{ U = (x, y) \in X : x, y \in C[I, E], \quad \|U\| \leq r \}.
\]
Let \( U \in Q_r \) be an arbitrary ordered pair, then we have from proposition 2
\[
\|A_1 y(t)\| = \phi(A_1 y(t)) = \phi(a(t)) + \int_0^t \phi(f(t, y(s)))ds
\]
\[
= \|a\| + \int_0^t \|f(t, y(s))\|ds \leq \|a\| + M_1 \int_0^t ds \leq \|a\| + M_1 T
\]
and
\[
\|A_2 x(t)\| = \|g(t, x(m(t)))\| \leq \|g(t, 0)\| + L\|x\| \leq M_2 + L\|x\|.
\]
Now
\[
\|AU(t)\| = \|A_1 y(t)\| + \|A_2 x(t)\|
\]
\[
\leq \|a\| + M_1 T + \{ M_2 + L\|x\| \} = r.
\]
Then \( \|AU\| \leq r \). Hence, \( AU \in Q_r \), which proves that \( AQ_r \subset Q_r \), i.e. \( A : Q_r \to Q_r \), and the class of functions \( \{ AQ_r \} \) is uniformly bounded.

Now \( Q_r \) is nonempty, closed, convex and uniformly bounded.
As a consequence of proposition 1, then \( \{ AQ_r \} \) is relatively weakly compact.
Now, we shall prove that \( A : X \to X \).
Let \( t_1, t_2 \in I, \quad t_1 < t_2 \) (without loss of generality assume that \( AU(t_2) - AU(t_1) \neq 0 \), then
\[
A_1 y(t_2) - A_1 y(t_1) = (a(t_2) - a(t_1)) + \int_0^{t_2} f(t_2, y(s))ds - \int_0^{t_1} f(t_1, y(s))ds
\]
\[
\leq (a(t_2) - a(t_1)) + \int_0^{t_1} f(t_2, y(s))ds + \int_0^{t_2} f(t_2, y(s))ds
\]
\[
- \int_0^{t_1} f(t_1, y(s))ds
\]
\[
\leq (a(t_2) - a(t_1)) + \int_0^{t_1} [f(t_2, y(s)) - f(t_1, y(s))]ds
\]
\[
+ \int_{t_1}^{t_2} f(t_2, y(s))ds.
\]
Therefore as a consequence of proposition 2, we obtain
\[
\|A_1 y(t_2) - A_1 y(t_1)\| = \phi(A_1 y(t_2) - A_1 y(t_1))
\]
\[
\leq \phi(a(t_2) - a(t_1)) + \int_0^{t_1} \phi(f(t_2, y(s)) - f(t_1, y(s)))ds
\]
\[
+ \int_{t_1}^{t_2} \phi(f(t_2, y(s)))ds
\]
Therefore,

\[
\|A\| = \left\|A_1y(t_2), A_2x(t_2) - (A_1y(t_1), A_2x(t_1))\right\| \\
\leq \left\|((A_1y(t_2) - A_1y(t_1)), (A_2x(t_2) - A_2x(t_1)))\right\| \\
= \left\|A_1y(t_2) - A_1y(t_1)\right\| + \left\|A_2x(t_2) - A_2x(t_1)\right\| \\
\leq \|a(t_2) - a(t_1)\| + \int_{t_1}^{t_1} \|f(t_2, y(s)) - f(t_1, y(s))\|ds \\
+ M_1|t_2 - t_1| \\
+ \|g(t_2, x(m(t_2))) - g(t_1, x(m(t_2)))\| + L\|x(m(t_2)) - x(m(t_1))\|
\]

which proves that \( A : X \to X \).

Finally, we want to prove that \( A \) is weakly sequentially continuous.

Let \( \{U_n\} \) be a sequence in \( Q_r \) converges weakly to \( U \) \( \forall t \in I \), then we have the two sequences \( \{y_n\}, \{x_n\} \) converges weakly to \( y, x \), respectively, i.e. \( y_n(t) \to y, \ x_n(t) \to x, \ \forall t \in I \) weakly.

Since \( f(t, y(t)) \) and \( g(t, x(m(t))) \) are weakly sequentially continuous in \( y \) and \( x \), then \( f(t, y_n(t)) \) and \( g(t, x_n(m(t))) \) converges weakly to \( f(t, y((t))) \) and \( g(t, x(m(t))) \) respectively.

Thus \( \phi(f(t, y_n(t))) \) and \( \phi(g(t, x_n(m(t)))) \) converges strongly to \( f(t, y(t)) \) and \( g(t, x(m(t))) \) respectively.

By applying Lebesgue dominated convergence theorem for Pettis integral, then we get

\[
\phi(\int_0^t f(t, y_n(s))ds) = \int_0^t \phi(f(t, y_n(s)))ds \\
\to \int_0^t \|f(t, y(s))\|ds, \ \forall \phi \in E^*, \ t \in I
\]

i.e. \( \phi(A_1y_n(t)) \to \phi(A_1y(t)) \), and then

\[
\|A_1y_n(t)\| \to \|A_1y(t)\|
\]
and
\[
\phi(A_2x_n(t)) = \phi(g(t, x_n(m(t)))) \\
\rightarrow \|g(t, x(m(t)))\|, \quad \forall \phi \in E^*, \ t \in I
\]
i.e. \(\phi(A_2x_n(t)) \rightarrow \phi(A_2x(t))\), and then
\[
\|A_2x_n(t)\| \rightarrow \|A_2x(t)\|
\]
Therefore,
\[
\|AU_n(t)\| = \|A(x_n(t), y_n(t))\| \\
= \|(A_1y_n(t), A_2x_n(t))\| \\
= \|A_1y_n(t)\| + \|A_2x_n(t)\| \\
\rightarrow \|A_1y(t)\| + \|A_2x(t)\| \\
\rightarrow \|(A_1y(t), A_2x(t))\| \\
\rightarrow \|AU(t)\|
\]
Hence, \(A\) is weakly sequentially continuous (i.e. \(AU_n(t) \rightarrow AU(t), \ \forall t \in I\) weakly).
Since all conditions of O’Regan theorem are satisfied, then the operator \(A\) has
at least one fixed point \(U \in Q_r\), and hence the system (2)-(3) and consequently
integral equations (1) has a weak solution.

\section*{2.2 Functional integral equation approach}

Assume that the functions \(f\) and \(g\) satisfies the following assumptions:
1- \(f(t, .)\) is weakly sequentially continuous for each \(t \in I\).
2- \(f(., x)\) is weakly measurable on \(I\).
3- There exists an integrable function \(k_1 \in L^1(I), \ \int_0^t |k_1(s)|ds < M_1\) and a constant \(b_1 > 0\) such that \(\|f(t, x)\| \leq k_1(t) + b_1\|x\|\).
4- \(g(t, .)\) is sequentially continuous in \(x \in E\) for each \(t \in I\).
5- \(g(., x)\) is weakly measurable on \(I\).
6- There exists an integrable function \(k_2 \in L^1(I), \ \int_0^t |k_2(s)|ds < M_2\) and a constant \(b_2 > 0\) such that \(\|g(t, x)\| \leq k_2(t) + b_2\|x\|\).
7- \(a \in C[I, E]\).

Now we have to solve the nonlinear functional integral equation (1) getting
the weak continuous solution \(x \in C[I, E]\) as follows.

\textbf{Definition 2.3} By a weak solutions of the integral equation (1) we mean that
a weakly continuous function \(x \in C[I, E]\) such that
\[
\phi(x(t)) = \phi(a(t)) + \int_0^t \phi(f(t, g(s, x(m(s))))))ds, \quad t \in [0, T]
\]
for all \(\phi \in E^*\).
Now for the existence of a weak continuous solution of (1) we have the following theorem

**Theorem 2.3** Let the assumptions 1-7 be satisfied. Then the nonlinear functional integral equations (1) has at least one weak continuous solution $x \in C[I, E]$.

**Proof.** Let $A$ be any operator defined by

$$Ax(t) = a(t) + \int_0^t (f(t, g(s, x(m(s)))) ds, \ t \in [0, T].$$

Define the set $Q_r$ as follows

$$Q_r = \{x \in C[I, E] : \|x\| \leq r\}.$$

Let $x \in Q_r$, then we have from proposition 2

$$\|Ax(t)\| = \phi(Ax(t)) = \phi(a(t) + \int_0^t f(t, g(s, x(m(s)))) ds$$

$$= \phi(a(t)) + \int_0^t \phi(f(t, g(s, x(m(s)))) ds$$

$$= \|a\| + \int_0^t \|f(t, g(s, x(m(s))))\| ds$$

$$\leq \|a\| + \int_0^t |k1(s)| ds + b1 \int_0^t \|g(s, x(m(s)))\| ds$$

$$\leq \|a\| + \int_0^t |k1(s)| ds + b1 \{\int_0^t (|k2(s)| + b2 \|x(m(s))\|) ds\}$$

$$\leq \|a\| + M_1 + b1\{M_2 + b2T\} \leq r$$

where, $r = \frac{\|a\| + M_1 + b1M_2}{1-b1b2T}$.

Hence, $Ax \in Q_r$, which proves that $AQ_r \subset Q_r$, i.e. $A : Q_r \rightarrow Q_r$, and the class of functions $\{AQ_r\}$ is uniformly bounded.

Now $Q_r$ is nonempty, closed, convex and uniformly bounded.

As a consequence of proposition 1, then $\{AQ_r\}$ is relatively weakly compact.

Now, we shall prove that $A : C[I, E] \rightarrow C[I, E]$. Let $t_1, t_2 \in I, \ t_1 < t_2$ (without loss of generality assume that $Ax(t_2) - Ax(t_1) \neq 0$), then

$$Ax(t_2) - Ax(t_1) = (a(t_2) - a(t_1)) + \int_0^{t_2} f(t_2, g(s, x(m(s)))) ds - \int_0^{t_1} f(t_1, g(s, x(m(s)))) ds$$

$$\leq (a(t_2) - a(t_1)) + \int_0^{t_1} f(t_2, g(s, x(m(s)))) ds$$

$$+ \int_0^{t_2} f(t_2, g(s, x(s))) - \int_0^{t_1} f(t_1, g(s, x(m(s)))) ds$$

$$\leq (a(t_2) - a(t_1)) + \int_0^{t_1} [f(t_2, g(s, x(m(s)))) - f(t_1, g(s, x(m(s))))] ds$$
+ \int_{t_1}^{t_2} f(t_2, g(s, x(m(s))))
\leq (a(t_2) - a(t_1)) + \int_{t_1}^{t_2} [f(t_2, g(s, x(m(s)))) - f(t_1, g(s, x(m(s))))] ds
+ \int_{t_1}^{t_2} f(t_2, g(s, x(m(s))))

Therefore as a consequence of proposition 2, we obtain

\| Ax(t_2) - Ax(t_1) \| = \phi(Ax(t_2) - Ax(t_1))
\leq \phi[a(t_2) - a(t_1)] + \int_{t_1}^{t_2} \phi[f(t_2, g(s, x(m(s)))) - f(t_1, g(s, x(m(s))))] ds
+ \int_{t_1}^{t_2} \phi[f(t_2, g(s, x(m(s))))]
\leq \|a(t_2) - a(t_1)\| + \int_{t_1}^{t_2} \|f(t_2, g(s, x(m(s)))) - f(t_1, g(s, x(m(s))))\| ds
+ \int_{t_1}^{t_2} k(s) ds + b \int_{t_1}^{t_2} \|g(s, x(m(s)))\| ds.
\leq \|a(t_2) - a(t_1)\| + \int_{t_1}^{t_2} \|f(t_2, g(s, x(m(s)))) - f(t_1, g(s, x(m(s))))\| ds
+ M_1|t_2 - t_1| + bM_2|t_2 - t_1|

which proves that \( A : C[T, E] \rightarrow C[I, E] \).

Finally, we want to prove that \( A \) is weakly sequentially continuous.

Let \( \{x_n\} \) be a sequence in \( Q_r \) converges weakly to \( x \) \( \forall t \in I \), i.e. \( x_n(t) \rightarrow x, \forall t \in I \) weakly.

Since \( f(t, g(t, x(m(t)))) \) is weakly sequentially continuous in \( g \), and \( g(t, x(m(t))) \) is weakly sequentially continuous in \( x \), then \( g(t, x_n(m(t))) \) converges weakly to \( g(t, x(m(t))) \) and consequently \( f(t, g(t, x_n(m(t)))) \) converges weakly to \( f(t, g(t, x(m(t)))) \).

Thus \( \phi(f(t, g(t, x_n(m(t)))))) \) converges strongly to \( \phi(f(t, g(t, x(m(t)))))) \).

By applying Lebesgue dominated convergence theorem for Pettis integral, then we get

\( \phi(\int_0^t f(t, g(s, x_n(m(s)))) ds) = \int_0^t \phi(f(t, g(s, x_n(m(s)))) ds \rightarrow \int_0^t \|f(t, g(s, x(m(s))))\| ds, \forall \phi \in E^*, t \in I \)

i.e. \( \phi(Ax_n(t)) \rightarrow \phi(Ax(t)) \), and then

\( \|Ax_n(t)\| \rightarrow \|Ax(t)\| \).

Hence, \( A \) is weakly sequentially continuous (i.e. \( Ax_n(t) \rightarrow Ax(t), \forall t \in I \) weakly).

Since all conditions of O’Regan theorem are satisfied, then the operator \( A \) has at least one fixed point \( x \in Q_r \), and hence the integral equation (1) has at least one weak solution \( x \in C[I, E] \).
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