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$q$-Harmonic Mappings for which
Analytic Part is $q$-Convex Functions

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Abstract

In the present article we will examine the subclass of planar harmonic mappings. Let $h(z)$ and $g(z)$ are analytic functions in the open unit disc $D = \{ z \mid |z| < 1 \}$ and having the power series representation $h(z) = z + a_2z^2 + \ldots$ and $g(z) = b_1z + b_2z^2 + \ldots$. If $f = h(z) + \overline{g(z)}$ be the solution of the non-linear partial differential equation $w_q(z) = \left( \frac{D_qg(z)}{D_qh(z)} \right) = \frac{f\bar{z}}{f_z}$ with $|w_q(z)| < 1$, $h(z)$ $q$-convex function, then this class is called $q$-harmonic mappings for which analytic part is $q$-convex functions and the class of such functions is denoted by $SHC(q)$, where $D_qh(z) = \frac{h(z) - h(qz)}{1 - qz} = f_z$, $D_qg(z) = \frac{g(z) - g(qz)}{1 - qz} = \bar{f}_\bar{z}$, $q \in (0, 1)$.

Mathematics Subject Classification: 3045

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1 Introduction

A planar harmonic mapping in the open unit disc $\mathbb{D}$ is a complex valued function $f$ which maps $\mathbb{D}$ onto the some planar domain $f(\mathbb{D})$. Since $\mathbb{D}$ is simply connected domain the mapping $f$ has a canonical representation $f = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic functions in $\mathbb{D}$ and have the following power series.

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

where $a_n, b_n \in \mathbb{C}$, $n = 0, 1, 2, \ldots$ and usual we call $h(z)$ the analytic part of $f$ and $g(z)$ is co-analytic part of $f$. An elegant and complete account theory of harmonic mappings is given Duren’s monograph [3]. Lewy [6] proved in 1936 that the harmonic function $f$ is locally univalent in $\mathbb{D}$, if and only if its Jacobian $J_f(z) = |h'(z)|^2 - |g'(z)|^2$ is different from zero in $\mathbb{D}$. In view of this result locally univalent harmonic mapping in the open unit disc $\mathbb{D}$ are either sense-preserving if $|h'(z)| > |g'(z)|$ in $\mathbb{D}$ or sense-reserving if $|g'(z)| > |h'(z)|$ in $\mathbb{D}$.

Through this paper we will restrict ourselves to the study of sense-preserving harmonic mappings. We also note that $f = h(z) + \overline{g(z)}$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z)$ does not vanish in $\mathbb{D}$ and the second dilatation $w(z) = \frac{g'(z)}{h'(z)}$ has the property $|w(z)| < 1$ for every $z \in \mathbb{D}$. (In other words $f$ is the solution of non-linear partial differential equation $w(z) = \frac{f_z}{h_z}$ with $|w(z)| < 1$.) Therefore the class of all sense-preserving harmonic mappings in the open unit disc with $a_0 = b_0 = 0$, $a_1 = 1$ will be denoted by $S_H$. Thus $S_H$ contains standart class of univalent functions. The family of all mappings $f \in S_H$ with the additional property $g'(0) = 0$, i.e. $b_1 = 0$ is denoted by $S_H^0$. Hence it is clear that $S \subset S_H^0 \subset S_H$.

In this paragraph of this paper we will give the concept of the $q$-calculus. If $q \in (0, 1)$ fixed, a subset $\mathbb{B}$ of $\mathbb{C}$ is geometric set if $qz \in \mathbb{B}$, whenever $z \in \mathbb{B}$. If a subset $\mathbb{B}$ of $\mathbb{C}$ is a geometric set, then it contains all geometric sequences $\{zq^n\}_{0}^{\infty}$, $qz \in \mathbb{B}$. Let $f$ be a function (real or complex valued) defined on geometric set $\mathbb{B}$, $|q| \neq 1$. The $q$-difference operator which was introduced by Jackson F.H. [1],[5] and E.Heine or Euler [1],[5], defined by

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \text{ for } z \in \mathbb{B} \setminus \{0\} \quad (1.1)$$

The $q$-difference operator (1.1) sometimes called Jackson difference operator if $0 \in \mathbb{B}$, the $q$-derivative at zero defined by for $|q| < 1$

$$D_q f(z) = \lim_{n \to \infty} \frac{f(zq^n) - f(0)}{zq^n}, \text{ for } z \in \mathbb{B} \setminus \{0\} \quad (1.2)$$
provided the limit exists and does not depend on \( z \), in addition \( q \)-derivative at zero defined by for \( |q| < 1 \)

\[
D_q f(0) = D_{q^{-1}} f(0)
\]  

(1.3)

Under the hypothesis of the definition of the \( q \)-difference operator, then we have the following rules [1],[5]

1) For a function \( f(z) = z^n \),

\[
D_q f(z) = D_q z^n = \frac{1 - q^n}{1 - q} z^{n-1}
\]

2) Let \( f(z) \) and \( g(z) \) be defined on a \( q \)-geometric set \( B \subset \mathbb{C} \) such that \( q \)-derivatives of \( f \) and \( g \) exists for all \( z \in B \), then

(i) \( D_q (af(z) \pm bg(z)) = aD_q f(z) \pm bD_q g(z) \), where \( a \) and \( b \) are real or complex constants.

(ii) \( D_q (f(z).g(z)) = g(z).D_q f(z) + f(qz)D_q g(z) \)

(iii) \[
D_q \left( \frac{f(z)}{g(z)} \right) = \frac{g(z)D_q f(z) - f(qz)D_q g(z)}{g(z).g(qz)}
= \frac{g(qz)D_q f(z) - f(qz)D_q g(z)}{g(z).g(qz)}, \quad g(z).g(qz) \neq 0
\]

(iv) As a right inverse Jackson [1], [5] introduced \( q \)-integral

\[
\int_{0}^{z} f(t) d_q t = z(1 - q) \sum_{n=0}^{\infty} q^n f(zq^n)
\]

provided that the series converges. The following theorem is analogue of the fundamental theorem of calculus.

**Theorem 1.1** ([1],[5]). Let \( f \) be a \( q \)-regular at zero, function defined on \( q \)-geometric set \( B \) containing zero. Define

\[
F(z) = \int_{c}^{z} f(\zeta) d_q \zeta, \quad (z \in \mathbb{B})
\]

where \( c \) is a fixed point in \( B \), then \( F \) is \( q \)-regular at zero, furthermore \( D_q F(z) \) exists for every \( z \in \mathbb{B} \) and

\[
D_q F(z) = f(z)
\]
for $z \in \mathbb{B}$.

Conversely: If $a$ and $b$ are two points in $\mathbb{B}$, then

$$\int_{a}^{b} D_q f(\zeta)d_q\zeta = f(b) - f(a)$$

3) The $q$-differential is defined as,

$$d_qf(z) = f(z) - f(qz),$$

therefore

$$D_q f(z) = \frac{d_q f(z)}{d_qz} = \frac{f(z) - f(qz)}{(1-q)z} \Rightarrow d_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} d_qz$$

4) The partial $q$-derivative of a multivariable real continuous function $f(x_1, x_2, \ldots, x_i, \ldots, x_n)$ to a variable $x_i$ defined by

$$D_{q_i} f(\vec{x}) = \frac{f(\vec{x}) - \varepsilon_{q_i} f(\vec{x})}{(1-q)x_i} \quad (x_i \neq 0, q \in (0, 1))$$

$$[D_{q_i} f(\vec{x})]_{x_i=0} = \lim_{x_i \to 0} D_{q_i} f(\vec{x})$$

where $\varepsilon_{q_i} f(\vec{x}) = f(x_1, x_2, \ldots, x_{i-1}, qx_i, x_{i+1}, \ldots, x_n)$ and we use $D^k_{q_i} x_k$ instead of the operator $\frac{\partial^k}{\partial q^q x^k}$ for some simplification.

**Lemma 1.2** (Jack Lemma). Let $\phi(z)$ be regular in the open unit disc $\mathbb{D}$ with $\phi(0) = 0$ and $|\phi(z)| < 1$ for every $z \in \mathbb{D}$. If $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0$, then we have

$$z_0 \phi'(z) = m \phi(z_0)$$

where $m \geq 1$ is a real number.

Finally, let $\Omega$ be the family of functions $\phi(z)$ regular in the open unit disc $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0, |\phi(z)| < 1$ for every $z \in \mathbb{D}$. Denote by $\mathcal{P}(q)$ the family of functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ which are regular in the open unit disc and satisfying

$$\left| p(z) - \frac{1}{1-q} \right| < \frac{1}{1-q}, \quad z \in \mathbb{D}, \quad q \in (0, 1)$$
Let $A$ be the family of functions $f$ which are regular in the open unit disc $D$ and satisfying the conditions $f(0) = 0$, $f'(0) = 1$, let $f(z)$ be an element of $A$ if $f(z)$ satisfies the condition

$$
\frac{D_q(D_qf(z))}{D_qf(z)} = p(z), \quad z \in D
$$

where $p(z) \in \mathcal{P}(q)$, then $f(z)$ is called $q$-convex function. The class of such functions is denoted by $C_q$.

Let $f_1(z)$ and $f_2(z)$ be elements of $A$. If there exists a function $\phi(z) \in \Omega$ such that $f_1(z) = f_2(\phi(z))$, then we say that $f_1(z)$ is subordinate to $f_2(z)$ and we write $f_1(z) \prec f_2(z)$. Thus $f_1(z) \prec f_2(z)$ if and only if $f_1(0) = f_2(0)$ and $f_1(D) \subseteq f_2(D)$ implies $f_1(D_r) \subseteq f_2(D_r)$, $D_r = \{z \mid |z| < r, \ 0 < r < 1\}$. (Subordination principle [4])

2 MAIN RESULTS

PRELIMINARY

We will need the following lemma and theorems for the aim of this paper.

**Lemma 2.1** ($q$-Jack’s Lemma[2]). Let $\phi(z)$ be analytic in $D$ with $\phi(0) = 0$. Then if $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in D$, then we have

$$z_0D_q\phi(z_0) = m\phi(z_0)$$

$m \geq 1$ real number.

**Proof.** Using the $q$-difference operator and Jack Lemma (1.2), then we have

$$D_q\phi(z) = \frac{\phi(z) - \phi(qz)}{(1-q)z} = \frac{\phi(z) - \phi(z_0)}{z - z_0}, \quad qz = z_0$$

If we take the limit for $z \to z_0$ we obtain

$$\lim_{z \to z_0} D_q\phi(z) = D_q\phi(z_0) = \lim_{z \to z_0} \frac{\phi(z) - \phi(z_0)}{z - z_0} = \phi'(z_0)$$

Therefore we have

$$z_0\phi'(z_0) = m\phi(z_0) = z_0D_q\phi(z_0)$$
Theorem 2.2 ([2]). Let $f(z)$ be a regular function in the open unit disc $\mathbb{D}$. Then

$$D_q(\log f(z)) = \frac{D_q f(z)}{f(z)}$$

Proof. Using the definition of $q$-difference operator, then we have

$$D_q(\log f(z)) = \frac{\log f(z) - \log f(qz)}{z - qz} = \log \left(1 + h \frac{D_q f(z)}{f(z)}\right)^{\frac{1}{h}}$$

taking limit for $h \to 0$ we obtain

$$D_q(\log f(z)) = \frac{D_q f(z)}{f(z)}$$

Theorem 2.3 ([2]). Let $f(z)$ be an element of $C_q$, then

$$\left(\frac{r}{1 + qr}\right)^{\frac{1-q}{\log q^{-1}}} \leq |f(z)| \leq \left(\frac{r}{1 - qr}\right)^{\frac{1-q}{\log q^{-1}}} \quad (2.1)$$

$$(1 + qr)^{-\frac{1-q^2}{q^2 \log q^{-1}}} \leq |D_q f(z)| \leq (1 - qr)^{-\frac{1-q^2}{q^2 \log q^{-1}}} \quad (2.2)$$

Proof. Using theorem (2.5), then since $\frac{z D_q f(z)}{f(z)} < \frac{1}{1-qz}$ and the transformation $w = \frac{1}{1-qz}$ maps $|z| = r$ onto the disc with the center $C(r) = \frac{1}{1-q^2 r^2}$, and the radius $\rho(r) = \frac{qr}{1-q^2 r^2}$. Thus we can write

$$\left|z \frac{D_q f(z)}{f(z)} - \frac{1}{1 - q^2 r^2}\right| \leq \frac{qr}{1 - q^2 r^2} \quad (2.3)$$

The inequality 2.3 can be written in the following form,

$$\frac{1}{1 + qr} \leq \Re ez \frac{D_q f(z)}{f(z)} \leq \frac{1}{1 - qr} \quad (2.4)$$

On the other hand we have (using the $q$-partial differential rule)

$$\Re ez \frac{D_q f(z)}{f(z)} = r \cdot \frac{\partial_q}{\partial_r} \log |f(z)| \quad (2.5)$$

Considering 2.4 and 2.5 together, we can write

$$\frac{1}{r(1 + qr)} \leq \frac{\partial_q}{\partial_r} \log |f(z)| \leq \frac{1}{r(1 - qr)} \quad (2.6)$$
If we take \( q \)-integral both side of 2.6 we get 2.1.

Since \( \lim_{q \to 1} \frac{1-q}{\log q} = 1 \) then 2.1 reduces to

\[
\frac{r}{1+r} \leq |f(z)| \leq \frac{r}{1-r}
\]

This is the growth theorem of convex functions.

And other side, since the transformation \( \left( \frac{1+z}{1-qz} \right) \) maps \( |z| = r \) onto the disc with the center \( C(r) = \left( \frac{1+q^{2}r^{2}}{1-q^{2}r^{2}}, 0 \right) \) and the radius \( \rho(r) = \frac{(1+q)r}{1-q^{2}r^{2}} \) and since \( f(z) \in C_q \), using the subordination principle, then we can write

\[
\left| \left( 1 + qz \frac{D_q(D_qf(z))}{D_qf(z)} \right) - \frac{1+q^{2}r^{2}}{1-q^{2}r^{2}} \right| \leq \frac{(1+q)r}{1-q^{2}r^{2}}
\]  
(2.7)

The inequality can be written in the following form,

\[
-\frac{1+q}{q} \cdot \frac{1}{1+qr} \leq \frac{\partial_q}{\partial_r} \log |D_qf(z)| \leq \frac{1+q}{q} \cdot \frac{1}{1-qr}
\]  
(2.8)

In this step, if we take \( q \)-integral both sides of 2.8 we get 2.2 \( \square \)

**Theorem 2.4 ([7]).** \( p(z) \in \mathcal{P}(q) \) if and only if \( p(z) \prec \frac{1+z}{1-qz} \).

**Theorem 2.5 ([2]).** If \( f(z) \in C_q \), then

\[
z \frac{D_qf(z)}{f(z)} \prec \frac{1}{1-qz}
\]

**Proof.** We define the function \( \phi(z) \) by

\[
z \frac{D_qf(z)}{f(z)} = \frac{1}{1-q\phi(z)}
\]  
(2.9)

Since

\[
f(z) = z + a_2z^2 + a_3z^3 + \ldots \quad \text{and} \quad zD_qf(z) = z + a_2 \frac{1-q^2}{1-q} z^2 + a_3 \frac{1-q^3}{1-q} z^3 + \ldots
\]

then \( \phi(z) \) is well defined and analytic at the same time

\[
\frac{zD_qf(z)}{f(z)} \bigg|_{z=0} = 1 = \frac{1}{1-q\phi(0)} \implies \phi(0) = 0
\]

We need to show that \( |\phi(z)| < 1 \) for all \( z \in \mathbb{D} \). Assume to the contrary, then there exists a \( z_0 \in \mathbb{D} \) such that \( |\phi(z_0)| = 1 \). The definition of the class of \( C_q \), using theorem (2.4) and subordination principle, then we write

\[
A_r = \left\{ f(z) \mid f(z) \in C_q, \left| \left( 1 + qz \frac{D_q(D_qf(z))}{D_qf(z)} \right) - \frac{1+q^{2}r^{2}}{1-q^{2}r^{2}} \right| \leq \frac{(1+q)r}{1-q^{2}r^{2}}, q \in (0, 1) \right\}
\]  
(2.10)
On the other hand, using the definition \( q \)-derivative, theorem (2.3), relation 2.9 and after the straightforward calculations we get

\[
1 + qz \frac{D_q(D_qf(z))}{D_qf(z)} = q \cdot \left( \frac{1}{1 - q\phi(z)} \right) + \frac{\log q^{-1}}{1 - q} \cdot \frac{zD_q\phi(z)}{1 - q\phi(z)} + \left( 1 - q \frac{\log q^{-1}}{1 - q} \right) \tag{2.11}
\]

Using (2.1) in 2.14, then we can write

\[
1 + qz_0 \frac{D_q(D_qf(z_0))}{D_qf(z_0)} = \left[ q \cdot \left( \frac{1}{1 - q\phi(z_0)} \right) + \frac{\log q^{-1}}{1 - q} \cdot \frac{m\phi(z_0)}{1 - q\phi(z_0)} + \left( 1 - q \frac{\log q^{-1}}{1 - q} \right) \right] \notin A_r
\]

But this is a contradiction with 2.10. Therefore \( |\phi(z)| < 1 \) for all \( z \in \mathbb{D} \). \( \square \)

In the present paper we will investigate the following subclass of \( q \)-harmonic mapping

\[
SHC(q) = \left\{ f = h(z) + \overline{g(z)} \mid w_q(z) = \frac{D_qg(z)}{D_qh(z)} \prec b_1 \frac{1 + z}{1 - qz}, \ h(z) \in C_q, \ q \in (0, 1) \right\},
\]

where

\[
D_qh(z) = \frac{h(z) - h(qz)}{(1 - q)z} = f(z), \quad D_qg(z) = \frac{g(z) - g(qz)}{(1 - q)z} = \tilde{f}_z
\]

**SOME REMARKS ON SHC(q)**

Since \( f = h(z) + \overline{g(z)} \Rightarrow h(z) = z + a_2 z^2 + a_3 z^3 + \ldots, \ g(z) = b_1 z + b_2 z^2 + b_3 z^3 + \ldots \), then we have

\[
D_qh(z) = 1 + a_2 \frac{1 - q^2}{1 - q} z + a_3 \frac{1 - q^3}{1 - q} z^2 + \ldots, \quad D_qg(z) = b_1 + b_2 \frac{1 - q^2}{1 - q} z + b_3 \frac{1 - q^3}{1 - q} z^2 + \ldots
\]

Thus, for \( q \to 1 \)

\[
D_qh(z) = h'(z), \quad D_qg(z) = g'(z), \quad w_q(z) = w(z) = \frac{g'(z)}{h'(z)}
\]

\[
J_{f_q} = |D_qh(z)|^2 - |D_qg(z)|^2 \to J_f(z) = |h'(z)|^2 - |g'(z)|^2
\]

**Theorem 2.6.** Let \( f = h(z) + \overline{g(z)} \) be an element of SHC\( (q) \), then

\[
\frac{g(z)}{h(z)} \prec b_1 \frac{1 + z}{1 - qz}
\]
Proof. We define the function
\[
g(z) = \frac{h(z) \left( 1 + \phi(z) \right)}{b_1 - q\phi(z)}
\]  
(2.12)
and therefore \(\phi(z)\) is analytic and \(\frac{g(z)}{h(z)}\big|_{z=0} = b_1 = b_1 \frac{1 + \phi(0)}{1 - q\phi(0)} \Rightarrow \phi(0) = 0\). We need to show that \(|\phi(z)| < 1\) for every \(z \in \mathbb{D}\). On the other hand the linear transformation \(w(z) = b_1 \frac{1 + z}{1 - qz}\) maps \(|z| = r\) onto the circle with the center \(C(r) = \left( \frac{\alpha_1(1 - r^2)}{1 - q^2 r^2}, \frac{\alpha_2(1 - q^2)}{1 - q^2 r^2} \right)\) and the radius \(\rho(r) = \left| \frac{b_1(1 - q)r}{1 - q^2 r^2} \right|\), where \(\alpha_1 = \Re b_1\), \(\alpha_2 = \Re b_2\). Thus, using the subordination principle and the definition of the class \(\text{SHC}(q)\), then we can write
\[
w_q(\mathbb{D}_r) = \left\{ \frac{D_q g(z)}{D_q h(z)} \left| \frac{D_q g(z)}{D_q h(z)} - \frac{b_1(1 + qr^2)}{1 - q^2 r^2} \right| \leq \left| \frac{b_1(1 - q)r}{1 - q^2 r^2} \right|, q \in (0, 1) \right\}
\]  
(2.13)
If we take \(q\)-derivative from (2.12) we get
\[
\frac{D_q g(z)}{D_q h(z)} = b_1 \frac{1 + \phi(qz)}{1 - \phi(qz)} + \frac{(1 + q)zD_q \phi(z)}{(1 - q\phi(z))(1 - q\phi(qz))} \cdot \frac{h(z)}{zD_q h(z)}
\]  
(2.14)
In this step, if we use theorem 2.5 and subordination principle we get
\[
\frac{D_q g(z)}{D_q h(z)} = b_1 \frac{1 + \phi(qz)}{1 - q\phi(qz)} + \frac{(1 + q)zD_q \phi(z)}{(1 - q\phi(qz))}
\]  
(2.15)
Now we assume that there exists \(z_0 \in \mathbb{D}_r\) such that \(|\phi(z_0)| = 1\), using \(q\)-Jack’s lemma we obtain
\[
\frac{D_q g(z_0)}{D_q h(z_0)} = \left( b_1 \frac{1 + \phi(qz_0)}{1 - q\phi(qz_0)} + \frac{(1 + q)m\phi(z_0)}{(1 - q\phi(qz_0))} \right) \notin w_q(\mathbb{D}_r).
\]
This is a contradiction with (2.13), therefore we have \(|\phi(z)| < 1\) for every \(z \in \mathbb{D}\), thus we have \(\frac{g(z)}{h(z)} < b_1 \frac{1 + z}{1 - qz}\).

\[\square\]

Corollary 2.7. Let \(f = h(z) + \overline{g(z)}\) be an element of \(\text{SHC}(q)\), then
\[
|b_1| \left( \frac{1 - r}{1 + qr} \right)^{\frac{1 - q^2}{q^2 \log q} - r} \leq |D_q g(z)| \leq |b_1| \left( \frac{1 + r}{1 - qr} \right)^{\frac{1 - q^2}{q^2 \log q} - r}.
\]
Proof. Since \( \frac{g(z)}{h(z)} < b_1 \frac{1 + z}{1 - qz} \), then using subordination principle

\[
\left| \frac{g(z)}{h(z)} - \frac{b_1 (1 + qr^2)}{1 - q^2 r^2} \right| \leq \frac{|b_1|(1 + q) r}{1 - q^2 r^2} \Rightarrow \left| \frac{b_1}{1 - qr} \right| \leq \frac{|g(z)|}{h(z)} \leq \frac{|b_1|(1 + r)}{1 - qr}.
\]

In this step if we use theorem 2.3 we get

\[
\left| b_1 \left( \frac{1 - r}{1 + qr} \right) \right| \left( \frac{r}{1 + qr} \right)^{\frac{1 + q}{\text{log } q - 2}} \leq |g(z)| \leq \left| b_1 \left( \frac{1 + r}{1 - qr} \right) \right| \left( \frac{r}{1 - qr} \right)^{\frac{1 + q}{\text{log } q - 2}}
\]

Similarly

\[
\left| b_1 \left( \frac{1 - r}{1 + qr} \right) \right| \left( \frac{r}{1 + qr} \right)^{\frac{1 + q}{\text{log } q - 2}} \leq \left| D_q g(z) \right| \leq \left| b_1 \left( \frac{1 + r}{1 - qr} \right) \right| \left( \frac{r}{1 - qr} \right)^{\frac{1 + q}{\text{log } q - 2}}
\]

then we have

\[
\left| b_1 \left( \frac{1 - r}{1 + qr} \right) \right| \left( \frac{r}{1 + qr} \right)^{\frac{1 + q}{\text{log } q - 2}} \leq \left| D_q g(z) \right| \leq \left| b_1 \left( \frac{1 + r}{1 - qr} \right) \right| \left( \frac{r}{1 - qr} \right)^{\frac{1 + q}{\text{log } q - 2}}
\]

\[\Box\]

Corollary 2.8. Let \( f = h(z) + \overline{g(z)} \) be an element of \( \text{SHC}(q) \), then

\[
F_2(q, |b_1|, r) \leq |J_{f_q}(z)| \leq F_1(q, |b_1|, r)
\]  

(2.16)

where

\[
F_1(q, |b_1|, r) = (1 - qr)^{-\frac{1 - q^2}{q^2 \text{log } q - 1}} \frac{(1 + |b_1| + (q - |b_1|) r)[(1 + |b_1|) - (q + |b_1|) r]}{(1 + qr)^2}
\]

\[
F_2(q, |b_1|, r) = (1 + qr)^{-\frac{1 - q^2}{q^2 \text{log } q - 1}} \frac{[(1 - |b_1|) - (q + |b_1|) r][(1 + |b_1|) + (|b_1| - q) r]}{(1 - qr)^2}
\]

Proof. Since

\[
J_{f_q}(z) = |D_q h(z)|^2 - |D_q g(z)|^2 = |D_q h(z)|^2 \left( 1 - |w_q(z)|^2 \right),
\]

and since

\[
\left| b_1 \left( \frac{1 - r}{1 + qr} \right) \right| \left( \frac{r}{1 + qr} \right) \leq |w_q(z)| \leq \left| b_1 \left( \frac{1 + r}{1 - qr} \right) \right| \left( \frac{r}{1 - qr} \right)
\]

then we have (2.16).  \[\Box\]
References


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