Positive Solution for the Integral Boundary Value Problem of a Class of Nonlinear Semipositone Fractional Differential Equations

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Abstract

In this paper, by using the Krasnosel’skii fixed-point theorem, we establish the existence of multiple positive solutions for the following integral boundary value problem of a class of nonlinear semipositone fractional differential equations

\[
\begin{cases}
D^{\alpha}_{0^+} u + \lambda[f(t,u) + g(t,u)] = 0, & 0 < t < 1, \quad n - 1 < \alpha \leq n, \\
u^{(j)}(0) = 0, & 0 \leq j \leq n - 2, \quad u(1) = \mu \int_0^1 u(s)ds,
\end{cases}
\]

where \(\lambda\) and \(\mu\) are positive parameters, \(\mu\) and \(\alpha\) are real numbers satisfying \(0 < \mu < \alpha\), \(n\) is a positive integer number and \(n \geq 3\), \(D^{\alpha}_{0^+}\) is the standard Riemann-Liouville derivative, \(f, g : (0, 1) \times [0, +\infty) \to (-\infty, +\infty)\) are continuous and may change sign. The conditions for the existence of positive solutions are obtained respectively under the condition that \(f\) and \(g\) are suplinear or sublinear with one being suplinear while the other being sublinear. The results obtained in this paper improve and generalize many well-known results.

Keywords: Riemann-Liouville’s fractional derivative; Fractional differential equation; Positive solution; Semipositone; Integral boundary value conditions

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1 Introduction

In this paper, we consider the following integral boundary value problem for a class of nonlinear semipositone fractional differential equations

\[
\begin{cases}
D_0^\alpha u + \lambda [f(t, u) + g(t, u)] = 0, \quad 0 < t < 1, \quad n - 1 < \alpha \leq n, \\
u^{(j)}(0) = 0, \quad 0 \leq j \leq n - 2, \quad u(1) = \mu \int_0^1 u(s)ds,
\end{cases}
\]

(1.1)

where \( \lambda \) and \( \mu \) are positive parameters, \( \mu \) and \( \alpha \) are real numbers satisfying \( 0 < \mu < \alpha \), \( n \) is a positive integer number and \( n \geq 3 \), \( D_0^\alpha \) is the standard Riemann-Liouville derivative, \( f, g : (0,1) \times [0, +\infty) \rightarrow \mathbb{R} \) are sign-changing continuous functions and may be singular at \( t = 0, 1 \). In this paper, a function \( u \in C^{n-1}[0,1] \cap C^n(0,1) \) is called a positive solution of the problem (1.1) if \( u(t) > 0 \) on \( t \in (0,1) \), \( D_0^\alpha u(t) \in L(0,1) \), and \( u \) satisfies (1.1) on \( t \in (0,1) \).

Recently, many results were obtained for the existence of solutions of nonlinear fractional differential equations by the use of nonlinear analysis techniques, and for details the reader is referred to [1-4] and the references therein.

In [3], the authors consider the existence of positive solutions of the following nonlinear fractional differential equation boundary value problem with changing sign nonlinearity

\[
\begin{cases}
D_0^\alpha u(t) + \lambda f(t, u(t)) = 0, \quad 0 < t < 1, \\
u(0) = u'(0) = u(1) = 0,
\end{cases}
\]

(1.2)

where \( 2 < \alpha \leq 3 \) is a real number, \( D_0^\alpha \) is the standard Riemann-Liouville derivative, \( \lambda \) is a positive parameter, \( f : (0,1) \times [0, +\infty) \rightarrow \mathbb{R} \) is a continuous sign-changing function and is allowed to be singular at \( t = 0, 1 \).

In [4], by using the Krasnosel’skii fixed-point theorem, Yuan consider the positive solution for the \((n-1, 1)\)-type semipositone conjugate boundary value problem

\[
\begin{cases}
D_0^\alpha u(t) + \lambda f(t, u(t)) = 0, \quad 0 < t < 1, \\
u^{(j)}(0) = 0, \quad 0 \leq j \leq n - 2, \\
u(1) = 0,
\end{cases}
\]

(1.3)

where \( \lambda \) is a positive parameter, \( \alpha \in (n - 1, n] \) is a real number, and \( n \) is a positive integer number with \( n \geq 3 \), \( D_0^\alpha \) is the standard Riemann-Liouville derivative. The author establishes the properties of the Green’s function of the boundary value problem, and derives an interval of \( \lambda \) such that for any \( \lambda \) lying in this interval, the semipositone boundary value problem has multiple positive solutions under the following assumptions:

\((A_1) : f : [0,1] \times [0, +\infty) \rightarrow \mathbb{R} \) is a continuous sign-changing function, moreover there exists a function \( g(t) \in L^1([0,1], (0, +\infty)) \) such that \( f(t, x) \geq -g(t) \), for any \( t \in (0,1), u \in [0, +\infty) \).
(A2): \( f(t, 0) > 0 \) for any \( t \in [0, 1] \).

(A3): There exists \([\theta_1, \theta_2] \in (0, 1)\) such that \( \lim_{x \to +\infty} \min_{t \in [\theta_1, \theta_2]} \frac{f(t, x)}{x} = +\infty \).

Inspired by the above mentioned work, the aim of this paper is to establish conditions for the existence of single and multiple positive solutions of the BVP \((1.1)\). Compared to the results in \([2-5]\), our work presented in this paper has the following new features. Firstly, the nonlinear term involves two semi-positone functions. Secondly, the single and multiple positive solutions for the BVP \((1.1)\) are obtained without using condition \((A2)\). Thirdly, the boundary conditions include an integral boundary value condition which is more general and covers multi-points boundary conditions as special cases.

2 Preliminaries and lemmas

**Definition 2.1.** The Riemann-Liouville fractional integral of order \( \alpha > 0 \) of a function \( u : (0, +\infty) \to \mathbb{R} \) is given by

\[
I^\alpha_{0+} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds
\]

provided that the right-hand side is pointwise defined on \((0, +\infty)\).

**Definition 2.2.** The Riemann-Liouville fractional derivative of order \( \alpha > 0 \) of a function \( u : (0, +\infty) \to \mathbb{R} \) is given by

\[
D^\alpha_{0+} u(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^{n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds
\]

where \( n = [\alpha] + 1 \), \([\alpha]\) denotes the integer part of number \( \alpha \), provided that the right-hand side is pointwise defined on \((0, +\infty)\).

**Lemma 2.1.** Let \( \alpha > 0 \), \( u \in C(0,1) \cap L(0,1) \), then the fractional differential equation

\[D^\alpha_{0+} u(t) = 0,\]

has the solution, \( u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n} \), \( c_i \in \mathbb{R} \) \( i = 1, 2, \ldots, n \), as the unique solution, where \( n \) is the smallest integer greater than or equal to \( \alpha \).

**Lemma 2.2 ([1]).** Let \( \alpha > 0 \). Assume that \( D^\alpha_{0+} u, u \in C(0,1) \cap L(0,1) \). Then the following equality holds

\[I^\alpha_{0+} D^\alpha_{0+} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n},\]

where \( c_i \in \mathbb{R} \) \( i = 1, 2, \ldots, n \), \( n - 1 < \alpha \leq n \).
Lemma 2.3 ([5,6]). Let \( X \) be a real Banach space, and let \( P \subset X \) be a cone in \( X \). Assume that \( \Omega_1, \Omega_2 \) are open subsets of \( X \) with \( \emptyset \in \Omega_1, \Omega_1 \subset \Omega_2 \), and \( S: P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P \) be a completely continuous operator such that either

(i) \( \|Su\| \leq \|u\|, u \in P \cap \partial \Omega_1 \) and \( \|Su\| \geq \|u\|, u \in P \cap \partial \Omega_2 \), or

(ii) \( \|Su\| \geq \|u\|, u \in P \cap \partial \Omega_1 \) and \( \|Su\| \leq \|u\|, u \in P \cap \partial \Omega_2 \).

Then \( S \) has a fixed point in \( P \cap (\overline{\Omega_2} \setminus \Omega_1) \).

Lemma 2.4. Let \( h \in C(0,1) \cap L(0,1) \) be a given function, \( \lambda > 0 \) be a real number, then the boundary value problem

\[
\begin{aligned}
D_{0+}^\alpha u(t) + \lambda h(t) &= 0, \quad 0 < t < 1, \quad 2 \leq n - 1 < \alpha \leq n, \\
u^{(j)}(0) &= 0, \quad 0 \leq j \leq n - 2, \quad u(1) = \mu \int_0^1 u(s)ds,
\end{aligned}
\tag{2.1}
\]

has a unique solution

\[
u(t) = \lambda \int_0^1 G(t,s)h(s)ds, \quad t \in [0,1],
\tag{2.2}
\]

where

\[
G(t,s) = \begin{cases}
\frac{\Gamma(\alpha)}{\Gamma(\alpha - \mu)} t^{\alpha-1}(1-s)^{\alpha-1}(1-\mu)^{\alpha-1}(s-t)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\
\frac{\Gamma(\alpha)}{\Gamma(\alpha - \mu)} t^{\alpha-1}(1-s)^{\alpha-1}(1-\mu)^{\alpha-1}, & 0 \leq t \leq s \leq 1.
\end{cases}
\tag{2.3}
\]

Lemma 2.5. The function \( G(t,s) \) defined by (2.3) has the following properties

\[
\mu t^{\alpha-1}q(s) \leq G(t,s) \leq \frac{M_0}{(\alpha - \mu)\Gamma(\alpha)} t^{\alpha-1}, \quad G(t,s) \leq M_0 q(s) \quad \text{for} \quad t,s \in [0,1],
\]

where \( M_0 = (\alpha - \mu)(\alpha - 1) + \mu \) and \( q(s) = \frac{1}{(\alpha - \mu)\Gamma(\alpha)} s(1-s)^{\alpha-1} \).

Lemma 2.6. Let \( \epsilon \in C(0,1) \cap L(0,1) \) with \( \epsilon(t) > 0 \) on \( t \in (0,1) \), and \( \lambda > 0 \) be a real number, then the boundary value problem

\[
\begin{aligned}
D_{0+}^\alpha u(t) + \lambda \epsilon(t) &= 0, \quad 0 < t < 1, \quad 2 \leq n - 1 < \alpha \leq n, \\
u^{(j)}(0) &= 0, \quad 0 \leq j \leq n - 2, \quad u(1) = \mu \int_0^1 u(s)ds,
\end{aligned}
\tag{2.4}
\]

has a unique solution \( \omega(t) = \lambda \int_0^1 G(t,s)\epsilon(s)ds \) with

\[
\omega(t) \leq \frac{\lambda M_0}{(\alpha - \mu)\Gamma(\alpha)} t^{\alpha-1} \int_0^1 \epsilon(s)ds, \quad 0 \leq t \leq 1.
\tag{2.5}
\]
3 Main Results

For the convenience in presentation, we denote:

\[
\sigma = \frac{M_0^2}{\mu(\alpha - \mu)} \int_0^1 e(s) \, ds; \quad \psi^*(r) = \max_{x \in [0,r]} \psi(x),
\]
and make the following assumptions here:

\[(H_1)\] \(f, g \in C((0,1) \times [0, +\infty), (-\infty, +\infty))\) and there exist \(e_1, e_2, \varphi \in C((0,1), [0, +\infty))\) and \(\psi \in C([0, +\infty), [0, +\infty))\) such that

\[-e_1(t) \leq f(t, x), \quad -e_2(t) \leq g(t, x), \quad t \in (0, 1), x \in [0, +\infty),
\]

\[f(t, x) + g(t, x) \leq \varphi(t)\psi(x), \quad t \in (0, 1), x \in [0, +\infty).\]

\[(H_2)\] \(0 < \int_0^1 q(s)(\varphi(s) + e(s)) \, ds < +\infty,\)
where \(e(s) = e_1(s) + e_2(s).\)

\[(H_3)\] There exists \([a, b] \subset (0, 1)\) such that

\[
\lim \inf \min_{x \to +\infty} \frac{f(t, x)}{x} = +\infty, \quad \text{or} \quad \lim \inf \min_{x \to +\infty} \frac{g(t, x)}{x} = +\infty.
\]

Moreover

\[f(t, x) + e_1(t) \geq L_1 x, \quad (t, x) \in (0, 1) \times [0, \sigma],\]

where

\[L_1 > \max \left\{ \frac{4M_0^2}{\mu^2 \int_0^1 q(s) s^\alpha \, ds}, \frac{2M_0^2 \psi^*(r_0) + 1}{\mu^2 \int_0^1 q(s) \varphi(s) + e(s) \, ds} \right\}, \]

in which \(r_0 > \sigma.\)

In order to overcome the difficulty associated with semipositone, we consider the following boundary value problem

\[
\begin{cases}
D_0^\alpha u(t) + \lambda \left[ f(t, [u(t) - \omega(t)]^+) + g(t, [u(t) - \omega(t)]^+) + e(t) \right] = 0, & 0 < t < 1, \\
u^{(j)}(0) = 0, & 0 \leq j \leq n - 2, \quad u(1) = \mu \int_0^1 u(s) \, ds,
\end{cases}
\]

(3.1)

where \(\lambda > 0, [u(t) - \omega(t)]^+ = \max\{u(t) - \omega(t), 0\},\) and \(\omega(t) = \lambda \int_0^1 G(t, s) e(s) \, ds\)
which is the solution of the following boundary value problem

\[
\begin{cases}
D_0^\alpha \omega(t) + \lambda e(t) = 0, & 0 < t < 1, \quad n - 1 < \alpha \leq n, \quad \lambda > 0, \\
\omega^{(j)}(0) = 0, & 0 \leq j \leq n - 2, \quad \omega(1) = \mu \int_0^1 u(s) \, ds.
\end{cases}
\]
We will show that there exists a solution \( u(t) \) for the boundary value problem (3.1) with \( u(t) > \omega(t), t \in (0, 1) \). If this is true, then \( \overline{u}(t) = u(t) - \omega(t) \) is a positive solution of the singular semipositone boundary value problem (1.1), because for any \( t \in (0, 1) \),

\[
-D_{0+}^{\alpha} \overline{u}(t) = -D_{0+}^{\alpha} u(t) + D_{0+}^{\alpha} \omega(t) = \lambda \left[ f(t, \overline{u}(t)) + g(t, \overline{u}(t)) + e(t) \right] - \lambda e(t) = \lambda \left[ f(t, \overline{u}(t)) + g(t, \overline{u}(t)) \right].
\]

Hence, we will concentrate our study on the boundary value problem (3.1).

Let \( E = C[0, 1] \) be endowed with the maximum norm \( \|u\| = \max_{0 \leq t \leq 1} |u(t)| \) for \( u \in E \). Define a cone \( P \) by

\[
P = \left\{ u \in E : u(t) \geq \frac{\mu^{\alpha-1}}{M_0} \|u\|, t \in [0, 1] \right\}.
\]

Set \( \Omega_r = \{ u \in E : \|u\| < r \} \), \( P_r = P \cap \Omega_r \), \( \partial P_r = P \cap \partial \Omega_r \).

Define an integral operator \( T : P \to E \) by

\[
Tu(t) = \lambda \int_0^1 G(t, s) \left[ f(s, [u(s) - \omega(s)]^+) + g(s, [u(s) - \omega(s)]^+) + e(s) \right] ds.
\]

**Lemma 3.1.** Suppose that \((H_1), (H_2)\) hold. Then \( T : P \to P \) is a completely continuous operator.

**Proof.** For any \( u \in P, t \in [0, 1] \), from Lemma 2.5 and conditions \((H_1)\) and \((H_2)\), we have

\[
Tu(t) = \lambda \int_0^1 G(t, s) \left[ f(s, [u(s) - \omega(s)]^+) + g(s, [u(s) - \omega(s)]^+) + e(s) \right] ds
\]

\[
\leq \lambda M_0 \int_0^1 q(s) \left[ \varphi(s) \psi([u(s) - \omega(s)]^+) + e(s) \right] ds
\]

\[
\leq \lambda M_0 \int_0^1 q(s) \left[ \varphi(s) \max_{x \in [0, \|u\|]} \psi(x) + e(s) \right] ds
\]

\[
\leq \lambda M_0 \left( \max_{x \in [0, \|u\|]} \psi(x) + 1 \right) \int_0^1 q(s) \left[ \varphi(s) + e(s) \right] ds < +\infty.
\]

Therefore, the operator \( T \) is well defined.

For any \( u \in P, t \in [0, 1] \), Lemma 2.5 implies that

\[
Tu(t) \leq \lambda M_0 \int_0^1 q(s) \left[ f(s, [u(s) - \omega(s)]^+) + g(s, [u(s) - \omega(s)]^+) + e(s) \right] ds,
\]

then

\[
\|Tu\| \leq \lambda M_0 \int_0^1 q(s)[f(s, [u(s) - \omega(s)]^+) + g(s, [u(s) - \omega(s)]^+) + e(s)] ds.
\]
On the other hand, we have

$$Tu(t) \geq \lambda \mu \int_0^1 q(s)[f(s, [u(s) - \omega(s)]^+) + g(s, [u(s) - \omega(s)]^+) + e(s)]ds.$$ 

Then $Tu(t) \geq \frac{\lambda \mu}{M_0}[Tu], t \in [0, 1]$, which implies $T : P \to P$.

According to the Ascoli-Arzela theorem, we can easily get that $T : P \to P$ is a completely continuous operator. The proof is completed. □

**Theorem 3.1.** Suppose that $(H_1) - (H_3)$ hold. Then there exists $\lambda^* > 0$ such that the boundary value problem (1.1) has at least one positive solution for any $\lambda \in (0, \lambda^*)$.

**Proof.** Choose $r_1 > \sigma$. Let

$$\lambda^* = \min \left\{ \frac{1}{2}, \frac{r_1}{M_0[\psi^*(r_1) + 1]} \int_0^1 q(s)[\varphi(s) + e(s)]ds \right\}.$$ 

In the following, we suppose $\lambda \in (0, \lambda^*)$.

For any $u \in \partial P_{r_1}$, noticing that $u(t) \geq \frac{\mu \int_0^1 q(s)[\varphi(s) + e(s)]ds}{M_0} r_1, t \in [0, 1]$, and using (2.5), we have

$$r_1 \geq u(t) - \omega(t) \geq t^{\alpha-1} \left[ \frac{\mu t^{\alpha-1}}{M_0} - \frac{M_0}{2(\alpha - \mu)\Gamma(\alpha)} \int_0^1 e(s)ds \right]$$ 

$$> \frac{t^{\alpha-1}M_0}{2(\alpha - \mu)\Gamma(\alpha)} \int_0^1 e(s)ds > 0, t \in (0, 1].$$

Therefore, for any $u \in \partial P_{r_1}, t \in [0, 1],

$$Tu(t) = \lambda \int_0^1 G(t, s)[f(s, [u(s) - \omega(s)]^+) + g(s, [u(s) - \omega(s)]^+) + e(s)]ds$$ 

$$\leq \lambda M_0 \int_0^1 q(s)[\varphi(s)\psi([u(s) - \omega(s)]^+) + e(s)]ds$$ 

$$< \lambda M_0[\psi^*(r_1) + 1] \int_0^1 q(s)[\varphi(s) + e(s)]ds$$ 

$$< \lambda^* M_0[\psi^*(r_1) + 1] \int_0^1 q(s)[\varphi(s) + e(s)]ds \leq r_1.$$ 

Thus

$$\|Tu\| < \|u\|, \quad \forall u \in \partial P_{r_1}. \quad (3.3)$$

On the other hand, set $\gamma = \min_{t \in [a, b]} t^{\alpha-1} > 0$. Now choose a real number

$$L > \frac{2M_0}{\lambda^2 \gamma^2 \int_a^b q(s)ds}. \quad (3.4)$$
By \((H_3)\), from \(\liminf_{x \to +\infty} \min_{t \in [a,b]} \frac{f(t,x)}{x} = +\infty\), there exists a constant \(C > 0\) such that for any \(t \in [a,b], x \geq C\), we have

\[
\frac{f(t,x)}{x} > L. \tag{3.5}
\]

Select \(r_2 = \max\left\{r_1, \frac{2M_0^2}{\mu(\alpha-\mu)\Gamma(\alpha)} \int_0^1 e(s)ds, \frac{2M_0^2C}{\mu}\right\}\). Then by the selection of \(r_2\) we have

\[
\frac{M_0}{(\alpha - \mu)\Gamma(\alpha)} \int_0^1 e(s)ds < \frac{\mu r_2}{2M_0}, \frac{\gamma \mu r_2}{2M_0} > C,
\]

Moreover, for any \(u \in \partial P_{r_2}\), by (2.5), we have

\[
u(t) - \omega(t) \geq \lambda \mu r_2 \int_0^1 e(s)ds \geq \lambda \int_a^b \mu t^{\alpha-1} q(s) f(s, [u(s) - \omega(s)])ds
\[
\geq \lambda L \mu \gamma \int_a^b q(s)u(s) - \omega(s)ds \geq \frac{\lambda L \mu \gamma}{2M_0} \int_a^b q(s)ds > r_2.
\]

Thus

\[
\|Tu\| > \|u\|, \quad \forall u \in \partial P_{r_2}. \tag{3.8}
\]

Obviously, if \(\liminf_{x \to +\infty} \frac{a(t,x)}{x} = +\infty\) holds, then (3.8) is still valid. It follows from Lemma 2.3, (3.3) and (3.8) that \(T\) has a fixed point \(u^* \in P\) such that \(r_1 < \|u^*\| < r_2\). Since \(\|u^*\| > r_1\), by (3.2), we have \(u^*(t) - \omega(t) > 0, t \in (0,1]\). Let \(\overline{u}(t) = u^*(t) - \omega(t)\), then \(\overline{u}(t)\) is a positive solution of the singular semipositone boundary value problem (1.1). The proof is completed.

References


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