Some Stochastic Properties of Cholera Model

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Abstract

In this paper, we study dynamical and stochastic models of cholera’s transmission. The global stability of the two steady states is proved by construction of Lyapunov functions for a dynamical model. Moreover, we give the conditions of existence of unique positive solution, and the global stability in probability and the exponential p-stability of the stochastic model, by taking account of the intensity of white noise perturbation.

Keywords: Nonlinear epidemic model; Lyapunov function; Itô’s formula; global stability, moment exponential stability

1 Introduction

Cholera is a severe diarrhoea disease caused by the bacterium Vibrio Cholerae. Transmission occurs to human when the food or water are contaminated, and also where one has a contact with cholera patient’s faeces, vomit and corpse. The purpose of this paper is to study the global stability of dynamical and stochastic models of cholera, according the following plan. In section 2 we present and study the global stability of dynamical model of Wang and Modnak [5]. In section 3, we give the condition of existence of unique solution and the global stability of equilibria of the stochastic model.
2 The basic model

The basic model of cholera transmission can be written as a combined dynamical system \((S(t), I(t), R(t) - B(t))\), where \(S(t)\), \(I(t)\), \(R(t)\) and \(B(t)\) denote the susceptible, the infected recovered human and the environmental component respectively. Hence the model is given by:

\[
\begin{align*}
\dot{S} &= \Lambda - \beta_e S \frac{B}{K + B} - \beta_h SI - \mu S \\
\dot{I} &= \beta_e S \frac{B}{K + B} + \beta_h SI - (\gamma + \mu)I \\
\dot{B} &= \epsilon I - \delta B \\
\dot{R} &= \gamma I - \mu R
\end{align*}
\]

with these parameters:

\(\Lambda\) is the recruitment into the population; \(\beta_e\) and \(\beta_h\) represent rates of ingesting vibrios from the contaminated water or through human to human interaction respectively; \(\mu\) denote the rate that an individual in the population died from reasons not related to the disease; \(\gamma\) is the rate that an infectious individual dies because of the disease; \(\xi\) is the rate of human contribution to \(V.\) cholerae, \(\delta\) is the natural death of \(V.\) cholerae, \(K\) is the pathogen concentration that yield 50% chance of catching cholera see [5].

**Proposition 1** Let \((S(t), I(t), R(t), B(t))\) be the solution of system(1) with initial conditions \((S(0), I(0), R(0))\) and the compact set:

\[
\Omega = \{(S, I, R) \in \mathbb{R}_+^3; B \in R); S + I + R \leq \frac{\Lambda}{\mu}; B < \frac{\xi \Lambda}{\mu \delta}\}
\]

Then, under the flow described by (1), \(\Omega\) is a positively set that attracts all solutions of \(\mathbb{R}_+^4\).

**Proof:** Consider the following Lyapounov function

\[
W(t) = (S(t) + I(t) + R(t))
\]

Its time derivative satisfies:

\[
\frac{dW(t)}{dt} = (\dot{S}(t) + \dot{I} + \dot{R}) = \Lambda - \mu W(t)
\]

Hence, \(\frac{dW(t)}{dt} \leq 0\) for \(W(t) > \frac{\Lambda}{\mu}\), which implies that \(\Omega\) is positively invariant set. Solving this differential equation one has that:

\[
0 \leq W(t) \leq \frac{\Lambda}{\mu} + W(0) e^{-\mu t}
\]

Where \(W(0)\) is the initial condition of \(W(t)\). Thus, as \(t \to +\infty\) one has that \(0 < W(t) < \frac{\Lambda}{\mu}\). In the same way one has \(\frac{dB(t)}{dt} = \xi I - \delta B \leq \frac{\xi \Lambda}{\mu} - \delta B \leq 0\) for \(B(t) \geq \frac{\xi \Lambda}{\mu \delta}\), this implies that:

\[
0 \leq B(t) \leq \frac{\xi \Lambda}{\mu \delta} + B(0) e^{-\delta t}
\]
Some stochastic properties of cholera model

at \( t \to \infty \), \( 0 \leq B(t) \leq \frac{\xi A}{\mu} \).

Then, one can conclude that \( \Omega \) is an attractive set. This achieves the proof.

\[
R_0 = \frac{\Lambda(K\beta_h \delta + \beta_e \xi)}{\mu \delta (\gamma + \mu) K}
\]

**Theorem 1** When \( R_0 < 1 \), the disease free equilibrium is globally asymptotically stable in \( \Omega \)

**Proof:** Define a Lyapunov function

\[
V(t) = I(t)
\]

then the derivative of \( V \) along the positive solution of (1) , we obtain

\[
\dot{V}(t) = \dot{I} = \beta_e S \frac{B}{K + B} + \beta_h SI - (\gamma + \mu)I
\]

Notice that \( \beta_e S \frac{B}{K + B} \leq \beta_e SI \frac{B}{K} \leq \beta_e \frac{\xi A}{\mu} I \) and \( R_0 \leq 1 \). From the above , we have

\[
\dot{V}(t) \leq \beta_e \frac{\xi A}{K \mu} I + \beta_h \frac{A}{\mu} I - (\gamma + \mu)I \leq (R_0 - 1)I \leq 0
\]

Hence by LaSalle ’s principe [3] the disease free equilibrium is globally asymptotically stable. System (1) has an endemic point \((\bar{S}, \bar{I}, \bar{B}, \bar{R})\) satisfying :

\[
\begin{align*}
\frac{\Lambda - \beta_e S}{K + B} - \beta_h SI - \mu S = 0 \\
\beta_e \frac{B}{K + B} + \beta_h SI - (\gamma + \mu)I = 0 \\
\xi I - \delta B = 0 \\
\gamma I - \mu R = 0
\end{align*}
\]

(3)

and we obtain: \( \bar{B} = \frac{\xi A}{\delta} \bar{I}; \bar{S} = \frac{\Lambda}{\mu} - \frac{\gamma + \mu}{\mu} \bar{I} \) et \( \bar{I} \) which is the solution of :

\[
E \bar{I}^2 + F \bar{I} + G = 0
\]

(4)

with

\[
E = -\beta_h (\gamma + \mu) \xi;
F = \beta_h A \xi - (\gamma + \mu) [\beta_e + \mu] \xi + \beta_h \delta K; \]

and \( G = \beta_e A \xi + \beta_h \Lambda K \delta - (\gamma + \mu)(\mu \delta K). \)

Then

- for

\[
R_0 = \frac{\Lambda (K \beta_h \delta + \beta_e \xi)}{\mu \delta (\gamma + \mu) K} > 1
\]

\( G \) is positive and the fact that \( E \) is negative, one has \( \bar{I}_1 \bar{I}_2 = \frac{G}{E} < 0 \) and the equation (4) has two solutions \( \bar{I}_1 < 0 \) et \( \bar{I}_2 > 0 \), one will take account only for \( \bar{I}_2 \) whom is in \( \Omega \)

\[
\bar{I}_2 = -\frac{F}{2E}
\]

- For

\[
R_0 = \frac{\Lambda (K \beta_h \delta + \beta_e \xi)}{\mu \delta (\gamma + \mu) K} = 1
\]
G=0, equation (4) has $I_1$ which is the disease free equilibrium and $I_2 = \frac{-F}{K}$.

The case

$$R_0 = \frac{\Lambda(K\beta h \delta + \beta e \xi)}{\mu \delta (\gamma + \mu) K} < 1$$

is not available, the two solutions are negatives and are not in $\Omega$.

And one has the following result on stability of endemic equilibrium:

**Theorem 2** If $R_0 > 1$ the endemic equilibrium is globally asymptotically stable.

**Proof:** Define a Lyapunov function

$$V(S, I, B, R) = c_1(S - S)^2 + c_2(I - I)^2 + c_3(B - B)^2$$

where $c_1$, $c_2$ and $c_3$ are positive constants to be chosen latter. Then the derivative of $V$ along the positive solution of (1), we obtain:

$$\dot{V} = -2c_1 \left( \frac{\bar{B}}{K + B} + \mu + \beta h I \right) (S - \bar{S})^2 - 2c_1 \frac{SK}{(K + B)(K + B)} (S - \bar{S})(B - \bar{B})$$

$$+ 2c_2 \left( \frac{SK}{(K + B)(K + B)} (I - I)(B - \bar{B}) + 2c_2 \frac{\bar{B}}{K + B} (I - I)(S - \bar{S}) + 2c_2 \beta h \bar{S}(I - I)^2 + 2c_2 \beta h I (S - \bar{S})(I - I) \right)$$

$$- 2c_2 (\gamma + \mu)(I - I)^2 + 2c_3 \xi (I - I)(B - \bar{B}) - 2c_3 (B - \bar{B})^2$$

$$= -2c_1 \left( \frac{\bar{B}}{K + B} + \mu + \beta h I \right) (S - \bar{S})^2 - 2c_2 \left( [\gamma + \mu - \beta h \bar{S}] (I - I)^2 \right) + 2c_2 \beta h \bar{S} (I - I)^2 - 2c_3 (B - \bar{B})^2$$

$$+ 2c_1 \beta h \bar{S} - c_2 (\beta h I + \frac{\bar{B}}{K + B})) (S - \bar{S})(I - I)$$

$$- 2c_1 \frac{SK}{(K + B)(K + B)} (S - \bar{S})(B - \bar{B}) + 2c_2 \frac{SK}{(K + B)(K + B)} + c_3 \xi) (I - I)(B - \bar{B})$$

For $c_1 \beta h \bar{S} - c_2 (\beta h I + \frac{\bar{B}}{K + B}) > 0$ for all $I > 0$ and $c_2 \frac{SK}{(K + B)(K + B)} + c_3 \xi) very small, we can deduce that

$$\dot{V} (S, I, B, R) \leq 0$$

Hence by LaSalle’s principe [3] the endemic equilibrium is globally asymptotically stable on $\Delta$

### 3 Stochastic model

Through this paper, unless otherwise specified, we let $(\Omega, F, \{F_t\}_{t \geq 0}, P)$ be a complete space with filtration $\{F_t\}$ satisfying the usual conditions (i.e. it is right continuous and increasing while $F_0$ contains all null sets). If we replace the contact rate $\beta h$ and $\beta e$ by $\beta h + \frac{\sigma_{1}dW_1}{dt}$ and $\beta e + \frac{\sigma_{1}dW_2}{dt}$
respectively, where $\frac{\sigma_1 dW_1}{dt}$ and $\frac{\sigma_2 dW_2}{dt}$ are white noise (i.e. $W_i$ is Brownian motion), the system (1) becomes:

$$
\begin{align*}
\dot{S} &= \Lambda - \beta_1 e^S \frac{B}{K+B} - \beta_h S I - \mu S - \sigma_1 S I dW_1 - \sigma_2 S \frac{B}{K+B} dW_2 \\
\dot{I} &= \beta_1 e^S \frac{B}{K+B} + \beta_h S I - (\gamma + \mu) I + \sigma_1 S I dW_1 + \sigma_2 S \frac{B}{K+B} dW_2 \\
\dot{B} &= \xi I - \delta B \\
\dot{R} &= \gamma I - \mu R
\end{align*}
$$

(5)

We define the differential operator $L$ associated with 3-dimensional stochastic differential equation:

$$
\begin{align*}
dx(t) &= f(x,t) dt + \phi(x,t) dB(t)
\end{align*}
$$

(6)
on $t \geq 0$ with initial value $x(0) = x_0$. The solution is denoted by $X(t,x_0)$. Assume that $f(0,t) = \phi(t,0) = 0$ for all $t \geq 0$, so (5) has the solution of $x(t)=0$ which is called the trivial solution.

**Definition 1** The solution of system (5) is stochastically ultimately bounded a.s. if for any $\epsilon \in (0,1)$, there exists a positive constant $\varrho = \varrho(\epsilon)$ such that for any initial value $X(t,x_0) \in \mathbb{R}_+^3$, the solution of system (5) has the property:

$$
\limsup_{t \to \infty} P\{|X(t)| \geq \varrho\} < \epsilon
$$

(7)

**Definition 2** The trivial solution $x(t)=0$ of (5) is said to be stable in probability if for all $\epsilon > 0$,

$$
\lim_{x_0 \to 0} \mathbb{P}(\sup_{t \geq 0} |x(t,x_0)| \geq \epsilon) = 0
$$

**Definition 3** The trivial solution $x(t)=0$ of (5) is said to be asymptotically stable if it is stable in probability and moreover

$$
\lim_{x_0 \to 0} \mathbb{P}(\lim_{t \to \infty} x(t,x_0) = 0) = 1
$$

**Definition 4** The trivial solution $x(t)=0$ of (5) is said to be globally asymptotically stable if it is stable in probability and moreover

$$
\mathbb{P}(\lim_{t \to \infty} x(t,x_0) = 0) = 1
$$

**Definition 5** The trivial solution $x(t)=0$ of (5) is said to be almost surely exponentially stable if for all $x_0 \in \mathbb{R}^n$,

$$
\lim_{t \to \infty} \frac{1}{t} \ln |x(t,x_0)| < 0 \quad \text{a.s.}
$$

**Definition 6** The trivial solution $x(t)=0$ of (5) is said to be exponentially $p$ stable if there is a pair of positive constants $C_1$ and $C_2$ such that for all $x_0 \in \mathbb{R}^n$, $\mathbb{E}(|x(t,x_0)|^p) \leq C_1 |x(t,x_0)|^p e^{-C_2 t}$ on $t \geq 0$

Suppose that $V(x,t)$ is a Lyapunov function, then we define the action of $L$ on $V$ by:

$$
LV(x,t) = V_t(x,t) + V_x(x,t) f(x,t) + \frac{1}{2} \text{trace}[\phi^T V_{xx}(x,t) \phi(x,t)].
$$
3.1 Existence and uniqueness of positive solutions

**Theorem 3** For any given \((S(0), I(0), B(0), R(0)) \in \mathbb{R}^4_+\), there is a unique solution \((S(t), I(t), B(t), R(t))\), on \(t \geq 0\) and will remain in \(\mathbb{R}^4_+\) with probability one. Namely, \((S(t), I(t), B(t), R(t)) \in \Omega\) for all \(t \geq 0\).

**Proof:** Since the coefficients of model (5) satisfy the local Lipchitz condition, then there exists a unique local solution on \([0, \tau]\), where \(\tau\) is the explosion time.

Proposition 1 show that \(0 \to \infty\) and \(B(t) < \frac{\xi}{\mu}\), on \(t \in [0, \tau]\).

We, now, want to show that this solution is global, i.e. \(\tau = +\infty\) a.s. Let \(n_0 > 0\) be sufficiently large for for any \((S(0), I(0), B(0), R(0))\) remaining in the interval \([\frac{1}{n_0}, n_0]\). For each integer \(n > n_0\), we define the stopping time:

\[
\tau_n = \inf\{t \in [0, \tau); S(t) \not\in (\frac{1}{n}, n), I(t) \not\in (\frac{1}{n}, n), B(t) \not\in (\frac{1}{n}, n)\text{ or } R(t) \not\in (\frac{1}{n}, n)\}
\]

By reduction to absurdity, we suppose that \(\tau = +\infty\) is false, there is a pair of constant \(T > 0\) and for any \(\varepsilon \in (0, 1)\) such that \(P\{\tau \leq T\} > \varepsilon\). Consequently, there is an integer \(n_1 \geq n_0\) such that

\[
P\{\tau_n \leq T\} \geq \varepsilon, n \geq n_1
\]  

Define a \(C^3\) function \(V : \mathbb{R}^4_+ \to \mathbb{R}\), by

\[
V(S, I, B, R) = (S - \ln S) + (I - \ln I) + (B - \ln B) + (R - \ln R)
\]

defined for \((S(t), I(t), B(t), R(t)) \in \Omega\).

Using Itô formula, we compute

\[
dV(S, I, B, R) = LVdt - \sigma_1(S - I)dW_1(t) - \sigma_2(S - I)\frac{B}{K + B}dW_2(t)
\]  

where

\[
LV = (1 - \frac{1}{S})(\Lambda - \beta_eS\frac{B}{K + B} - \beta_hSI - \mu S - \sigma_1SI dW_1 - \sigma_2 S - \frac{B}{K + B}dW_2)
\]

\[
+ (1 - \frac{1}{I})(\beta_eS\frac{B}{K + B} + \beta_hSI - (\gamma + \mu)I + \sigma_1SI dW_1 + \sigma_2 S - \frac{B}{K + B}dW_2)
\]

\[
+ (1 - \frac{1}{B})(\xi I - \delta B) + (1 - \frac{1}{R})(\gamma I - \mu R) + \frac{1}{2}(\sigma_1^2I^2 + \sigma_2^2\frac{B^2}{(B + K)^2}) + \sigma_1^2S^2 + \sigma_2^2 \frac{S^2B^2}{(B + K)^2}
\]

\[
= \Lambda + 3\mu + \delta + \gamma + \beta_e\frac{B}{K + B} + (\beta_h + \xi)I - \mu(S + I + R) - \beta_hS - \beta_e\frac{B}{K + B}I - \delta B
\]

\[
- \frac{1}{B}\xi I + \frac{1}{2}(\sigma_1^2I^2 + \sigma_2^2\frac{B^2}{(B + K)^2}) + \sigma_1^2S^2 + \sigma_2^2 \frac{S^2B^2}{(B + K)^2}
\]

from(2), we have:

\[
LV \leq \Lambda + 3\mu + \delta + \gamma + \beta_e + (\beta_h + \xi)\frac{\lambda}{\mu} + \sigma_1\frac{(\Lambda/\mu)^2}{\mu^2} + \sigma_2^2 = C
\]

Therefore, we obtain

\[
dV \leq Cdt - \sigma_1(S - I)dW_1(t) - \sigma_2(1 - 1)\frac{B}{K + B}dW_2(t)
\]
By integrating both sides of (10) from 0 to $\tau_k \wedge T$ yields that:
\[
\int_0^{\tau_k \wedge T} dV(S(t), I(t), B(t), R(t)) = \int_0^{\tau_k \wedge T} C dt - \int_0^{\tau_k \wedge T} \sigma_1 (S-I) dW_1(t) - \int_0^{\tau_k \wedge T} \sigma_2 (I/R-1) \frac{B}{K+B} dW_2(t)
\]
where $\tau_n \wedge T = \min \{\tau_n, T\}$. Whence taking expectations of the above inequality leads to
\[
EV(S(\tau_n \wedge T), I(\tau_n \wedge T), B(\tau_n \wedge T), R(\tau_n \wedge T)) \leq V(S(0), I(0), B(0), R(0)) + CT \quad (11)
\]
Set $\Omega_n = \{\tau_n \leq T\}$ for $n > n_1$ by inequality (8), we obtain $P(\Omega_n) \geq \varepsilon$. Note that every $\omega \in \Omega_n$, there exists at least one of $S(\tau_n, \omega)$, $I(\tau_n, \omega)$, $B(\tau_n, \omega)$ and $R(\tau_n, \omega)$ equals either $n$ or $\frac{1}{n}$, hence
\[
V(S(\tau_n, \omega), E(\tau_n, \omega), I(\tau_n, \omega)) \geq (n-1 - \ln n) \wedge \left( \frac{1}{n} - 1 - \ln \frac{1}{n} \right)
\]
as a consequence from (11) one has:
\[
V(S(0), I(0), B(0), R(0)) + CT \geq E[1_{\Omega_n(\omega)} V(S(\tau_n, \omega), E(\tau_n, \omega), I(\tau_n, \omega))] \geq \varepsilon (n-1-\ln n) \wedge \left( \frac{1}{n} - 1 - \ln \frac{1}{n} \right)
\]
where $1_{\Omega_n}$ is the indicator function of $\Omega_n$. Let $n \to +\infty$ lead to the contradiction
\[
+\infty > V(S(0), I(0), B(0), R(0)) + CT = +\infty \quad (12)
\]
So we must have $\tau_\infty = \infty$. Therefore, the solution $S(t), I(t), B(t), R(t)$ of model will not explode at a finite time with probability one. This completes the proof.

**Theorem 4** The solutions of System (5) are stochastically ultimately bounded for any initial value $(S(0), I(0), B(0), R(0)) \in \Omega$

**Proof:** From theorem 3 we know that the solution $(S(t), E(t), I(t))$ will remains in $\mathbb{R}_+^3$ for all $t \geq 0$ with probability 1. Define the functions $V_1 = e^t (S^\theta), V_2 = e^t I^\theta$ with $0 < \theta < 1$. By Itô’s formula, we have:
\[
dV_1 = LV_1 dt - \sigma_1 \theta e^t S^\theta I dW_1(t) - \sigma_2 e^t S^\theta \frac{B}{K+B} dW_2(t)
\]
\[
dV_2 = LV_2 dt + \sigma_1 \theta e^t S^\theta I dW_1(t) + \sigma_2 e^t S^\theta \frac{B}{K+B} dW_2(t) \quad (13)
\]
where
\[
LV_1 = e^t S^\theta (1 + \theta (\frac{A}{S} - \beta_T l - \mu) + \frac{\theta (1-1)S^\theta \sigma_1^2 I^2}{2} + \frac{\theta (1-1)S^\theta \sigma_2^2}{2} \frac{B^2}{K+B})
\]
\[
LV_2 = e^t I^\theta (1 + \theta (\beta_T \frac{SB}{(K+B)} + \beta h S - (\gamma + \mu)) + \frac{\theta (1-1)S^\theta \sigma_2^2 I^2}{2} + \frac{\theta (1-1)S^\theta \sigma_2^2}{2} \frac{B^2}{K+B}) \quad (14)
\]
Thus, there exists the positive constants $C_1$ and $C_2$ such that we have $LV_1 < C_1 e^t$ and $LV_2 < C_2 e^t$. It follows that $e^t E(S^\theta(t)) - E(S^\theta(0)) \leq C_1 e^t$ and $e^t E(I^\theta(t)) - E(I^\theta(0)) \leq C_2 e^t$. Then we get
\[
\limsup_{t \to \infty} E(S^\theta(t)) \leq C_1 < \infty
\]
\[
\limsup_{t \to \infty} E(I^\theta(t)) \leq C_2 < \infty \quad (15)
\]
for \( X(t) = (S(t), I(t)) \in \mathbb{R}_+^2 \), note that

\[
|X(t)|^\theta = (S^2(t) + I^2(t))^\frac{\theta}{2} \leq 2^\frac{\theta}{2} \max\{S^\theta(t), I^\theta(t)\} \\
\leq 2^\frac{\theta}{2} \max\{S^\theta(t) + I^\theta(t)\}
\]  

(16)

consequently

\[
\limsup_{t \to \infty} E|X(t)| \leq 2^\theta (c_1 + c_2)
\]

as a result, there exists a positive constant \( \delta_1 \) such that

\[
\limsup_{t \to \infty} E|\sqrt{X(t)}| < \delta_1
\]  

(17)

now for any \( \varepsilon > 0 \), let \( \delta = \frac{\delta_1^2}{\varepsilon^2} \), then by the Chebychev’s inequality,

\[
P\{|X(t)| > \varepsilon\} \leq \frac{E|\sqrt{X(t)}|}{\sqrt{\delta}}
\]  

(18)

Hence

\[
\limsup_{t \to \infty} P\{|X(t)| > \varepsilon\} \leq \frac{\delta_1}{\sqrt{\delta}} = \varepsilon
\]  

(19)

which gives us the desired assertion

\section{Moment exponential stability}

**Lemma 1** Suppose that there exists a function \( V(t,x) \in C^{1,2}(\mathbb{R}_+^+, \mathbb{R}^n) \), satisfying the following inequalities:

\[
K_1|x| \leq V(t,x) \leq K_2|x|^p
\]

and

\[
LV(t,x) \leq -K_3|x|^p, t \geq 0
\]

where \( p, K_1, K_2 \) and \( K_3 \) are positive constants. Then the equilibrium of (5) is \( p \)th moment exponentially stable. When \( p=2 \), it is usually said to be exponentially stable in mean square and the disease free equilibrium is globally asymptotically stable.

**Lemma 2** Set \( p \geq 2 \) and \( \varepsilon,x,y > 0 \). Then

\[
x^{p-1}y \leq \frac{(p-1)\varepsilon}{p}x^p + \frac{1}{p}y^{p-2}y^p
\]

and

\[
x^{p-2}y^2 \leq \frac{(p-2)\varepsilon}{p}x^p + \frac{2}{p}y^{(2-p)/2}y^p
\]

**Theorem 5** Set \( p \geq 0 \), if \( R_0 < 1 \), the disease free equilibrium is \( p \)th moment exponentially stable in \( \Gamma \)
**Proof:** Set $p \geq 2$ and $(S(0), I(0), B(0), R(0)) \in \Gamma$, in view of theorem 3, the solution of system remains in $\Gamma$. We consider the following Lyapounov function

$$V = \left(\frac{\Lambda}{\mu} - S\right)^p + \frac{1}{p}I^p + B^p + R^p$$

By Ito’s formula; we obtain:

$$LV = -p\left(\frac{\Lambda}{\mu} - S\right)^{p-1}(\Lambda - \beta e S - \beta h S I - \mu S) - \frac{1}{2}p(p-1)\left(\frac{\Lambda}{\mu} - S\right)^{p-2}\left\{\frac{\sigma_1^2 I^2 + \sigma_2^2 S^2}{(K + B)^2}\right\}$$

$$+ \frac{1}{2}p(p-1)\left(\beta e S + \beta h S I - (\gamma + \mu)I\right) + \frac{1}{2}p(p-1)\left(\sigma_1^2 I^2 + \sigma_2^2 S^2\right)\frac{B^2}{(K + B)^2}$$

$$+ pB^{p-1}(\xi I - \delta B) + BR^{p-1}(\gamma I - \mu R)$$

$$= -p\mu\left(\frac{\Lambda}{\mu} - S\right)^p + p\left(\frac{\Lambda}{\mu} - S\right)^{p-1}(\beta e S - \beta h S I)$$

$$- \frac{1}{2}p(p-1)\left(\frac{\Lambda}{\mu} - S\right)^{p-2}\frac{B^2}{(K + B)^2} + \frac{1}{2}(p-1)\left(\sigma_1^2 I^2 + \sigma_2^2 S^2\right)\frac{B^2}{(K + B)^2}$$

$$+ \frac{1}{2}(p-1)\left(\sigma_1^2 I^2 + \sigma_2^2 S^2\right)\frac{B^2}{(K + B)^2} + c_2 p B^{p-1}(\xi I - \delta B)$$

$$+ pR^{p-1}(\gamma I - \mu R)$$

from (2) $S \leq \frac{\Lambda}{\mu}$ and using the fact that $\frac{B}{K + B} < 1$ it exists $\alpha \in [0, 1]$ such that $\frac{\alpha^2}{\mu^2}(x^n + y^n) \leq S^2 x^{n-1} y^2$, we obtain:

$$LV \leq -p\mu\left(\frac{\Lambda}{\mu} - S\right)^p + p\left(\frac{\Lambda}{\mu} - S\right)^{p-1}(\beta e S - \beta h S I)$$

$$- \frac{1}{2}p(p-1)\alpha^2 I^p - \frac{1}{2}p(p-1)\alpha^2 S^p + \frac{1}{2}(p-1)\left(\sigma_1^2 I^2 + \sigma_2^2 S^2\right)\frac{B^2}{(K + B)^2}$$

$$+ \frac{1}{2}(p-1)\left(\sigma_1^2 I^2 + \sigma_2^2 S^2\right)\frac{B^2}{(K + B)^2} + c_2 p B^{p-1}(\xi I - \delta B)$$

$$+ pR^{p-1}(\gamma I - \mu R)$$

$$\leq -p\mu\left(\frac{\Lambda}{\mu} - S\right)^p + p\beta e\left(\frac{\Lambda}{\mu} - S\right)^{p-1} + p\beta h\left(\frac{\Lambda}{\mu} - S\right)^{p-1} I - \frac{1}{2}p(p-1)\alpha^2 I^p$$

$$- \frac{1}{2}p(p-1)\alpha^2 S^p + \frac{1}{2}(p-1)\left(\sigma_1^2 I^2 + \sigma_2^2 S^2\right)\frac{B^2}{(K + B)^2}$$

$$+ \frac{1}{2}(p-1)\left(\sigma_1^2 I^2 + \sigma_2^2 S^2\right)\frac{B^2}{(K + B)^2} + c_2 p B^{p-1}(\xi I - \delta B) + \gamma p R^{p-1} I - \mu R p$$

now, applying lemma 1, we obtain:

$$LV \leq -[p\mu + \frac{1}{2}(p-1)\alpha^2] - (p-1)\alpha^2 I^p - p\beta e\left(\frac{\Lambda}{\mu} - S\right)^p$$
\[-\frac{1}{2}p(p-1)\alpha\sigma^2_1 \mu^2 + (\gamma + \mu) - \beta_h \Lambda \varepsilon^{1-p} - (\beta_e + \beta_c) \Lambda \mu - (\xi + c_3\gamma)\varepsilon^{1-p}]I_p
\]
\[-(p\delta - (p-1)\varepsilon\xi)B^p\]

We choose \(\varepsilon\) sufficiently small such that the coefficients of \((\Lambda - S)B^p\) and \(R^p\) be negative. We also, can choose \(c_1, c_2\) and \(c_3\) positive such that the coefficient of \(I^p\) be negative. according to lemma 1 the proof is complete.

### 4.1 Almost sure exponential stability of cholera model

In this subsection we investigate stochastic stability of the disease free equilibrium \(E_0 = (\Lambda, 0, 0, 0)\) almost sure exponential stability. The following result gives a sufficient condition for the almost surely exponential stability.

**Theorem 6** If \(R_0 < 1\) and \(2\sigma^2_1\mu < \beta^2_h\) then the disease free equilibrium is almost surely exponential stable in \(\Omega\)

**Proof:** Define

\[V = \ln((\Lambda/\mu - S) + I + B + R)\]

Using the Ito formula, we have

\[LV = \frac{1}{\Lambda/\mu - S + I + B + R} \left[ -\Lambda + 2(\beta_e S \frac{B}{K + B} + \beta_h S I) + \mu S - \mu I - \delta B + \xi I - \mu R \right] \]

\[= \frac{1}{\Lambda/\mu - S + I + B + R} \left[ -\Lambda + \beta_h S I \right] + \mu S - \mu R \]

\[\leq \frac{1}{\Lambda/\mu - S + I + B + R} \left[ -\Lambda + \beta_h S I \right] + \mu S - \mu R \]

\[\leq \frac{2\sigma^2_1 S^2 I^2}{\Lambda/\mu - S + I + B + R} \]

\[\leq \frac{2\sigma^2_1 S^2 I^2}{\Lambda/\mu - S + I + B + R} \]

Set \(U = \frac{2\sigma_1 SI}{\Lambda/\mu - S + I + B + R}\), on the fact that

\[2\beta_h U - 2\sigma^2_1 U - \mu = -2\sigma^2_1(U - (\frac{\beta_h}{2\sigma^2_1} + (\frac{\beta^2_h - 2\sigma^2_1\mu}{2\sigma^2_1})/2\sigma^2_1)\]

we have

\[dV = (-2\sigma^2(U - (\frac{\beta_h}{2\sigma^2_1} + (\frac{\beta^2_h - 2\sigma^2_1\mu}{2\sigma^2_1})/2\sigma^2_1) dt + 2\sigma_1 U dW(t)\]

\[\leq (+\frac{\beta^2_h - 2\sigma^2_1\mu}{2\sigma^2_1}/2\sigma^2_1) dt + 2\sigma_1 U dW_1(t)\]

Integrating both sides from 0 to \(t\), we have

\[\log(\Lambda/\mu - S + I + B + R) \leq \log(\Lambda/\mu - S(0) + I(0) + B(0) + R(0)) + (\frac{\beta^2_h - 2\sigma^2_1\mu}{2\sigma^2_1})t + G(t) \quad (20)\]

From theorem 3
4.2 Almost sure convergence

4.2.1 Extinction

**Theorem 7** If \( R_0 < 1 \), then \((I(t), R(t))\) converge almost surely exponentially to \((0,0)\), i.e. the disease dies out with probability 1.

**Proof:** Let \((S_0, I_0, B_0, R_0) \in \Delta\). Since \( R_0 < 1 \), let us define the Lyapunov function

\[
V(S, I, B, R) = \ln(I(t) + \theta R(t))
\]

such that \( \theta > 0 \)

By Ito’s formula and the fact that \((S(t), I(t), R(t)) \in \Delta\) we have:

\[
dV = \frac{1}{(I(t) + \theta R(t))} \left( \beta c S \frac{B}{K + B} + \beta h S I - (\gamma + \mu)I + \theta(\gamma I + \mu R) \right) dt - \frac{1}{(I(t) + \theta R(t)^2} \left( \sigma^2 S^2 I^2 + \sigma^2 S \frac{B^2}{(K + B)^2} \right) + \frac{1}{(I(t) + \theta R(t))} (\sigma_1 S I dW(t) + \sigma_2 S \frac{B}{K + B} dW_2(t))
\]

\[
\leq \frac{1}{(I(t) + \theta R(t))} \left( \frac{\beta c S}{K + B} + \beta h S I - (\gamma + \mu)I + \theta(\gamma I + \mu R) \right) dt + \frac{1}{(I(t) + \theta R(t))} (\sigma_1 S I dW(t) + \sigma_2 S \frac{B}{K + B} dW_2(t))
\]

\[
\leq \frac{1}{(I(t) + \theta R(t))} \left( \frac{\beta c S}{K + B} + \beta h S I - (\gamma + \mu)I + \theta(\gamma I + \mu R) \right) dt + \frac{1}{(I(t) + \theta R(t))} (\sigma_1 S I dW(t) + \sigma_2 S \frac{B}{K + B} dW_2(t))
\]

\[
\leq \frac{1}{(I(t) + \theta R(t))} \left( \beta c + \beta h \frac{\lambda}{\mu} - \beta \frac{\lambda}{\mu} \gamma - \beta \frac{\lambda}{\mu} \mu \right) dt + \frac{1}{(I(t) + \theta R(t))} (\sigma_1 S I dW(t) + \sigma_2 S \frac{B}{K + B} dW_2(t))
\]

\[
\leq -\bar{\theta} dt + \frac{1}{(I(t) + \theta R(t))} (\sigma_1 S I dW(t) + \sigma_2 S \frac{B}{K + B} dW_2(t))
\]

where \( \bar{\theta} = \min\{\gamma + \beta h \frac{\lambda}{\mu} - \gamma \theta, \gamma \mu\} \). By integrating we check

\[
\ln(I(t) + \theta R(t)) \leq \ln(I_0 + \theta R_0) - \bar{\theta} t + \int_0^t \frac{1}{(I(t) + \theta R(t))} (\sigma_1 S I dW(t) + \sigma_2 S \frac{B}{K + B} dW_2(t)) \quad (21)
\]

Let \( G(t) \) a martingale defined by \( G(t) = \int_0^t \frac{1}{(I(t) + \theta R(t))} (\sigma_1 S I dW(t) + \sigma_2 S \frac{B}{K + B} dW_2(t)) \). In virtue of theorem (3), the solution of model (5) remains in \( \Delta\) then by the strong law of large number for local martingale, we have:

\[
\lim_{t \to +\infty} \frac{1}{t} \int_0^t \frac{1}{(I(t) + \theta R(t))} (\sigma_1 S I dW(t) + \sigma_2 S \frac{B}{K + B} dW_2(t)) = 0
\]

we deduce from (21) that

\[
\lim_{t \to +\infty} \sup \frac{1}{t} \ln(I(t) + \theta R(t)) \leq -\bar{\theta} < 0
\]

This completes the proof.
5 Persistence

**Definition 7** System (5) is said to be persistent in the mean, if

\[
\lim_{t \to +\infty} \inf \frac{1}{t} \int_0^t I(t) \, dt > 0
\]

**Theorem 8** If \( \beta_h \frac{\Lambda}{\mu(\gamma + \mu)} > 1 \) system (5) is said to be persistent in the mean, moreover we have:

\[
\begin{align*}
\lim_{t \to +\infty} \inf \frac{1}{t} \int_0^t I(t) \, dt &> (\gamma + \mu)(\beta_h \frac{\Lambda}{\mu(\gamma + \mu)} - 1) \frac{\Lambda(\Lambda - 1) + \mu^2}{\beta_h \mu \Lambda}; \\
\lim_{t \to +\infty} \inf \frac{1}{t} \int_0^t R(t) \, dt &\geq (\gamma(\gamma + \mu)(\beta_h \frac{\Lambda}{\mu(\gamma + \mu)} - 1) \frac{\Lambda(\Lambda - 1) + \mu^2}{\beta_h \mu I}; \\
\lim_{t \to +\infty} \inf \frac{1}{t} \int_0^t \left( \frac{\Lambda}{\mu} - S(t) \right) dt &> (\gamma + \mu)^2 (\beta_h \frac{\Lambda}{\mu(\gamma + \mu)} - 1) \frac{\Lambda(\Lambda - 1) + \mu^2}{\beta_h \mu I}
\end{align*}
\]

The proof is based on the following lemma

**Lemma 3** [2] Let \( g \in C([0, \infty) \times \Omega, [0, \infty)) \) and \( G \in C([0, \infty) \times \Omega, [0, \infty)) \). If there exists positive constants \( \lambda_0 \) and \( \lambda \) such that:

\[
\ln g(t) \geq \lambda_0 t - \lambda \int_0^t g(s) ds + G(t) \quad \text{a.s.}
\]

for all \( t \geq 0 \), and \( \lim_{t \to +\infty} \frac{G(t)}{t} = 0 \) a.s., then

\[
\lim_{t \to +\infty} \inf \frac{1}{t} \int_0^t g(t) dt \geq \frac{\lambda_0}{\lambda} \quad \text{a.s.}
\]

**Proof of theorem:** We consider the following Lyapunov function

\[
V(S, I, R) = \alpha_1 (S + I + R) + \alpha_2 S + \ln I
\]

where \( \alpha_1 \) and \( \alpha_2 \) are defined below By Ito’s formula we have:

\[
\begin{align*}
dV &= \alpha_1 [\Lambda - \mu (S + I + R)] + \alpha_2 [(\Lambda - \beta_c S) \frac{B}{K + B} - \beta_h SI - \mu S] dt - \sigma_1 S dt dW_1 - \sigma_2 S \frac{B}{K + B} dt dW_2 \\
&\quad + (\beta_c S \frac{B}{I(K + B)} + \beta_h S - (\gamma + \mu) + \sigma_2^2 S^2) dt + \sigma_1 S dW_1 + \sigma_2 S \frac{B}{I(K + B)} dW_2 \\
&\geq \alpha_1 [(\Lambda - \mu S) + I + R)] + \alpha_2 [(\Lambda - \mu S) - \beta_c S \frac{B}{K + B} - \beta_h \frac{\Lambda}{\mu} I] dt - \sigma_1 S dt dW_1 - \sigma_2 S \frac{B}{K + B} dt dW_2 \\
&\quad + (\beta_c S \frac{B}{I(K + B)} + \beta_h \Lambda \frac{1}{\mu} - \beta_h (\frac{\Lambda}{\mu} - S) - (\gamma + \mu) + \sigma_2^2 S^2) dt + \sigma_1 S dW_1 + \sigma_2 S \frac{B}{I(K + B)} dW_2 \\
&\geq (\gamma + \mu)(\beta_h \Lambda \frac{1}{\mu(\gamma + \mu)} - 1) + [(\alpha_1 + \alpha_2) \mu - \beta_h] (\frac{\Lambda}{\mu} - S) - (\alpha_2 - \frac{\mu}{\Lambda}) \beta_c S \frac{B}{K + B} + (\alpha_1 - \beta_h \frac{\Lambda}{\mu}) I
\end{align*}
\]
Some stochastic properties of cholera model

\[ + (1 - \alpha_2 I) S \sigma_1 S dW_1 + \left( \frac{1}{I} - \alpha_2 \right) \sigma_2 S \frac{B}{K + B} dW_2 \]

with \( \alpha_2 = \frac{\mu}{K} \) and \( \beta_h = (\alpha_1 + \alpha_2) \mu \) one has:

\[ dV \geq (\gamma + \mu)(\beta_h \frac{A}{\mu(\gamma + \mu)} - 1) dt \]

\[ - \beta_h (\frac{A - 1}{\mu} + \frac{\mu}{A}) I \]

\[ + (1 - \alpha_2 I) S \sigma_1 S dW_1 + \left( \frac{1}{I} - \alpha_2 \right) \sigma_2 S \frac{B}{K + B} dW_2 \]

and integrating both sides, one obtains:

\[ V(S, I, R) \geq V(S_0, I_0, R_0) + (\gamma + \mu)(\beta_h \frac{A}{\mu(\gamma + \mu)} - 1) t - \beta_h (\frac{A - 1}{\mu} + \frac{\mu}{A}) \int_0^t I \]

\[ + (1 - \alpha_2 I) \sigma_1 \int_0^t S dW_1 + \left( \frac{1}{I} - \alpha_2 \right) \sigma_2 \int_0^t S \frac{B}{K + B} dW_2 \]

Hence

\[ \ln I \geq (\gamma + \mu)(\beta_h \frac{A}{\mu(\gamma + \mu)} - 1) t - \beta_h (\frac{A - 1}{\mu} + \frac{\mu}{A}) \int_0^t I \]

where

\[ D(t) = V(S_0, I_0, R_0) - (\alpha_1 + \alpha_2) S - \alpha_1 I - \alpha_1 R + (1 - \alpha_2) \sigma_1 \int_0^t S dW_1 + \left( \frac{1}{I} - \alpha_2 \right) \sigma_2 \int_0^t S \frac{B}{K + B} dW_2 \]

From the theorem of the large numbers for martingales, we deduce:

\[ \lim_{t \to +\infty} \frac{D(t)}{t} = 0 \]

By using lemma 2, we obtain:

\[ \ln I \geq (\gamma + \mu)(\beta_h \frac{A}{\mu(\gamma + \mu)} - 1) \frac{\Lambda(\Lambda - 1) + \mu^2}{\beta_h \mu \Lambda} \]

From (2) we have:

\[ \hat{R} = \gamma I - \mu R \]

hence

\[ \liminf_{t \to \infty} R(t) = \frac{\gamma}{\mu} \liminf_{t \to \infty} (I(t)) + \liminf_{t \to \infty} \frac{R_0 - R}{\mu t} \geq \gamma (\gamma + \mu)(\beta_h \frac{A}{\mu(\gamma + \mu)} - 1) \frac{\Lambda(\Lambda - 1) + \mu^2}{\beta_h \mu \Lambda} \]

for the last inequality, one has:

\[ dN = d(S + I + R) = (\mu(\frac{\Lambda}{\mu} - S) - \mu I - \mu R) dt \]

Hence

\[ \liminf_{t \to \infty} \frac{\Lambda}{\mu} - S = \lim_{t \to \infty} \frac{N - N_0}{\mu t} + \mu \liminf_{t \to \infty} I(t) + \mu \liminf_{t \to \infty} R(t) \]

\[ \geq \mu (\gamma + \mu)^2 (\beta_h \frac{A}{\mu(\gamma + \mu)} - 1) \frac{\Lambda(\Lambda - 1) + \mu^2}{\beta_h \mu \Lambda} ; \]
References


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