Stability of Some Thermoelastic Systems with Internal Time-Varying Delay

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Abstract

In this paper we consider a thermoelastic type system with Cattaneo’s law and internal time-varying delay. Under suitable assumption on the weigh of delay, we prove the exponential stability of this system by using suitable energy functionals. More we show that the exponential stability of thermoelastic system with Cattaneo’s law implies a polynomial stability of the corresponding thermoelastic system with the Fourier’s law, where the result for the polynomial stability was proved recently by Borichev- Tomilov in [1].

Keywords: Thermoelastic of second sound, time-varying delay, Fourier’s law, Cattaneo’s law, exponential stability, polynomial stability, energy method.

1 Introduction

The theory of thermoelasticity with Fourier’s law for heat conduction was studied by several authors see for example [10, 14]. Rivera et al [15] proved the uniform rate of decay for the solution in two or three dimensions. Lebeau and Zuazua [8] proved that the decay rate is never uniform when the domain is convex. It is known that the classical thermoelastic system

\[
\begin{aligned}
\ddot{w}_1(t) - \alpha \partial_{xx} w_1(t) + \beta \partial_x \theta(t) &= 0, \quad t \geq 0, \\
\dot{\theta}(t) - k \partial_{xx} \theta(t) + \delta \partial_x w_1(t) &= 0, \quad t \geq 0, \\
w_1(0) = w_1^0, \quad \dot{w}_1(0) = w_1^1, \quad \theta(0) = \theta^0,
\end{aligned}
\]

(1.1)
with various boundary conditions is exponentially stable. Thus, the impact of dissipation is given through heat conduction, modeled by Fourier’s law and connected to the displacement by system (1.1), is strong enough also to damp the displacement in an exponential manner. But it’s not true in higher dimensions. In general for n-dimensional case it is not true that the total energy associated with the solution of the thermoelastic system decay to zero as was shown in pioneering work of Dafermos [3]. For more literature on the subject, we can see the book of Jiang and Racke [6]. To overcome the physical paradox of infinite speed for the propagation of heat signal in the classical equation, one approach is to introduce the Cattaneo’s law [9, 12] to replace the Fourier’s law for heat conduction. Moreover, introducing a delay term in the internal feedback of the thermoelastic system, Racke [13] proved that this system is unstable. In many cases, the presence of a small delay terms may be a source of instability (see [4]) and for stabilizing this system, we need additional conditions or control terms. In this paper we characterize the stability of thermoelastic system with second sound and internal time delay. The delay function is admitted to be time-varying delay with a priority given upper bound on its derivative (see [11]). Our approach is to use the energy method, to show that if the coefficient of the delay damping terms is smaller than the one of undelay damping terms, we have the exponential decay of some energy function for thermoelastic of second sound system. Moreover, in this paper we prove that the exponential stability of this system implies a polynomial stability of the corresponding thermoelastic system with the Fourier’s law, using a polynomial estimation of the resolvent of its generator obtained by Borichev-Tomilov [1].

Let us begin by the following system

\[
\ddot{w}_1(t, x) - \lambda \partial_{xx} w_1(t, x) + k_1 \dot{w}_1(t, x) + k_2 \dot{w}_1(t - \tau(t)) + \mu \partial_x w_2(t, x) = 0,
\]

(1.2)

\[
\dot{w}_2(t, x) + \partial_x w_3(t, x) + \mu \partial_x \dot{w}_1(t, x) = 0,
\]

(1.3)

\[
\rho \dot{w}_3(t, x) + w_3(t, x) + \partial_x w_2(t, x) = 0,
\]

(1.4)

\[
w_1(t, 0) = w_1(t, 1) = 0, w_2(t, 0) = w_2(t, 1) = 0,
\]

(1.5)

\[
w_1(0, x) = w_1^0(x), \dot{w}_1(0, x) = w_1^1(x), w_2(0, x) = w_2^0, w_3(0, x) = w_3^0(x),
\]

(1.6)

\[
\dot{w}_1(s) = f_0(s), \quad s \in (-\tau(0), 0),
\]

(1.7)

where \((t, x) \in (0, +\infty) \times (0, 1)\) and \(\lambda, \mu, \rho, k_1, k_2\) are positive constants with \(k_2 < k_1\), and \(f_0\) the history function. The function

\[
\tau \in W^{2,\infty}([0, T]), \quad T > 0,
\]

(1.8)

is the time-varying delay satisfies

\[
0 < m \leq \tau(t) \leq M, \quad t > 0,
\]

(1.9)

\[
\dot{\tau}(t) < 1, \quad t > 0,
\]

(1.10)

where \(m\) and \(M\) are positive constants, and \(\rho > 0\) is a constant called the thermal relaxation time. In the absence of this time-lag i.e., \(\rho = 0\), the thermoelastic system with Cattaneo’s law is reduced to the classical thermoelastic system with Fourier’s law. Now we introduce \(\overline{w}_2(x, t) := w_2(x, t) - \int_0^1 w_2^0(x) dx\). Then by (1.3) and boundary conditions (1.6), we have \(\int_0^1 \overline{w}_2(x, t) dx = 0\).
\((w_1, \bar{w}_2, w_3)\) satisfies the same equations (1.2)-(1.7). In what follows we will work with \(\bar{w}_2\), but we denote it by \(w_2\) for simplicity.

Let us introduce a general abstract system which includes system (1.2)-(1.7) as a particular example. For this, let \(A_1 : \mathcal{D}(A_1) \subset H \to H\) is positive self adjoint with compact inverse and \(H\) be Hilbert space (which will be identified to its dual space) equipped with norm and inner product denoted respectively by \(\|\cdot\|_H\), \(\langle \cdot, \cdot \rangle_H\). We introduce the scale of Hilbert spaces \(H_\alpha\), \(\alpha \in \mathbb{R}\), as follows: for every \(\alpha \geq 0\), \(H_\alpha = \mathcal{D}(A_1^{\alpha})\), with the norm \(\|z\|_\alpha = \|A_1^{\alpha/2}z\|_H\). The space \(H_{-\alpha}\) is defined by duality with respect to the pivot space \(H\) as follows: \(H_{-\alpha} = [H_\alpha]^*\), for \(\alpha > 0\). The operator \(A_1\) can be extended (or restricted) to each \(H_\alpha\), such that it becomes a bounded operator

\[
A_1 : H_\alpha \to H_{\alpha-1}, \quad \forall \alpha \in \mathbb{R}. \tag{1.11}
\]

We assume that the operator \(A_1\) can be written as \(A_1 := A_2^*A_2\), where \(A_2 \in \mathcal{L}(H, H_{-\frac{1}{2}})\), which can be extended (or restricted) to \(H_\alpha\), such that it becomes an operator of \(\mathcal{L}(H_\alpha, H_{\alpha-\frac{1}{2}})\), \(\alpha \in \mathbb{R}\). The control operator \(B\) be a bounded linear from \(U\) to \(H_{-\frac{1}{2}}\), and \(B^* \in \mathcal{L}(H_{\frac{1}{2}}, U)\), where \(U\) be a Hilbert space (which will be identified to its dual space). The coupling operator \(C \in \mathcal{L}(H, H_{-\frac{1}{2}})\) can be extended or restricted to \(H_\alpha\), such that it belongs to \(\mathcal{L}(H_\alpha, H_{\alpha-\frac{1}{2}})\), \(\alpha \in \mathbb{R}\).

We consider the following abstract system of thermoelasticity type

\[
\ddot{w}_1(t) + A_1w_1(t) + k_1BB^*\dot{w}_1(t) + k_2BB^*\dot{w}_1(t - \tau(t)) + Cw_2(t) = 0, \tag{1.12}
\]

\[
\dot{w}_2(t) + A_2w_2(t) - C^*\dot{w}_1(t) = 0, \tag{1.13}
\]

\[
\rho \ddot{w}_3(t) + w_3(t) - A_2^*w_2(t) = 0, \tag{1.14}
\]

\[
w_1(0) = w_1^0, \quad \dot{w}_1(0) = w_1^1, \quad w_2(0) = w_2^0, \quad w_3(0) = w_3^0, \quad \dot{w}_1(s) = f_0(s), s \in (-\tau(0), 0). \tag{1.15}
\]

In order to see that abstract system (1.12)-(1.15) includes system (1.2)-(1.7) as a particular example, we set \(H_1 = H_2 = U = L^2(0,1), \quad H_{1,\frac{1}{2}} = H_0^1(0,1), \quad \text{and } A_1 = -\lambda \partial_{xx}, \quad \mathcal{D}(A_1) = H^2(0,1) \cap H_0^1(0,1), \quad A_2 = \partial_x, \quad C = \mu \partial_x, \quad \mathcal{D}(A_2) = \mathcal{D}(C) = H^1(0,1), \quad A_2^* = -\partial_x, \quad C^* = -\mu \partial_x, \quad \mathcal{D}(A_2^*) = \mathcal{D}(C^*) = H_0^1(0,1), \quad BB^* = I, \quad f_0 \in L^2(-\tau(0), 0; U).

The paper is organized as follows. In section 2, we show the well-posedness of the evolution system (1.12)-(1.15). Then in section 3, we devote to the proof of the exponential stability of the thermoelastic system with Cattaneo’s law. In the last section, we proof the polynomial stability of the corresponding thermoelastic system with the Fourier’s law.

### 2 Well posedness of the problem

Setting \(z(\delta, t) = B^*\dot{w}_3(t - \tau(t)\delta), \quad \delta \in (0,1), \quad t > 0\). Then the problem (1.12)-(1.15) is equivalent to the following one

\[
\ddot{w}_1(t) + A_1w_1(t) + k_1BB^*\dot{w}_1(t) + k_2Bz(1) + Cw_2(t) = 0, \tag{2.16}
\]
\[
\dot{w}_2(t) + A_2w_3(t) - C^* \dot{w}_1(t) = 0, \\
\rho \ddot{w}_3(t) + w_3(t) - A_2^* w_2(t) = 0, \\
\tau(t) \ddot{z}(\delta, t) + (1 - \dot{\tau}(t)\delta) \partial_\delta z(\delta, t) = 0, \\
z(0, t) = B^* \dot{w}_1(t), \\
w_1(0) = w_1^0, \dot{w}_1(0) = w_1^1, w_2(0) = w_2^0, w_3(0) = w_3^0, \\
z(\delta, 0) = z^0 = B^* \dot{w}_1(-\delta\tau(0)) = B^* f_0(-\delta\tau(0)),
\]

where \( t > 0 \) and \( \delta \in (0, 1) \).

Taking \( \rho = 0 \) in the above system, then we obtain the following thermoelastic system with Fourier's law

\[
\dot{w}_1(t) + A_1w_1(t) + k_1BB^* \dot{w}_1(t) + k_2Bz(1) + Cw_2(t) = 0, \\
\dot{w}_2(t) + A_2A_2^* w_2(t) - C^* \dot{w}_1(t) = 0, \\
\tau(t) \ddot{z}(\delta, t) + (1 - \dot{\tau}(t)\delta) \partial_\delta z(\delta, t) = 0, \\
z(0, t) = B^* \dot{w}_1(t), \\
w_1(0) = w_1^0, \dot{w}_1(0) = w_1^1, w_2(0) = w_2^0, \\
z(\delta, 0) = z^0 = B^* \dot{w}_1(-\delta\tau(0)) = B^* f_0(-\delta\tau(0)).
\]

Now, we introduce the Hilbert space \( \mathcal{H} := H_{1/2} \times H \times H \times H \times L^2(0, 1; U) \), equipped with the usual inner product

\[
\begin{pmatrix}
w_1 \\
v \\
w_2 \\
w_3 \\
z_1
\end{pmatrix}
\cdot
\begin{pmatrix}
\dot{w}_1 \\
\dot{v} \\
\dot{w}_2 \\
\dot{w}_3 \\
\dot{z}_1
\end{pmatrix} = \langle w_1, \dot{w}_1 \rangle_{H_{1/2}} + \langle v, \dot{v} \rangle_H + \langle w_2, \dot{w}_2 \rangle_H + \rho \langle w_3, \dot{w}_3 \rangle_H + \int_0^1 \langle z_1(s), \dot{z}_1(s) \rangle_U \, ds.
\]

The space \( \mathcal{H} \) is endowed with this inner product is a Hilbert space, and the associated norm of this inner product is equivalent to the canonical norm of \( \mathcal{H} \). Let the Hilbert space \( \mathcal{H}_F := H_{1/2} \times H \times H \times L^2(0, 1; U) \), equipped with the usual inner product

\[
\begin{pmatrix}
w_1 \\
v \\
w_2 \\
z_1
\end{pmatrix}
\cdot
\begin{pmatrix}
\dot{w}_1 \\
\dot{v} \\
\dot{w}_2 \\
\dot{z}_1
\end{pmatrix} = \langle w_1, \dot{w}_1 \rangle_{H_{1/2}} + \langle v, \dot{v} \rangle_H + \langle w_2, \dot{w}_2 \rangle_H + \int_0^1 \langle z_1(s), \dot{z}_1(s) \rangle_U \, ds.
\]

To study the well-posedness of the system (2.16)-(2.22) and (2.23)-(2.28), we write them as the first order evolution equation in \( \mathcal{H} \) and in \( \mathcal{H}_F \) respectively

\[
\frac{d\eta}{dt} = A(t)\eta, \quad \eta \in \mathcal{H}, \\
\eta^0 = (w_1^0, w_1^1, w_2^0, w_3^0, z^0),
\]

and

\[
\frac{d\tilde{\eta}}{dt} = A_F(t)\tilde{\eta}, \quad \tilde{\eta} \in \mathcal{H}_F, \\
\tilde{\eta}^0 = (w_1^0, w_1^1, w_2^0, z^0),
\]

\[
\text{where } t > 0 \text{ and } \delta \in (0, 1).
\]
where \( A(t) \) is the unbounded linear operator defined by

\[
A(t) : \mathcal{D}(A(t)) \subset \mathcal{H} \rightarrow \mathcal{H}, \quad A(t) \begin{pmatrix} w_1 \\ v \\ w_2 \\ w_3 \\ z \end{pmatrix} = \begin{pmatrix} v \\ -A_1w_1 - k_1BB^*v - k_2Bz(1) - Cw_2 \\ -A_2w_3 + C^*v \\ \frac{1}{p}(-w_3 + A_3^*w_2) \\ \frac{\tau(t)\delta - 1}{\tau(t)} \partial_\delta z \end{pmatrix},
\]

with \( \mathcal{D}(A(t)) = \{(w_1, v, w_2, w_3, z) \in H_1 \times H_{1/2} \times H_{1/2} \times H^1, (w_1, v, w_2, w_3, z) \in H \}, \)

and \( A_F(t) \) is the unbounded linear operator defined by

\[
A_F(t) : \mathcal{D}(A_F(t)) \subset \mathcal{H} \rightarrow \mathcal{H}, \quad A_F(t) \begin{pmatrix} w_1 \\ v \\ w_2 \\ z \end{pmatrix} = \begin{pmatrix} v \\ -A_1w_1 - k_1BB^*v - k_2Bz(1) - Cw_2 \\ -A_2A_3w_2 + C^*v \\ \frac{\tau(t)\delta - 1}{\tau(t)} \partial_\delta z \end{pmatrix},
\]

with \( \mathcal{D}(A_F(t)) = \{(w_1, v, w_2, z) \in H_1 \times H_{1/2} \times H_{1/2} \times H^1, (w_1, v, w_2, z) \in H \}, \)

Notice that the domain of the operators \( A(t) \) and \( A_F(t) \) are independent of the time \( t \), i.e.

\[
\mathcal{D}(A(t)) = \mathcal{D}(A(0)), \quad t > 0,
\]

\[
\mathcal{D}(A_F(t)) = \mathcal{D}(A_F(0)), \quad t > 0.
\]

We define on the Hilbert space \( \mathcal{H} \) the following time inner product

\[
\left\langle \begin{pmatrix} w_1 \\ v \\ w_2 \\ w_3 \\ z_1 \end{pmatrix}, \begin{pmatrix} \bar{w}_1 \\ \bar{v} \\ \bar{w}_2 \\ \bar{w}_3 \\ \bar{z}_1 \end{pmatrix} \right\rangle = \langle w_1, \bar{w}_1 \rangle_{H_{1/2}} + \langle v, \bar{v} \rangle_{H} + \langle w_2, \bar{w}_2 \rangle_{H} + \rho \langle w_3, \bar{w}_3 \rangle_H + \xi \tau(t) \int_0^1 \langle z_1(s), \bar{z}_1(s) \rangle_U ds,
\]

with associated norm \( \| \|_t \), where \( \xi \) is a fixed positive constant satisfying

\[
\frac{k_2}{\sqrt{1 - d}} \leq \xi \leq 2k_1 - \frac{k_2}{\sqrt{1 - d}}.
\]

and \( d \) is a constant such that

\[
\bar{\tau}(t) \leq d < \frac{3}{4}, \quad t > 0,
\]

note that from \( k_2 < k_1 \) and (2.36), the above constant \( \xi \) exists.

The following fundamental theorem give existence and uniqueness results (for the proof see [7, Theorem 1.9]).

**Theorem 2.1.** Assume that

(1) \( \mathcal{D}(A(0)) \) is a dense subset of \( \mathcal{H} \),
(2) The condition (2.33) holds,
(3) for all \( t \in [0, T] \), \( \mathcal{A}(t) \) generates a strongly continuous semigroup on \( \mathcal{H} \) and the family \( \mathcal{A} = \{ \mathcal{A}(t) : t \in [0, T] \} \) is stable with stability constants \( C \) and \( \gamma \) independent of \( t \), i.e., the semigroup \( (S_t(s))_{s \geq 0} \) generated by \( \mathcal{A}(t) \) satisfies \( \| S_t(s)Y \|_\mathcal{H} \leq C e^{\gamma s} \| Y \|_\mathcal{H} \), for all \( Y \in \mathcal{H} \) and \( s \geq 0 \),
(4) \( \partial_t \mathcal{A}(t) \) belongs to \( L^\infty([0, T], B(\mathcal{D}(\mathcal{A}(0)), \mathcal{H})) \), the space of equivalent classes of essentially bounded, strongly measure functions from \([0, T]\) into the set \( B(\mathcal{D}(\mathcal{A}(0)), \mathcal{H}) \) of bounded operators from \( \mathcal{D}(\mathcal{A}(0)) \) into \( \mathcal{H} \).

Then, problem (2.29) has a unique solution \( Y \in C(0, \infty; \mathcal{D}(\mathcal{A}(0))) \cap C^1(0, \infty; \mathcal{H}) \), for all initial datum in \( \mathcal{D}(\mathcal{A}(0)) \).

Now to prove the existence and uniqueness result, as in [11] we check the above assumptions for our problem (2.29), and we show the following variable norm technique of Kato [7].

\[
\frac{\| Y \|_t}{\| Y \|_s} \leq e^{\frac{c}{m}|t-s|} \quad \forall t, \ s \in [0, T],
\]

(2.37)

where \( Y = (w_1, v, w_2, w_3, z)_T \) and \( c \) is positive constant. For all \( s, t \in [0, T] \), we have
\[
\| Y \|^2_t - \| Y \|^2_s e^{\frac{c}{m}|t-s|} = (1 - e^{\frac{c}{m}|t-s|}) [\| w_1 \|^2 + \| v \|^2 + \| w_2 \|^2 + \rho \| w_3 \|^2]
\]
\[+\xi (\tau(t) - \tau(s) e^{\frac{c}{m}|t-s|}) \int_0^T \| z(\delta) \|^2 d\delta.
\]

From (1.8), (1.9) and (1.10), \( \tau \) and \( \tau \) are bounded and
\[
\tau(t) = \tau(s) + \dot{\tau}(s_0)(t-s), \quad \text{where} \ s_0 \in (s, t),
\]

then
\[
\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{c}{m} |t-s| \leq e^{\frac{c}{m}|t-s|},
\]

this archives the proof of (2.37).

By (2.33), the domain \( \mathcal{D}(\mathcal{A}(t)) \) is independent of the time \( t \), and the density of \( \mathcal{D}(\mathcal{A}(0)) \) is obvious. We take \( Y = (w_1, v, w_2, w_3, z)_T \in \mathcal{D}(\mathcal{A}(t)) \), for a fixed \( t \), then

\[
\mathcal{A}(t)Y = \begin{pmatrix}
  v \\
  -A_1w_1 - Cw_2 - k_2B^*v - k_2Bz(1) \\
  -A_2w_3 + C^*v \\
  \frac{1}{\tau(t)}(-w_3 + A^*_2w_2) \\
  \frac{1}{\tau(t)}(1 - \delta_5z)
\end{pmatrix}
\begin{pmatrix}
  w_1 \\
  v \\
  w_2 \\
  w_3 \\
  z
\end{pmatrix}
\]

\[
= - k_1 ||B^*v||^2_U - k_2 < B^*v, z(1)>_U - ||w_3||^2_H + \xi \int_0^1 (\dot{\tau}(t) - 1) \langle \partial_5 z(\delta), z(\delta) \rangle_U d\delta
\]

\[
= - k_1 ||B^*v||^2_U - k_2 < B^*v, z(1)>_U - ||w_3||^2_H + \xi \int_0^1 \frac{1}{2} \frac{\partial}{\partial \delta} z^2(\delta) (\dot{\tau}(t) - 1) d\delta
\]

\[
= - k_1 ||B^*v||^2_U - k_2 < B^*v, z(1)>_U + \xi \frac{1}{2} ||B^*v||^2_U - ||w_3||^2_H - \frac{\xi}{2} ||z(1)||^2_U
\]

\[
+ \frac{\xi \dot{\tau}(t)}{2} ||z(1)||^2 - \frac{\xi \dot{\tau}(t)}{2} \int_0^1 ||z(\delta)||^2 d\delta,
\]
by Cauchy-Schwarz’s inequality and (1.10) follows
\[
\langle A(t)Y, Y \rangle_t \leq \left( -k_1 + \frac{k_2}{2\sqrt{1-d}} + \frac{\xi}{2} \right) \left\| B^*v \right\|_U^2 - \left\| w_3 \right\|_H^2 + \left( \frac{k_2\sqrt{1-d}}{2} - \frac{\xi}{2} (1-d) \right) \left\| z(1) \right\|_U^2 \\
- \frac{\xi \dot{\tau}(t)}{2} \int_0^1 \| z(\delta) \|_U^2 d\delta,
\]
\[
\leq \left( -k_1 + \frac{k_2}{2\sqrt{1-d}} + \frac{\xi}{2} \right) \left\| B^*v \right\|_U^2 - \left\| w_3 \right\|_H^2 + \left( \frac{k_2\sqrt{1-d}}{2} - \frac{\xi}{2} (1-d) \right) \left\| z(1) \right\|_U^2 \\
+ \beta(t) < Y, Y >_t,
\]
where
\[
\beta(t) = \frac{(\dot{\tau}(t)^2 + 1)^{\frac{1}{2}}}{2\tau(t)}.
\] (2.38)

Finally, by (2.35) the operator
\[
A(t) - \beta(t)I \text{ is dissipative.}
\] (2.39)

Moreover
\[
\frac{d}{dt} A(t)Y = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
\ddot{\tau}(t) \tau d - \dot{\tau}(t) (\dot{\tau}(t) \delta - 1)
\end{pmatrix},
\]
and
\[
\dot{\beta}(t) = \frac{\ddot{\tau}(t) \dot{\tau}(t)}{2\tau(t) (\dot{\tau}(t)^2 + 1)^{\frac{1}{2}}} - \frac{\ddot{\tau}(t) (\dot{\tau}(t)^2 + 1)^{\frac{1}{2}}}{2\tau(t)^2},
\]
then from (1.8), (1.9) and (1.10), \( \beta(t) \) is bounded on \([0, T]\). Thus
\[
\frac{d}{dt} (A(t) - \beta(t)I) \in L^\infty_\mathbb{R} ([0, T], B(D(A(0)), H)).
\] (2.40)

Next, we are going to show that \( I - A(t) \) is surjective for fixed \( t \).
Given a vector \( F = (f, g, h, k, l)^T \in H \), we need \( Y = (w_1, v, w_2, w_3, z)^T \in D(A(t)) \) such that
\[
(I - A(t)) \begin{pmatrix}
w_1 \\
v \\
w_2 \\
w_3 \\
z
\end{pmatrix} = \begin{pmatrix}
f \\
g \\
h \\
k \\
l
\end{pmatrix}.
\]
This is equivalent to
\[
w_1 - v = f, \quad \text{(2.41)}
\]
\[
A_1 w_1 + v + C w_2 + k_1 B B^* v + k_2 B z(1) = g, \quad \text{(2.42)}
\]
Suppose that we have found $w_1$ with the appropriate regularity, then $v = w_1 - f \in H_{1,1/2}$. The function $z \in L^2(0,1;U)$ given by

$$z(\delta) = e^{-\delta \tau(t)} B^* v + e^{-\delta \tau(t)} \int_0^\delta e^{\sigma \tau(t)} l(\sigma) d\sigma, \quad \text{if } \dot{\tau}(t) = 0,$$

$$z(\delta) = e^{\tau(t) \ln(1-\delta \tau(t))} B^* v + e^{\tau(t) \ln(1-\delta \tau(t))} \int_0^\delta e^{\sigma \tau(t)} l(\sigma) d\sigma, \quad \text{if } \dot{\tau}(t) \neq 0,$$

is a solution to the equation (2.45) and verifies $z(0) = B^* v$. This means that once $w_3$ is found with the appropriate properties, we can find $z \in H^1(0,1;U)$.

By substituting $v$ and $z$ from (2.41) and (2.46) respectively into (2.42), (2.43) and (2.44), we obtain

$$w_2 + A_2 w_3 - C^* w_1 = h - C^* f,$$

and

$$(\rho + 1) w_3 - A_2^* w_2 = \rho k.$$

Let $\dot{\tau}(t) = 0$, we take then the duality brackets $(.,.)_{H_{1/2}^2 H_{1/2}}$ and we define a linear operator $\mathfrak{B}$ by

$$\mathfrak{B}(w_1, w_2, w_3) = (I + A_1 + k_1 BB^* + k_2 BB^* e^{-\tau(t)}) w_1 + C w_2,$$

$$(\rho + 1) w_3 - A_2^* w_2.$$

Obviously, to solve (2.47), (2.48) and (2.49), by Lax-Milgram lemma, it suffices to show that $\mathfrak{B}$ maps $H_{1/2}^2 \times H \times H$ onto $[H_{1/2}^2 \times H \times H]$ is coercive. This is true since, for $(w_1, w_2, w_3) \in H_{1/2}^2 \times H \times H$, we have

$$\langle \mathfrak{B}(w_1, w_2, w_3), (w_1, w_2, w_3) \rangle = (w_1, w_1)_{H_{1/2}^2 H_{1/2}} + (A_1 w_1, w_1)_{H_{1/2}^2 H_{1/2}} + \langle w_2, w_2 \rangle_{H^2} + \langle w_3, w_3 \rangle_{H^2}$$

$$+ (\rho + 1) \langle w_3, w_3 \rangle_{H_{1/2}^2} + k_1 \langle B^* w_1, w_1 \rangle_U + k_2 \langle B^* w_2, w_2 \rangle_U$$

$$= \langle w_1, w_1 \rangle_H + \langle A_1 w_1, w_1 \rangle_H + \langle w_2, w_2 \rangle_H + (1 + \rho) \langle w_3, w_3 \rangle_H$$

$$+ k_1 \langle B^* w_1, w_1 \rangle_U + k_2 \langle B^* w_2, w_2 \rangle_U$$

$$\geq \langle \mathfrak{B}(w_1, w_2, w_3), (w_1, w_2, w_3) \rangle \geq k \langle w_1, w_1 \rangle_{H_{1/2}^2} + \langle w_2, w_2 \rangle_{H^2} + \langle w_3, w_3 \rangle_{H^2},$$
then the Lax-Milgram lemma leads to the existence and uniqueness of
\((w_1, w_2, w_3) \in H_{\frac{1}{2}} \times H \times H\) solution to the equation (2.47), (2.48) and (2.49). Moreover \((\psi_1 - Cw_2) \in H\), then
\[ w_1 = [I + A_1 + k_1 BB^* + k_2 BB^*e^{-\tau(t)}]^{-1}(\psi_1 - Cw_2) \in D(A_1). \]
Since,
\[ A_2^*A_2 w_3 = -A_2^* w_2 + A_2^* C^* w_1 + A_2^* h - A_2^* C^* f \in H_{-\frac{1}{2}} \text{ and } A_2^* A_2 = A_1, \]
then \( w_3 = A_1^{-1}(-A_2^* w_2 + A_2^* C^* w_1 + A_2^* h - A_2^* C^* f) \in H_{\frac{1}{2}} \). Let the operator \( \Lambda_2 := I + (\rho + 1)^{-1} A_2 A_2^* \), we have \( w_2 = A_2^{-1}(C_1^* w_1 + h - C^* f - \rho A_2 k) \in H_{\frac{1}{2}} \).
Furthermore, \( A_1 w_1 + C w_2 + k_1 BB^* v + k_2 B z(1) = (g - v) \in H \). In summary, we have found \( (w_1, v, w_2, w_3, z)^T \in D(A(t)) \) satisfying (2.16)-(2.22).

Consequently, \( I - A(t) \) is surjective for fixed \( t \). In the similar way, we can find the same result if \( \bar{\tau}(t) \neq 0 \). Since \( \beta(t) > 0 \), then we have
\[ I - (A(t) - \beta(t) I) = (I + \beta(t) I) - A(t) \text{ is surjective, for } t > 0. \] \hspace{1cm} (2.50)
Finally from (2.37), (2.39) and (2.50) we have \( \{ A(t) - \beta(t) I / t \in [0, T] \} \) is a stable family of generators in \( \mathcal{H} \) with stability constants independent of \( t \), then the assumption (1)-(4) of Theorem 2.1 are verified. Thus, the problem
\[
\begin{cases}
\frac{d\bar{\eta}}{dt} = (A(t) - \beta(t) I)\bar{\eta}, & \bar{\eta} \in \mathcal{H}, \\
\bar{\eta}^0 = (w_1^0, w_2^0, w_3^0),
\end{cases}
\]
has unique solution \( \bar{\eta} \in C(0, \infty; D(A(0))) \cap C^1(0, \infty; \mathcal{H}) \). Consequently \( \eta(t) = e^{\int_0^t \beta(s) ds} \bar{\eta}(t) \) is solution of (2.29), i.e., \( \eta \) verified
\[
\begin{cases}
\frac{d\eta}{dt} = A(t)\eta, & \eta \in \mathcal{H}, \\
\eta^0 = (w_1^0, w_2^0, w_3^0, z^0).
\end{cases}
\]
Now, with a similar argument we can show that the problem (2.30) has a unique solution \( \eta \in C(0, \infty; D(A_F(0))) \cap C^1(0, \infty; \mathcal{H}_F) \), for all initial datum in \( D(A_F(0)) \).

### 3 Uniform decay for a thermoelastic system with Cattaneo’s law

Now to obtain the uniform stability of the system (1.2)-(1.7), we use the energy method to produce a suitable Lyapunov functional.

We define the functional energy of the solution of problem (1.2)-(1.7) as
\[ E(t) = \frac{1}{2} \int_0^1 \left( w_1^2 + \lambda (\partial_x w_1)^2 + w_2^2 + \rho w_3^2 \right) dx + \frac{\xi \tau(t)}{2} \int_0^1 \int_0^1 z^2(\rho, t) d\rho dx, \] \hspace{1cm} (3.1)
where \( \xi \) is a positive constant satisfying
\[ \frac{k_2}{\sqrt{1 - d}} < \xi < 2k_1 - \frac{k_2}{\sqrt{1 - d}}, \] \hspace{1cm} (3.2)
Proposition 3.1. Assume that the assumptions (3.2) and (2.36) are satisfied, then there exists a positive constant $C_0$ such that for any regular solution $(w_0^1, w_1^1, w_0^2, w_0^3, f_0) \in \mathcal{D}(A)$ of (1.2)-(1.7), we have
\[
\dot{E}(t) \leq -C_0 \left\{ \int_0^1 \left( \dot{w}_1^2 + w_3^2 + z^2(1,t) \right) dx \right\}.
\] (3.3)

Proof. Using (2.16)-(2.22) and integration by parts, we get
\[
\dot{E}(t) = (-k_1 + \frac{\xi}{2}) \int_0^1 w_1^2 dx - k_2 \int_0^1 \dot{w}_1 z(1,t) dx - \int_0^1 w_3^2 dx + \left( \frac{\xi'(t)}{2} - \frac{\xi}{2} \right) \int_0^1 z^2(1,t) dx.
\]
Hence by (2.36), the Cauchy-Schwarz inequality provides
\[
\dot{E}(t) \leq \left( -k_1 + \frac{\xi}{2} + \frac{k_2}{2(1-d)} \right) \int_0^1 w_1^2 dx - \int_0^1 \dot{w}_1 z(1,t) dx + \left( \frac{k_2(1-d)}{2} - \frac{\xi}{2} (1-d) \right) \int_0^1 z^2(1,t) dx.
\]
Then by (3.2) our conclusion holds. \qed

The first fundamental main result of this paper is

Theorem 3.2. Let $(w_0^1, w_1^1, w_0^2, w_0^3, f_0) \in \mathcal{D}(A)$ and assume that $k_2 < k_1$. Then there exist two positive constants $M_0$ and $\nu$ independent of $t$ such that for any solution of system (1.2)-(1.7), we have
\[
E(t) \leq M_0 e^{-\nu t}, \quad \forall t \geq 0.
\]

The proof of the above Theorem relies on the construction of a Lyapunov functional $\mathcal{L}$ equivalent to energy $E$, satisfying
\[
\dot{\mathcal{L}}(t) \leq -\Lambda \mathcal{L}(t),
\]
for some $\Lambda > 0$. In order to obtain such a functional $\mathcal{L}$, we establish several Lemmas.

Lemma 3.3. The functional $I_1 := \int_0^1 (w_1 \dot{w}_1 - \mu \rho w_1 w_3) dx$, satisfies, for some positive constant $a_1$, the estimate
\[
I_1(t) \leq -a_1 \int_0^1 (\partial_x w_1)^2 dx + C_1 \int_0^1 (\dot{w}_1^2 + w_3^2 + z^2(1,t)) dx.
\] (3.4)

Proof. By taking a derivative of $I_1$, using (2.16)-(2.22) and by integrating by parts, we have
\[
\dot{I}_1(t) = \int_0^1 \dot{w}_1^2 dx - \lambda \int_0^1 (\partial_x w_1)^2 dx - k_1 \int_0^1 w_1 \dot{w}_1 dx - k_2 \int_0^1 w_1 z(1,t) dx
\]
\[
- \mu \rho \int_0^1 \dot{w}_1 w_3 dx - \mu \int_0^1 w_1 w_3 dx.
\]
We use Young’s and Poincaré inequalities to get
\[
\dot{I}_1(t) \leq -\lambda + \epsilon_1 + \epsilon_2 + \epsilon_3 \int_0^1 (\partial_x w_1)^2 dx + (C_{\epsilon_1} + \frac{\mu\rho}{2}) \int_0^1 \dot{w}_1^2 dx \\
+ (C_{\epsilon_3} + \frac{\mu\rho}{2}) \int_0^1 w_3^2 dx + C_{\epsilon_2} \int_0^1 z^2(1,t) dx.
\]

By choosing \(\epsilon_i, i = 1, 2, 3\) small enough, (3.4) is established. \(\square\)

**Lemma 3.4.** The functional \(I_2 := \int_0^1 \left(\int_0^x w_2(y,t) dy\right) \dot{w}_1 dx\), satisfies, for some positive constant \(a_2\), the estimate
\[
\dot{I}_2(t) \leq -a_2 \int_0^1 \dot{w}_1^2 dx + C_2 \int_0^1 \left((\partial_x w_1)^2 + w_2^2 + w_3^2 + z^2(1,t)\right) dx.
\] (3.5)

**Proof.** By taking a derivative of \(I_2\) and using integrating by parts, we have
\[
\dot{I}_2(t) = -\mu \int_0^1 \dot{w}_1^2 dx - \int_0^1 \dot{w}_1 w_3 dx - \lambda \int_0^1 (\partial_x w_1) w_2 dx + \mu \int_0^1 \dot{w}_2^2 dx \\
- k_1 \int_0^1 \left(\int_0^x w_2(y,t) dy\right) \dot{w}_1 dx - k_2 \int_0^1 \left(\int_0^x w_2(y,t) dy\right) z(1,t) dx.
\]

We use Young’s inequality to get
\[
\dot{I}_2(t) \leq -\mu + \epsilon_4 + \epsilon_5 \int_0^1 \dot{w}_1^2 dx + C_{\epsilon_1} \int_0^1 w_3^2 dx + (C_{\epsilon_4} + C_{\epsilon_5} + \mu + \frac{k_2}{2}) \int_0^1 \dot{w}_2^2 dx \\
+ \epsilon_5 \int_0^1 (\partial_x w_1)^2 dx + \frac{k_2}{2} \int_0^1 z^2(1,t) dx.
\]

By choosing \(\epsilon_4, \epsilon_5\) small, (3.5) is established. \(\square\)

**Lemma 3.5.** The functional \(I_3 := -\rho \int_0^1 \left(\int_0^x w_2(y,t) dy\right) w_3 dx\), satisfies, for some positive constant \(a_3\), the estimate
\[
\dot{I}_3(t) \leq -a_3 \int_0^1 w_2^2 dx + C_3 \int_0^1 \left(\dot{w}_1^2 + w_3^2\right) dx.
\] (3.6)

**Proof.** By taking a derivative of \(I_3\) and using integrating by parts, we have
\[
\dot{I}_3(t) = \rho \int_0^1 \dot{w}_3^2 dx + \mu \rho \int_0^1 \dot{w}_1 w_3 dx \\
+ \int_0^1 \left(\int_0^x w_2(y,t) dy\right) w_3 dx - \int_0^1 \dot{w}_2^2 dx.
\]

We use Young’s inequality to get
\[
\dot{I}_3(t) \leq -1 + \epsilon_6 \int_0^1 w_2^2 dx + (C_{\epsilon_6} + \rho + \frac{\rho\mu}{2}) \int_0^1 \dot{w}_2^2 dx \\
+ \frac{\rho\mu}{2} \int_0^1 \dot{w}_1^2 dx.
\]

By choosing \(\epsilon_6\) small enough, (3.6) is established. \(\square\)
Lemma 3.6. The functional \( I_4 := \tau(t) \int_0^1 \int_0^1 e^{-\tau(t)\rho}d\rho dx \), satisfies, for some positive constant \( a_4 \), the estimate

\[
\dot{I}_4(t) \leq -a_4 \int_0^1 \int_0^1 z^2(\rho,t)d\rho dx + \int_0^1 \dot{w}_1^2 dx. \tag{3.7}
\]

Proof. By taking a derivative of \( I_4 \) and using integrating by parts, we have

\[
\dot{I}_4(t) = \int_0^1 \dot{w}_1^2 dx - [1 - \dot{\tau}(t)]e^{-\tau(t)} \int_0^1 z^2(1,t)dx \\
- \tau(t) \int_0^1 \int_0^1 e^{-\tau(t)\rho}z^2(\rho,t)d\rho dx.
\]

Therefore, by (2.36), we have

\[
\dot{I}_4(t) \leq \int_0^1 \dot{w}_1^2 dx - \tau(t) \int_0^1 \int_0^1 e^{-\tau(t)\rho}z^2(\rho,t)d\rho dx.
\]

Finally, (1.9) leads to the claim.

\[
\square
\]

Proof of Theorem 3.2. To finalize the proof of Theorem 3.2, we define the Lyapunov functional \( \mathcal{L} \) as follows

\[
\mathcal{L}(t) := N_0 E(t) + I_1 + I_2 + N_3 I_3 + I_4, \tag{3.8}
\]

where \( N_0 \) and \( N_3 \) are positive real numbers which will be chosen later. Consequently, the estimates (3.3), (3.4), (3.5), (3.6) and (3.7) together lead to

\[
\dot{\mathcal{L}}(t) \leq \left( -N_0 C_0 - a_2 + C_{\epsilon_1} + (N_3 + 1) \frac{\rho \mu}{2} + 1 \right) \int_0^1 \dot{w}_1^2 dx + \left( -a_1 + \epsilon_5 \right) \int_0^1 (\partial_x w_1)^2 dx \\
+ \left( -N_3 a_3 + C_{\epsilon_4} + C_{\epsilon_5} + \mu + \frac{k_2}{2} \right) \int_0^1 w_2^2 dx \\
+ \left( -N_0 C_0 + N_3 C_{\epsilon_6} + \frac{N_3 \rho \mu}{2} + C_{\epsilon_3} + C_{\epsilon_4} + \frac{\rho \mu}{2} \right) \int_0^1 w_3^2 dx \\
+ (-N_0 C_0 + \frac{k_2}{2}) \int_0^1 z(1,t)_0^1 z^2(\rho,t)d\rho dx. \tag{3.9}
\]

At this point, we have to choose our constants very carefully. First, let us choose \( \epsilon_5 \) small enough such that

\[
\epsilon_5 < a_1.
\]

Then, we fix \( N_3 \) large enough so that

\[
C_{\epsilon_4} + C_{\epsilon_5} + \mu + \frac{k_2}{2} < N_3 a_3.
\]

Also, we select \( N_0 \) large enough so that

\[
\frac{k_2}{2} < N_0 C_0.
\]
\[ C_{\epsilon_1} + (N_3 + 1) \frac{\rho \mu}{2} + 1 < N_0 C_0 + a_2, \]

and

\[ N_3 C_{\epsilon_6} + \frac{N_3 \rho \mu}{2} + C_{\epsilon_6} + C_{\epsilon_4} + \frac{\rho \mu}{2} < N_0 C_0. \]

Finally, there exists a positive constant \( \eta_1 \), such that (3.9) becomes

\[ \dot{\mathcal{L}}(t) \leq -\eta_1 \int_0^1 \left( \dot{w}_1^2 + (\partial_x w_1)^2 + w_2^2 + w_3^2 \right) dx - \eta_1 \int_0^1 \int_0^1 z^2(\rho, t) d\rho dx, \]

which implies, that there exists also \( \eta_2 > 0 \), such that

\[ \dot{\mathcal{L}}(t) \leq -\eta_2 E(t). \]

**Lemma 3.7.** There exist two positive constants \( M_1, M_2 \), such that

\[ M_1 E(t) \leq \mathcal{L}(t) \leq M_2 E(t). \]  

(3.11)

From (3.8), we have

\[ |\mathcal{L}(t) - N_0 E(t)| \leq \int_0^1 \left( |w_1 \dot{w}_1| + \mu \rho |w_1 w_3| \right) dx + \int_0^1 \int_0^x w_2(y, t) dy \dot{w}_1 dx \]

\[ + N_3 \rho \int_0^1 \int_0^x w_2(y, t) dy w_3 dx + \tau(t) \int_0^1 \int_0^1 e^{-\tau(t)\rho} d\rho dx. \]

By Young’s and Poincaré’s inequalities we get

\[ |\mathcal{L}(t) - N_0 E(t)| \leq C \left( \int (\dot{w}_1^2 + (\partial_x w_1)^2 + w_2^2 + w_3^2) dx + \tau(t) \int_0^1 \int_0^1 e^{-\tau(t)\rho} d\rho dx \right), \]

\[ \leq C E(t). \]

Therefore, we can choose \( N_0 \) large so that (3.11) is satisfied.

Combining (3.10) and (3.11), we conclude

\[ \dot{\mathcal{L}}(t) \leq -\Lambda \mathcal{L}(t), \quad \forall t \geq 0, \]

(3.12)

for some \( \Lambda > 0 \). A simple integration of (3.12) leads to

\[ \mathcal{L}(t) \leq \mathcal{L}(0) e^{-\Lambda t}, \quad \forall t \geq 0. \]

(3.13)

Again, the use of (3.13) and (3.11) completes the proof of Theorem 3.2.

## 4 Polynomial decay for a thermoelastic system with Fourier’s law

In this section, we show that the exponential stability of the thermoelastic system with Cattaneo’s law (2.16)-(2.22) implies the polynomial stability of the following thermoelastic system...
with Fourier’s law
\[
\begin{align*}
\dot{w}_1(t) + A_1 w_1(t) + k_1 B B^* \dot{w}_1(t) + k_2 B z(1) + C w_2(t) &= 0, \\
\dot{w}_2(t) + A_2 A_2^* w_2(t) - C^* \dot{w}_1(t) &= 0, \\
\tau(t) \dot{z}(\delta, t) + (1 - \tau(t) \delta) \partial_\delta z(\delta, t) &= 0, \\
z(0, t) &= B^* \dot{w}_1(t), \\
w_1(0) &= w_1^0, \quad w_1(0) = w_1^1, \quad w_2(0) = w_2^0, \\
z(\delta, 0) &= z_0 = B^* f_0(-\delta \tau(0)).
\end{align*}
\] (4.14)-(4.19)

For regular solutions, the energy of the system (4.14)-(4.19) is defined by
\[
E_F(t) = \frac{1}{2} \| (w_1, \dot{w}_1, w_2, \dot{w}_1(t + .)) \|^2_{H_F}, \quad t \geq 0.
\]

To prove the polynomial stability of the system (4.14)-(4.19), we need the following assumption

**H**: \( \rho(A_F) \supset \{ i\beta \mid \beta \in \mathbb{R} \} \equiv i\mathbb{R} \).

The second fundamental main result of this paper is

**Theorem 4.1.** Assume that the assumption \( \textbf{H} \) is verified.
Then the solution \((w_1, w_2, \dot{w}_1(t + .))\) of (4.14)-(4.19) is polynomially stable, i.e., there exists a constant \( C > 0 \) such that for all \((w_1^0, w_1^1, w_2^0, f_0) \in D_F \) we have
\[
E_F(t) \leq \frac{C}{t} \| (w_1^0, w_1^1, w_2^0, f_0) \|^2_{D_F}, \quad \forall t > 0.
\]

**Proof.** In the last section, we have shown that the system (2.16)-(2.22) is exponential stable, then by Gearhart-Prüs theorem we have
\[
\rho(A) \supset \{ i\beta \mid \beta \in \mathbb{R} \} \equiv i\mathbb{R},
\] (4.20)
and
\[
\limsup_{|\beta| \to \infty} \| (i\beta - A)^{-1} \| < \infty.
\] (4.21)

To prove the polynomial stability of the system (4.14)-(4.19), we show that \( A_F \) satisfies the following two conditions (see Borichev-Tomilov[1])
\[
\rho(A_F) \supset \{ i\beta \mid \beta \in \mathbb{R} \} \equiv i\mathbb{R},
\] (4.22)
and
\[
\limsup_{|\beta| \to \infty} \frac{1}{\beta^2} \| (i\beta - A_F)^{-1} \| < \infty,
\] (4.23)

where \( \rho(A_F) \), respectively \( \rho(A) \), denotes the resolvent set of the operator \( A_F \), respectively of \( A \).
By assumption \( \textbf{H} \) the condition (4.22) is satisfied. Now suppose that the condition (4.23) is
false. By the Banach-Steinhaus Theorem, there exist a sequence of real numbers \( \beta_n \to \infty \) and a sequence of vectors \( y_n = (u_n, v_n, \phi_n, z_n, \psi_n)^T \in \mathcal{D}_F \) with \( \|y_n\|_{\mathcal{H}_F} = 1 \) such that

\[
\|\beta_n^2 (i\beta_n I - A_F)y_n\|_{\mathcal{H}_F} \to 0 \quad \text{as} \quad n \to \infty, \tag{4.24}
\]

i.e.,

\[
\beta_n^2 (i\beta_n u_n - v_n) \to 0 \quad \text{in} \quad H^1_2,
\]

\[
\beta_n^2 (i\beta_n v_n + A_1 u_n + k_1 BB^* v_n + k_2 BB^* z_n(1) + C \phi_n) \to 0 \quad \text{in} \quad H,
\]

\[
\beta_n^2 (i\beta_n \phi_n + A_2 A^*_2 \phi_n - C^* v_n) \to 0 \quad \text{in} \quad H,
\]

\[
\beta_n^2 (i\beta_n z_n - \frac{\dot{\tau}(t)\delta - 1}{\tau(t)} \partial_\delta z_n) \to 0 \quad \text{in} \quad L^2(-\tau(0), 0; H).
\]

We notice that we have by the Cauchy-Schwarz inequality

\[
\|\beta_n^2 (i\beta_n I - A_F)y_n\|_{\mathcal{H}_F} \geq |\Re\langle \beta_n^2 (i\beta_n I - A_F)y_n, y_n \rangle_{\mathcal{H}_F}|.
\]

Then, by (4.24) and

\[
-\Re \langle A_F y_n, y_n \rangle \geq \left( k_1 - \frac{k_2}{2\sqrt{1-d}} - \frac{\xi}{2} \right) \|B^* v_n\|_U^2 + \|A^*_2 \phi_n\|^2_H + \left( -\frac{k_2 \sqrt{1-d}}{2} + \frac{\xi}{2}(1-d) \right) \|z_n(1)\|^2_U,
\]

we have

\[
\beta_n A^*_2 \phi_n \to 0 \quad \text{in} \quad H, \quad \beta_n B^* v_n \to 0 \quad \text{in} \quad H \quad \text{and} \quad \beta_n z_n(1) \to 0 \quad \text{in} \quad H.
\]

Letting \( \psi_n = A^*_2 \phi_n \), this implies

\[
i\beta_n u_n - v_n \to 0 \quad \text{in} \quad H^1_2,
\]

\[
i\beta_n v_n + A_1 u_n + k_1 BB^* v_n + k_2 BB^* z_n(1) + C \phi_n \to 0 \quad \text{in} \quad H,
\]

\[
i\beta_n \phi_n - C^* v_n + A_2 \psi_n \to 0 \quad \text{in} \quad H,
\]

\[
i\beta_n \psi_n + \psi_n - A^*_2 \phi_n \to 0 \quad \text{in} \quad H,
\]

\[
i\beta_n z_n - \frac{\dot{\tau}(t)\delta - 1}{\tau(t)} \partial_\delta z_n \to 0 \quad \text{in} \quad L^2(-\tau, 0; H).
\]

i.e. \( \tilde{y}_n = (u_n, v_n, \phi_n, \psi_n, z_n)^T \in \mathcal{D}(A) \) with \( \|\tilde{y}_n\|_H \) bounded such that

\[
\|(i\beta_n I - A)\tilde{y}_n\|_H \to 0 \quad \text{as} \quad n \to \infty,
\]

which implies that (4.21) is false and ends the proof of the Theorem 4.1. \( \square \)
References


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