A Steady State Heat Conduction Problem in
a Thick Annular Disc Due to Arbitrary
Axisymmetric Heat Flux

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Abstract

In this paper attempt has been made to discuss steady state quasi static thermal stresses in a thick annular disc \( a \leq r \leq b \), \(-h \leq z \leq h\) subjected to arbitrary heat flux at upper and lower surface of annular disc while inner and outer circular surface of thick annular disc are maintained at zero degree temperature. The governing heat conduction have been solved by using integral transform technique .The results are obtained in series form in term of Bessel’s functions, The result of displacement and stresses have been computed numerically and illustrated graphically.

Keywords: Quasi-static, Thermoelastic problem, Thermal Stresses, Axisymmetric Thermal Stresses

1. Introduction

loads and obtained the results for radial and axial displacements and temperature change moreover Sharma et al (2004) [10] studied the behavior of thermoelastic thick plate under lateral loads and obtained the results for radial and axial displacements and temperature change have been computed numerically and illustrated graphically for different theories of generalized thermoelasticity. Recently Ruhi et al (2005) [9] did thermoelastic analysis of thick walled finite length cylinders of functionally graded materials and obtained the results for stress, strain and displacement components through the thickness and along the length are presented due to uniform internal pressure and thermal loading. V. S. Kulkarni, K. C. Deshmukh [12] considered a thick annular disc which is subjected to a transient axisymmetric temperature field on the radial and axial directions of the cylindrical coordinate system and determined the expression for temperature, displacement and stress functions due to arbitrary heat flux on the upper and lower surface. Deshmukh et al [3] studied two dimensional nonhomogeneous boundary value problem of heat conduction and it’s thermal deflection of a semi infinite circular plate on the outer curved surface for an infinite length. Deshmukh et al [4] determined the thermal stresses induced by a point heat source in a circular plate by quasi static approach.

2. Formulation of the Problem

Consider a thick annular disc defined by \(a \leq r \leq b\), \(-h \leq z \leq h\). Let the disc be subjected to a axisymmetric temperature field on the radial direction in the cylindrical coordinate system. Initially the plate kept at zero temperature the arbitrary the heat flux \(Q(r)\) is prescribed over the upper surface \((z = h)\) and the lower surface \((z = -h)\) the fixed circular edges \((r = a, r = b)\) are at zero temperature. Assume the lower and upper surface of thick annular disc are traction free. Under this more realistic prescribed condition, the quasi state thermal stresses are required to be determined.

The differential equation governing displacement Potential function \(\phi(r, z)\) is given as

\[
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = K \tau
\]  

(1)

Where \(K\) is restraint coefficient and temperature changes \(\tau = T - T_i\), \(T_i\) is the initial temperature. Displacement function \(\phi\) is known as Goodier’s thermoelastic displacement Potential.

Heat Conduction Equations

The temperature of the disc at a time \(t\) satisfies the heat conduction equation,

\[
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0
\]  

(2)
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With the boundary condition

\[ \frac{\partial T}{\partial z} = \pm \frac{Q(r)}{\lambda} \quad \text{For} \quad z = \pm h \quad a \leq r \leq b \]  

(3)

\[ T=0 \quad \text{at} \quad r=a \quad -h \leq z \leq h \]  

(4)

\[ T=0 \quad \text{at} \quad r=b \quad -h \leq z \leq h \]  

(5)

Displacement Potential and Thermal Stresses

The displacement function in a cylindrical co-ordinate system is represented by the Michell’s function defined as in [5].

\[ U_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z} \]  

(6)

\[ U_z = \frac{\partial \phi}{\partial z} + 2(1-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2} \]  

(7)

The Michell’s function M must satisfy

\[ \nabla^2 \nabla^2 M = 0 \]  

(8)

Where

\[ \nabla^2 = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \]  

(9)

The component of the stresses are represented by the Thermo-elastic displacement Potential \( \phi \) and Michell’s function M as

\[ \sigma_{rr} = 2G \left[ \frac{\partial^2 \phi}{\partial r^2} - K \tau + \frac{\partial}{\partial z} \left( \nabla^2 M - \frac{\partial^2 M}{\partial z^2} \right) \right] \]  

(10)

\[ \sigma_{\theta\theta} = 2G \left[ \frac{1}{r} \frac{\partial \phi}{\partial r} - K \tau + \frac{\partial}{\partial z} \left( \nabla^2 M - \frac{\partial M}{\partial r} \right) \right] \]  

(11)

\[ \sigma_{zz} = 2G \left[ \frac{\partial^2 \phi}{\partial z^2} - K \tau + \frac{\partial}{\partial z} \left( (2-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2} \right) \right] \]  

(12)

and

\[ \sigma_{rz} = 2G \left[ \frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial r} \left( (1-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2} \right) \right] \]  

(13)

G and \( \nu \) are shear modulus and Poisson ratio respectively.

For traction free surface of the stress function
The equation (1) to (14) constitute mathematical formulation of the problem.

3. Solution

The temperature change Introducing the finite the Hankel transform over the variable \( r \) and its inverse its transform defined as in [11]

\[
\bar{T}(\alpha_m, z) = \int_a^b r K_0(\alpha_m r) T(r, z) \, dr
\]  

(15)

\[
T(r, z) = \sum_{m=1}^{\infty} \bar{T}(\alpha_m, z) K_0(\alpha_m r)
\]  

(16)

Where \( K_0(\alpha_m r) = \frac{R_0(\alpha_m r)}{\sqrt{N}} \)  

(17)

\[
R_0(\alpha_m r) = \begin{bmatrix} J_0(\alpha_m r) & Y_0(\alpha_m r) \\ J_1(\alpha_m b) & Y_1(\alpha_m b) \end{bmatrix}
\]  

(18)

the Normality constant

\[
N = \frac{b^2}{2} R_0^2(\alpha_m b) - \frac{a^2}{2} R_0^2(\alpha_m a)
\]  

(19)

Where \( \cdot \) Represents differentiation w.r.t space variable \( r \), and \( \alpha_1, \alpha_2, \ldots \) are roots of the transcendental equation

\[
\begin{bmatrix} J_n(aa) & Y_n(aa) \\ J_n(ab) & Y_n(ab) \end{bmatrix} = 0
\]  

(20)

\( J_n(x) \) is the Bessel function of the first kind of order \( n \) and \( Y_n(x) \) is the Bessel function of the second kind of order \( n \).

The transform satisfies the relation

\[
H \begin{bmatrix} \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \\ \frac{\partial T}{\partial z} \end{bmatrix} = -\alpha_m^2 \bar{T}(\alpha_m, z)
\]  

(21)

and

\[
H \begin{bmatrix} \frac{\partial^2 T}{\partial z^2} \\ \frac{\partial^2 T}{\partial z^2} \end{bmatrix} = \frac{d^2 T}{dz^2}
\]  

(22)

On applying the finite Hankel transform defined in the equation (15) to equation (2) one obtains

\[
\frac{\partial^2 \bar{T}}{\partial z^2} - \alpha_2^2 \bar{T} = 0
\]  

(23)
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Here $\tilde{T}$ is the Hankel transform of $T$ on solving the equation (23) one obtains

$$\tilde{T} = Ae^{\alpha_m z} + Be^{-\alpha_m z}$$  \hspace{1cm} (24)

On solving the equation (24) under the condition given in the equation (3) one obtains

$$A = B = \frac{Q\tilde{T}(\alpha_m)}{\alpha_m^2} \left[ \cosh(\alpha_m h) \right]$$

Using the equation (24) one obtains

$$\tilde{T} = \frac{2Q\tilde{T}(\alpha_m)}{\alpha_m^2} \left[ \cosh(\alpha_m h) \cosh(\alpha_m z) \right]$$  \hspace{1cm} (25)

$\tilde{T}(\alpha_m)$ is the Hankel transform of $f(r)$

On Appling the inverse Hankel transform defined is the equation (16) one obtains

$$T = \frac{Q}{\lambda} \sum_{n=1}^{\infty} \left[ \tilde{f}(\alpha_m) \frac{J_0(\alpha_m r)}{J_0(\alpha_m b)} \frac{Y_0(\alpha_m r)}{Y_0(\alpha_m b)} \right] \left[ \cosh(\alpha_m z) \right]$$  \hspace{1cm} (26)

Where $\tilde{f}(\alpha_m)$ is Hankel transform of $f(r)$

Since the initial temp $T_i=0 \quad \tau = T - Ti \quad \Rightarrow T = \tau$ \hspace{1cm} (27)

Michell’s Function M

Now assume Michell’s function $M$ which satisfy the condition (8)

$$M = \frac{Q \lambda}{\alpha_m} \sum_{n=1}^{\infty} \left[ \tilde{f}(\alpha_m) \frac{J_0(\alpha_m r)}{J_0(\alpha_m b)} \frac{Y_0(\alpha_m r)}{Y_0(\alpha_m b)} \right] \left[ H_n \cosh(\alpha_m z) + R_n \sinh(\alpha_m z) \right]$$  \hspace{1cm} (28)

Where $H_n$ and $R_n$ are arbitrary function which can be determined finally using the condition (14)

Goodier’s Thermoelastic Displacement Potential $\phi$

To obtains the displacement potential $\phi$ using the equation (26) and (27) in equation (1) one have

$$\phi = \frac{Q \lambda}{\alpha_m} \sum_{n=1}^{\infty} \left[ \tilde{f}(\alpha_m) \frac{J_0(\alpha_m r)}{J_0(\alpha_m b)} \frac{Y_0(\alpha_m r)}{Y_0(\alpha_m b)} \right] \left[ \frac{\cosh(\alpha_m z)}{z \alpha_m \sinh(\alpha_m h)} \right]$$  \hspace{1cm} (29)
Displacement and Thermal Stresses

Now using the equation (26), (27), (28) and (29) in the equation (6), (7), (10), (11), (12) and (13) one obtains the expression for displacement and stresses respectively as

\[
U_r = \frac{QK}{\lambda} \sum_{m=1}^{\infty} \left[ \frac{f(a_m)}{\sqrt{N}} \right] \left[ \frac{J_1(a_mr)}{J_0(a_m b)} - \frac{Y_1(a_mr)}{Y_0(a_m b)} \right] \left\{ \left[ -\alpha \sinh(a_m z) \right] + \frac{\alpha^2 H_n \sinh(a_m z)}{2 \alpha_m \sinh(a_m b)} + \alpha^2 R_n \left( \sinh(a_m z) + \alpha m \cosh(a_m z) \right) \right\} 
\]

\[
U_z = \frac{QK}{\lambda} \sum_{m=1}^{\infty} \left[ \frac{f(a_m)}{\sqrt{N}} \right] \left[ \frac{J_0(a_m r)}{J_0(a_m b)} - \frac{Y_0(a_m r)}{Y_0(a_m b)} \right] \left\{ \frac{\sinh(a_m z) + \alpha m \cosh(a_m z)}{2 \alpha_m^2 \sinh(a_m b)} \right\} - \alpha^2 H_n \cosh(a_m z) + \alpha^2 R_n \left( 2(1 - \nu) \cosh(a_m z) - \alpha_m z \sinh(a_m z) \right) 
\]

\[
\sigma_{rr} = 2GQK \sum_{m=1}^{\infty} \left[ \frac{f(a_m)}{\sqrt{N}} \right] \left[ \frac{J_1(a_m r)}{J_0(a_m b)} - \frac{Y_1(a_m r)}{Y_0(a_m b)} \right] \left\{ \alpha \sinh(a_m z) \right\} - \sum_{m=1}^{\infty} \left[ \frac{J_0(a_m r)}{J_0(a_m b)} - \frac{Y_0(a_m r)}{Y_0(a_m b)} \right] \left[ \cosh(a_m z) \right] \left[ \alpha_m \sinh(a_m b) \right] + H_n \alpha^2 m \left[ \frac{J_1(a_m r)}{J_0(a_m b)} - \frac{Y_1(a_m r)}{Y_0(a_m b)} \right] \sinh(a_m z) + \alpha^2 m R_n 
\]

\[
\sigma_{zz} = 2GQK \sum_{m=1}^{\infty} \left[ \frac{f(a_m)}{\sqrt{N}} \right] \left[ \frac{J_0(a_m r)}{J_0(a_m b)} - \frac{Y_0(a_m r)}{Y_0(a_m b)} \right] \left\{ \sinh(a_m z) + (a_m z) \cosh(a_m z) \right\} - \sum_{m=1}^{\infty} \left[ \frac{J_0(a_m r)}{J_0(a_m b)} - \frac{Y_0(a_m r)}{Y_0(a_m b)} \right] \left[ \cosh(a_m z) \right] \left[ \alpha_m \sinh(a_m b) \right] + H_n \alpha^2 m \left[ \frac{J_1(a_m r)}{J_0(a_m b)} - \frac{Y_1(a_m r)}{Y_0(a_m b)} \right] \sinh(a_m z) + \alpha^2 m R_n 
\]

\[
\sigma_{00} = 2GQK \sum_{m=1}^{\infty} \left[ \frac{f(a_m)}{\sqrt{N}} \right] \left[ \frac{J_1(a_m r)}{J_0(a_m b)} - \frac{Y_1(a_m r)}{Y_0(a_m b)} \right] \left\{ \sinh(a_m z) \right\} - \frac{1}{r} \sum_{m=1}^{\infty} \left[ \frac{J_0(a_m r)}{J_0(a_m b)} - \frac{Y_0(a_m r)}{Y_0(a_m b)} \right] \left[ \cosh(a_m z) \right] \left[ \alpha_m \sinh(a_m b) \right] + h_n \alpha^2 m \left[ \frac{J_1(a_m r)}{J_0(a_m b)} - \frac{Y_1(a_m r)}{Y_0(a_m b)} \right] \sinh(a_m z) + \frac{1}{r} R_n \alpha^2 m \left( 2 (1 - \nu) \cosh(a_m z) - \alpha_m z \sinh(a_m z) \right) 
\]
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\[
\begin{aligned}
&\left( J_0(\alpha_m r) - \frac{Y_0(\alpha_m r)}{Y_0(\alpha_m b)} J_0(\alpha_m b) \right) \\
&\sinh(\alpha_m z) + \frac{1}{r} \left[ \frac{J_1(\alpha_m r)}{J_0(\alpha_m b)} \frac{Y_1(\alpha_m r)}{Y_0(\alpha_m b)} \left( \sinh(\alpha_m z) + (\alpha_m z) \cosh(\alpha_m z) \right) \right]
\end{aligned}
\]

(33)

\[\sigma_{zz} = \frac{2GQK}{\lambda} \sum_{m=1}^{\infty} \left[ \frac{\tilde{f}(\alpha_m)}{\sqrt{N}} \left( J_0(\alpha_m r) - \frac{Y_0(\alpha_m r)}{Y_0(\alpha_m b)} J_0(\alpha_m b) \right) \right] \left( \frac{z \sinh(\alpha_m z)}{2 \sinh(\alpha_m h)} \right) \]

(34)

\[\sigma_{zz} = \frac{2GQK}{\lambda} \sum_{m=1}^{\infty} \left[ \frac{\tilde{f}(\alpha_m)}{\sqrt{N}} \left( J_0(\alpha_m r) - \frac{Y_0(\alpha_m r)}{Y_0(\alpha_m b)} J_0(\alpha_m b) \right) \right] \times
\]

\[
\left[ \frac{-\sinh(\alpha_m z) - (\alpha_m z) \cosh(\alpha_m z)}{2 \alpha_m \sinh(\alpha_m h)} \right]
\]

(35)

Determination of unknown arbitrary function \( H_n \) and \( R_n \). In order to satisfy the condition (14) solving the equation (21) and (22) for \( H_n \) and \( R_n \) one obtains.

\[H_n = \frac{(1 - 2\nu) \sinh(\alpha_m h) \cosh(\alpha_m h)}{2 \alpha_m^4 \sinh(\alpha_m h) \left[ \sinh(\alpha_m h) \cosh(\alpha_m h) - (\alpha_m h) \right]} \]

(36)

\[R_n = \frac{\sinh(\alpha_m h)}{2 \alpha_m^4 \left[ \sinh(\alpha_m h) \cosh(\alpha_m h) - (\alpha_m h) \right]} \]

(37)

Using the value of \( H_n \) and \( R_n \) in the equation (15) to (22) one obtains the expression for displacement and stresses

4. Special Case

Setting \( f(r) = (r^2 - a^2)^2 \)

(38)

Applying finite Hankel transform to (38), one obtains

\[
\tilde{f}(\alpha_m) = \frac{1}{\pi \sqrt{N}} \int_{a}^{b} \left( J_0(\alpha_m r) \frac{Y_0(\alpha_m r)}{Y_0(\alpha_m b)} J_0(\alpha_m b) \right) (r^2 - a^2)(r^2 - b^2)dr,
\]

(39)
5. Numerical Calculations

The numerical calculations have been carried out for steel (SN 50C) plate with the parameters $a = 1m$, $b = 2m$, $h = 0.3$, thermal diffusivity $K = 15.9 \times 10^{-6}(m^2s^{-1})$ and Poisson ratio $\nu = 0.281$ with $\alpha_1 = 3.120$, $\alpha_2 = 6.2734$, $\alpha_3 = 9.4182$, $\alpha_4 = 12.5614$, $\alpha_5 = 15.7040$ being the positive roots of transcendental equation

\[
\begin{bmatrix}
J_0(\alpha a) & Y_0(\alpha a) \\
J_0(\alpha b) & Y_0(\alpha b)
\end{bmatrix} = 0.
\]

In order to examine the influence of heat flux on the upper and lower surface of thick plate, one performed the numerical calculations $r = 1, 1.2, 1.4, 1.6, 1.8, 2m$ and $z = -0.3, -0.15, 0, 0.15, 0.3m$. Numerical variations in radial and axial directions are shown in the figures with the help of computer program.

6. Concluding Remarks

In this problem, a thick annular disc is considered which is free from traction and determined the expressions for temperature, displacement and stress function due to arbitrary heat flux under steady state. As a special case mathematical model is constructed for

\[
f(r) = (r^2 - a^2)(r^2 - b^2)^2
\]

and performed numerical calculations. The thermoelastic behavior is examined such as temperature, displacement and stresses with the help of arbitrary heat applied.

**Figure 1:** The temperature is maximum at the middle of the thick annular disc and symmetrical to words outer and inner circular surfaces.

**Figure 2:** The stress function $\sigma_{rr}$ is maximum in the $\frac{1}{4}$ the the $\frac{3}{4}$ middle region of the thick annular disc and reduces to zero towards inner and outer surfaces of the thick annular disc.

**Figure 3:** The stress function $\sigma_{zz}$ is maximum at $r = 1.4$ and varies in the thickness.

**Figure 4:** The stress function $\sigma_{\theta\theta}$ is maximum near to 1.5 and varies with the thickness of the plate.
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Fig. 1: Temperature $T$ versus $r$ for different values of $z$ at $z = -0.3, -0.15, 0, 0.15, 0.3$

Fig. 2: Stress functions $\sigma_{rr}$ versus $r$ for different values of $z$ at $z = -0.3, -0.15, 0, 0.15, 0.3$

Fig. 3: Stress function $\sigma_{zz}$ versus $r$ for different values of $z$ at $z = -0.3, -0.15, 0, 0.15, 0.3$
Fig. 4: Stress function $\sigma_{\theta \theta}$ versus $r$ for different values of $z$ at $z = -0.3, -0.15, 0, 0.15, 0.3$

References


[8] S. K. Roy Choudhary, A note of quasi static stress in a thin circular plate due to transient temperature applied along the circumference of a circle over the


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