Asymptotics of the Solution of a Boundary Value Problem in an Infinite Semi-Strip for One-Characteristic Differential Equation Degenerating into a Parabolic Equation

Mahir M. Sabzaliev

Department of Mathematics
Azerbaijan State University of Oil and Industry, Azerbaijan

Mahbuba E. Kerimova

Department of Mathematics
Azerbaijan State University of Oil and Industry, Azerbaijan

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Abstract

In an infinite semi-strip we consider a boundary value problem for a third order nonclassical type equation degenerating into a parabolic equation. The complete expansion of the solution of the problem under consideration in small parameter was constructed and the remainder term was estimated.

Keywords: Asymptotics, Boundary layer function, Remainder term

1 Introduction

Singularly perturbed boundary value problems have attracted attention of many mathematicians. But a great majority of the studied perturbed partial differential equations referred to one of three classical types. In the paper
In $D = \{(x, y)|0 \leq x \leq 1, 0 \leq y \leq 1\}$ the following boundary value problem was considered:

$$
\varepsilon^2 \frac{\partial}{\partial x}(\Delta u) - \varepsilon \Delta u + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u = f(x, y),
$$

(1)

$$
u|_\Gamma = 0, \frac{\partial u}{\partial x}|_{x=1} = 0,
$$

(2)

where $\Gamma$ is the boundary of domain $D$. Assuming that $f(x, y)$ for $x = y$ together with its derivatives of corresponding order vanishes, only the first terms of the asymptotics of the solution of boundary value problem (1), (2) were constructed.

In the paper [1] M.G. Javadov and M.M. Sabzaliev rejecting from the condition of vanishing of $f(x, y)$ for $x = y$, constructed the first terms of the asymptotics of the solution of boundary value problem (1), (2) with regard to internal layers arising near $x = y$. In the same paper, the asymptotics of the solution of this problem to within any degree of a small parameter was also constructed.

Complete asymptotics in a small parameter of the solution of a boundary value problem for a third order one-characteristic equation degenerating into an elliptic equation was constructed by M.M. Sabzaliev in [2].

Boundary value problems for nonclassical equations that are not arbitrary odd order one-characteristic equations degenerating into hyperbolic and parabolic equations, were studied in the papers [5]-[7].

In the papers [3] and [4] in an infinite strip

$$
P = \{(t, x)|0 \leq t \leq 1, -\infty < x < +\infty\}
$$

we have studied a boundary value problem for the equation

$$
L_\varepsilon u \equiv \varepsilon^2 \frac{\partial}{\partial t}(\Delta u) - \varepsilon \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + au = f(t, x),
$$

(3)

where $\varepsilon > 0$ is a small parameter, $\Delta \equiv \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}, a > 0$ is a constant, $f(t, x)$ is a given function.

In the present paper, in a semi-infinite semi

$$
P_+ = \{(t, x)|0 \leq t \leq 1, 0 \leq x < +\infty\}
$$

we consider a boundary value problem for equation (3) with the following boundary conditions:

$$
u|_{t=0} = 0, \quad u|_{t=0} = 0, \quad \frac{\partial u}{\partial t}|_{t=1} = 0,
$$

(4)
Asymptotics of the solution of a boundary value problem

\[ u|_{x=0} = 0, \quad \lim_{x \to +\infty} u = 0. \]  \hspace{1cm} (5)

Our goal is to construct the asymptotic expansion of the solution of boundary value problem (3)-(5) in a small parameter. When constructing the asymptotics we use the M.I. Vishik-L.A. Lusternik technique represented in [8] and [9]. For constructing the asymptotics, we carry out iterative processes.

2 The first iterative process

In the first iterative process we look for the solution of equation (3) in the form

\[ W = W_0 + \varepsilon W_1 + \ldots + \varepsilon^n W_n, \]  \hspace{1cm} (6)

and the functions \( W_i(t,x); i = 0, 1, \ldots, n \) will be chosen so that the following condition be fulfilled

\[ L_\varepsilon W = 0(\varepsilon^{n+1}). \]  \hspace{1cm} (7)

Substituting (7) in (3) and equating the terms with identical powers of \( \varepsilon \), for determining \( W_i; i = 0, 1, \ldots, n \) we get the following recurrently connected equations:

\[ \frac{\partial W_0}{\partial t} - \frac{\partial^2 W_0}{\partial x^2} + aW_0 = f(t,x), \]  \hspace{1cm} (8)

\[ \frac{\partial W_1}{\partial t} - \frac{\partial^2 W_1}{\partial x^2} + aW_1 = \frac{\partial^2 W_0}{\partial t^2}, \]  \hspace{1cm} (9)

\[ \frac{\partial W_k}{\partial t} - \frac{\partial^2 W_k}{\partial x^2} + aW_k = \frac{\partial^2 W_{k-1}}{\partial t^2} - \frac{\partial}{\partial t}(\Delta W_{k-1}); k = 2, 3, \ldots, n. \]  \hspace{1cm} (10)

Equation (8) is obtained from equation (3) for \( \varepsilon = 0 \) and is called a degenerate equation corresponding to equation (3).

Obviously, it is impossible to use all boundary conditions (4) and (5) for equations (8)-(10). For equations (8), (9), (10) with respect to \( t \) the first condition from (4) and with respect to \( x \) both conditions from (5) should be used, i.e.

\[ W_i|_{t=0} = 0, \]  \hspace{1cm} (11)

\[ W_i|_{x=0} = 0, \quad \lim_{x \to +\infty} W_i = 0; i = 0, 1, \ldots, n. \]  \hspace{1cm} (12)

Thus, from (8) and (11), (12) for \( i=0 \) we get that the function \( W_0(t,x) \) is the solution of equation (8) satisfying the conditions

\[ W_0|_{t=0} = 0, (0 \leq x < +\infty), \]  \hspace{1cm} (13)

\[ W_0|_{x=0} = 0, \quad \lim_{x \to +\infty} W_0 = 0; (0 \leq t \leq 1). \]  \hspace{1cm} (14)
It is known that the bounded solution of equation (8) satisfying initial condition (11) and the first boundary condition from (12) is determined by the formula

$$W_0(t, x) = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-\tau}} \left[ e^{-\frac{(x-\xi)^2}{4(t-\tau)}} - e^{-\frac{(x+\xi)^2}{4(t-\tau)}} \right] e^{-a(t-\tau)} f(\tau, \xi) d\tau d\xi. \quad (15)$$

On the function $f(t, x)$ standing at the right side of equation (8) we should impose such a condition that it could provide not only fulfilment of the second condition from (12) but also as a corollary vanishing as $x \to +\infty$ of the constructed functions $W_1, W_2, ..., W_n$.

Introduce the denotation:

$$C^{\alpha, \beta}(P_+) = \{ f(t, x) \mid \partial^i f(t, x) \in C(D); i = i_1 + i_2; i_1, i_2 = 0, 1, ..., \alpha; i_2 = 0, 1, ..., \beta \}.$$

It holds the following statement.

**Lemma 2.1** Let the function $f(t, x) \in C^{n+3, n+2}(P_+)$ and for any value of $t$ from $[0, 1]$ satisfy the condition

$$\left| \frac{\partial^k f(t, x)}{\partial t^{k_1} \partial x^{k_2}} \right| \leq c_1 e^{-c_2 x}; \quad c_1 > 0, c_2 > 0; \quad k = k_1 + k_2; \quad k_1 = 0, 1, ..., n + 3; \quad k_2 = 0, 1, ..., n + 2. \quad (16)$$

Then the function $W_0(t, x)$ determined by formula (15) is the solution of problem (8), (13), (14), and the function $W_0(t, x) \in C^{n+4, n+2}(P_+)$ and for any value of $t$ from $[0, 1]$ satisfies the condition

$$\left| \frac{\partial^k W_0(t, x)}{\partial t^{k_1} \partial x^{k_2}} \right| \leq c_3 e^{-c_2 x}; \quad c_3 > 0; \quad k = k_1 + k_2; \quad k_1 = 0, 1, ..., n + 4; \quad k_2 = 0, 1, ..., n + 2. \quad (17)$$

**Proof.** The fact that the function $W_0(t, x)$ determined by formula (15) is the solution of problem (8), (13), (14) is obvious. Prove that this function satisfies condition (17).

Introduce the following denotation:

$$J_1(t, x, \tau) = \frac{1}{\sqrt{t-\tau}} \int_0^t \frac{1}{\sqrt{t-\tau}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} f(t, \xi) d\xi; \quad (18)$$

$$J_2(t, x, \tau) = \frac{1}{\sqrt{t-\tau}} \int_0^t \frac{1}{\sqrt{t-\tau}} e^{-\frac{(x+\xi)^2}{4(t-\tau)}} f(t, \xi) d\xi. \quad (19)$$
Using these denotations, we can write down formula (15) in the form
\[ W_0(t, x) = \frac{1}{2\sqrt{\pi}} \int_0^t e^{-a(t-\tau)} [J_1(t, x, \tau) - J_2(t, x, \tau)] d\tau. \] (20)

By substituting \(-\frac{x-\xi}{2\sqrt{t-\tau}} = y\) the function \(J_1(t, x, \tau)\) takes the form
\[ J_1(t, x, \tau) = 2 \int_{-\frac{x-\xi}{2\sqrt{t-\tau}}}^{+\infty} e^{-y^2} f(\tau, x + 2\sqrt{t-\tau}y) dy. \] (21)

Having used condition (16), for \(k = 0\), we estimate the right side of (21) in the form:
\[ |J_1(t, x, \tau)| \leq 2c_1 \int_{-\frac{x-\xi}{2\sqrt{t-\tau}}}^{0} e^{-y^2} e^{-c_2(x+2\sqrt{t-\tau}y)} dy = \]
\[ = 2c_1 e^{-c_2x} \int_{-\frac{x-\xi}{2\sqrt{t-\tau}}}^{0} e^{-y^2} e^{-2c_2\sqrt{t-\tau}y} dy + \]
\[ + 2c_1 e^{-c_2x} \int_{0}^{+\infty} e^{-y^2} e^{-2c_2\sqrt{t-\tau}y} dy \leq \]
\[ \leq 2c_1 e^{-c_2x} \int_{0}^{\frac{x+\xi}{2\sqrt{t-\tau}}} e^{-y^2} e^{-2c_2\sqrt{t-\tau}y} dy + 2c_1 \sqrt{\pi} e^{-c_2x} = \]
\[ = 2c_1 e^{-c_2x} \left[ \sqrt{\pi} + e^{-4c_2^2(t-\tau)} \int_{0}^{z} e^{-\alpha^2} d\alpha \right] \leq c_3 e^{-c_2x}, \]
where \(z = \frac{x+\xi}{2\sqrt{t-\tau}}, \quad \alpha = y - 2c_2\sqrt{t-\tau}, \quad c_3 = 3\sqrt{\pi}c_1\).

By substitution \(\frac{x+\xi}{2\sqrt{t-\tau}} = y\) the function \(J_2(t, x, \tau)\) determined by formula (19) is reduced to the integral
\[ J_2(t, x, \tau) = 2 \int_{\frac{x+\xi}{2\sqrt{t-\tau}}}^{+\infty} e^{-y^2} f(\tau, -x + 2\sqrt{t-\tau}y) dy. \] (22)

In a similar way, we estimate \(J_2(t, x, \tau)\). Knowing the estimates for \(J_1\) and \(J_2\), from (20) we easily get the validity of estimates for \(W_0(t, x)\). The proof of the validity of estimation (17) for the derivatives \(W_0(t, x)\) differs negligibly from the proof of satisfaction of this estimation for \(W_0\). The process of differentiation with respect to \(x\) shows that the expressions for the derivatives from \(W_0\) with respect to \(x\) will differ from (20) (see also (21), (22)) by the fact that under the integral with respect to \(y\) there stand the derivatives \(f(t,x)\) with respect to the second argument, and the terms standing outside the integrals will be products of \(\exp\left[-\frac{x^2}{4(t-\tau)}\right]\) by degree \(x\). Therefore, estimation (17) is valid for the derivatives \(W_0(t, x)\) with respect to \(x\) to \((n+2)\)-th order, inclusively.

The validity of estimations (17) for the derivatives \(W_0(t, x)\) with respect to \(t\) and for mixed derivatives is based on their obvious expressions that we don’t cite.
Lemma 1 is proved.

Lemma 1 ensuring the smoothness of the function $W_0$ and satisfaction of estimations (17) allows to construct the remaining functions $W_1, W_2, ..., W_n$, that enter into the right hand side of (6).

It follows from (6), (11), (12) that the constructed function $W$ satisfies the following boundary conditions:

$$W|_{t=0} = 0, (0 \leq x < +\infty), W|_{x=0} = 0, \lim_{x \to +\infty} W = 0; (0 \leq t \leq 1).$$

This function doesn’t satisfy, generally speaking, the second and third boundary conditions from (4) for $t=1$. Therefore we should construct the function $V$ of boundary layer type near the boundary $t=1$ so that the obtained sum $W+V$ satisfies the boundary conditions

$$(W + V)|_{t=1} = 0, \quad \frac{\partial}{\partial t}(W + V)|_{t=1} = 0. \quad (24)$$

For constructing the function $V$ of boundary layer type it is necessary to carry out the second iterative process.

3 The second iterative process-construction of boundary layer functions

To carry out the second iterative process, at first we should write a new decomposition of the operator $L_\varepsilon$ near the boundary $t=1$. We make a change of variables $1-t = \varepsilon \tau, x = x$. A new decomposition of the operator $L_\varepsilon$ in the coordinates $(\tau, x)$ has the form

$$L_{\varepsilon,1} \equiv \varepsilon^{-1} \left[ -\left( \frac{\partial^3}{\partial \tau^3} + \frac{\partial^2}{\partial \tau^2} + \frac{\partial}{\partial \tau} \right) + \varepsilon \left( -\frac{\partial^2}{\partial x^2} + a \right) - \varepsilon^2 \frac{\partial^3}{\partial \tau \partial x^2} \right]. \quad (25)$$

We look for the boundary layer function $V$ near the boundary $t=1$ in the form

$$V = V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + ... + \varepsilon^{n+1} V_{n+1}, \quad (26)$$

as an approximate solution of the equation

$$L_{\varepsilon,1} V = 0. \quad (27)$$

Having substituted the expression for $V$ from (26) into (27), taking into account (25) and comparing the terms at identical powers of $\varepsilon$, we have:

$$\frac{\partial^3 V_0}{\partial \tau^3} + \frac{\partial^2 V_0}{\partial \tau^2} + \frac{\partial V_0}{\partial \tau} = 0, \quad (28)$$
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\[
\frac{\partial^3 V_1}{\partial \tau^3} + \frac{\partial^2 V_1}{\partial \tau^2} + \frac{\partial V_1}{\partial \tau} = - \frac{\partial^2 V_0}{\partial x^2} + aV_0, \quad (29)
\]

\[
\frac{\partial^3 V_k}{\partial \tau^3} + \frac{\partial^2 V_k}{\partial \tau^2} + \frac{\partial V_k}{\partial \tau} = - \frac{\partial^2 V_{k-1}}{\partial x^2} + aV_{k-1} - \frac{\partial^3 V_{k-2}}{\partial \tau \partial x^2}, k = 2, 3, ..., n + 1. \quad (30)
\]

For finding boundary conditions for equations (28), (29), (30) we should put expansions (26) in (24) and compare the terms at the same powers of \( \varepsilon \).

Then we get:

\[
V_i |_{\tau=0} = \varepsilon W_i |_{t=1}; i = 0, 1, ..., n; \quad V_{n+1} |_{\tau=0} = 0, \quad (31)
\]

\[
\left. \frac{\partial V_0}{\partial \tau} \right|_{\tau=0} = 0, \quad \left. \frac{\partial V_j}{\partial \tau} \right|_{\tau=0} = \left. \frac{\partial W_{j-1}}{\partial t} \right|_{t=1}; j = 1, 2, ..., n + 1. \quad (32)
\]

When constructing the functions \( V_j; j = 0, 1, ..., n + 1 \), we use the following statement.

**Lemma 3.1** The function \( V_j \) that is the solutions of boundary layer type equations (28), (29), (30) and satisfying corresponding conditions from (31), (32) is determined by the formula

\[
V_j(\tau, x) = \left[ \sum_{i=0}^{j} a_{ji}(x) \tau^i \right] e^{\lambda_1 \tau} + \left[ \sum_{i=0}^{j} b_{ji}(x) \tau^i \right] e^{\lambda_2 \tau}, j = 0, 1, ..., n + 1, \quad (33)
\]

where \( \lambda_{1,2} = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2} \), the coefficients \( a_{ji}(x), b_{ji}(x) \) are expressed uniformly by the functions \( W_0(t,x), W_1(t,x), ..., W_j(t,x) \) and their derivatives with respect to \( t \) are of first order for \( t=1 \), while the derivatives with respect to \( x \) are only of even order, \( W_{n+1} \equiv 0 \).

The proof of lemma 1 is given in [3].

Multiply the all functions \( V_j \) by a smoothing function and denote the obtained new functions again by \( V_j; j = 0, 1, ..., n + 1 \). It follows from (33) that all the functions \( V_j(\tau, x) \) satisfy the conditions

\[
V_j |_{x=0} = 0, \quad \lim_{x \to +\infty} V_j = 0; j = 0, 1, ..., n + 1.
\]

Then it follows from (23) and (26) that the sum \( \tilde{u} = W + V \) constructed by us, besides (24) satisfies also the boundary conditions

\[
(W + V) |_{t=0} = 0, \quad (W + V) |_{x=0} = 0, \quad \lim_{x \to \infty} (W + V) = 0. \quad (34)
\]

Denote the difference of the solution of problem (3), (4), (5) and \( \tilde{u} \) by

\[
u - \tilde{u} = \varepsilon^{n+1} z, \quad (35)
\]

and call \( \varepsilon^{n+1} z \) a remainder term. Now we should estimate the remainder term.
4 Estimation of remainder term and conclusion

The following statement holds.

**Lemma 4.1** For the function $z$ the following estimation is valid

$$
\varepsilon^2 \left\| \frac{\partial z}{\partial t} \right\|_{L^2(0;+\infty)}^2 + \varepsilon \left\| \frac{\partial z}{\partial t} \right\|_{L^2(P_+)}^2 + \left\| \frac{\partial z}{\partial x} \right\|_{L^2(P_+)}^2 + c_1 \|z\|_{L^2(P_+)}^2 \leq c_2, \tag{36}
$$

where $c_1 > 0, c_2 > 0$ are the constants independent of $\varepsilon$.

The proof of lemma 3 is in [3].

From $\tilde{u} = W + V$ and from (6), (26), (35) we have

$$
u = \sum_{i=0}^{n} \varepsilon^i W_i + \sum_{j=0}^{n+1} \varepsilon^j V_j + \varepsilon^{n+1} z. \tag{37}
$$

The obtained results may be generalized in the form of the following statement.

**Theorem 4.2** Assume that the function $f(t, x) \in C^{n+3, n+2}(P_+)$ satisfies condition (16). Then for the solution of boundary value problem (3)-(5), asymptotic representation (37) is valid. In the representation the functions $W_i$ are determined by the first iterative process, $V_j$ are boundary layer type functions near the boundary $t = 1$, and are determined by the second iterative process, $\varepsilon^{n+1} z$ is a remainder term, and for the function $z$ estimation (36) is valid.

References


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