Matched Asymptotic Expansions
Valid on a Neighborhood of a River Side

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Abstract
The river silting process is complex and represents one of the greatest challenges faced in the future. In this paper, we propose an asymptotic approach for describing silting of rivers to study asymptotic solutions governing a coupled (hydro-sedimentary) model. We focus our paper on the inner and outer asymptotic expansions of solutions. The proposed approach is based on the hypothesis that rivers silting can be explain by an asymptotic behavior near shock curves of the governing equations of sediment deposits. We look for an inner and outer asymptotic expansion which gives us expressions of the solution in the neighborhood of the shock zone. Once both asymptotic expansions found, we give matching property of both expansions.

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1 Introduction

Siltation of rivers is one of the most important phenomena in the hydrogeological field. It specifically contributes to the modification of riverbeds and, has significant impacts as well as on the environment. Sedimentation of rivers and morphological processes are among the most complex and least understood phenomena of nature.

We can estimate that the first modeling attempts of the modern era in the field of sedimentation of rivers go back to the 1950s. With the development of computer tools, this research took off in the 1970s and 1980s from the work of Cunge and al. [6, 14, 16] in 1-D models. Since then, considerable progress has been made in improving 1-D, 2-D and 3-D mathematical models. One can refer to [8, 9, 12, 17].

In the literature, two types of models for describing the river silting phenomenon are considered: sedimentary models and coupled hydro-sedimentary models. The hydro-sedimentary models reflect the coupling of the sedimentation process and fluid flow.

Our specific goal in this paper is to formally give analytically an asymptotic approach using an Inner [2] and outer Asymptotic expansion for describing silting process near river banks.

The paper is organized as follows. In section 2 we give the description of the problem. In section 3 we develop an inner and outer asymptotic expansion to describe the silting behavior near river banks, we study the solution of the problem and we give a matching property.

2 A matched asymptotic expansion to describe river silting phenomenon in a shock layer

2.1 Problem statement

As it is shown in [1] Let us consider a portion of a river whose free surface is denoted by a $\mathbb{R}^2$ - domain $\Omega$ in a Cartesian axis system $(x,y)$. The boundary of $\Omega$ is as follows:

$$\partial \Omega = \Gamma^r_1 \cup \Gamma^r_2 \cup \Gamma^s_1 \cup \Gamma^s_2$$

(1)
where boundaries $\Gamma^r_i; i = 1, 2$ represent the river banks, while boundaries $\Gamma^s_i; i = 1, 2$ delimit upstream and downstream the length of the river and are defined by equations $x = \pm L$. To study the dynamics of the silting up through

![Figure 1: Schematization of the free surface a portion of the studied river](image)

the river bed we consider $S(t, x, y)$ the height of sediments at a point $(x, y)$ at time $t$. To be complete in the description of the studied problem, the flow is assumed to be incompressible and is oriented in the $x$ direction. So to describe the sediment evolution, we denote by $S(t, x, y)$ the height of sediments deposit at the point $(x, y)$ at the time $t$ as it is shown in [1]. The sediments deposit depend of course of the flow dynamic of the river which we shall describe its Cartesian velocity field vector at the point $(x, y)$ and at the time $t$ by

$$U = (u(t, x, y), v(t, x, y), w(t, x, y))$$

To describe the silting phenomena near rivers banks one can neglected the third component $w$. Thus the equations system of the sediment dynamic are then

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{1}{Re} \Delta u + g_x$$

(3)

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \frac{1}{Re} \Delta v + g_y$$

(4)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

(5)

$$\frac{\partial S}{\partial t} + k_x(u, v) \frac{\partial S}{\partial x} + k_y(u, v) \frac{\partial S}{\partial y} = \Delta(\phi(S)) + f(t, x, y)$$

(6)

The equations (3), (4) and (5) are the well known Navier Stoke equations which describe incompressible flows, while the equation 6 describes sediment deposit [18, 20]. Functions $k_x(u, v)$ and $k_y(u, v)$ are parameters which describe
transport of sediments in $x$, $y$ directions, and the term $\Delta(\phi(S))$ describes dispersion of sediment particles moving in the river, $f(t,x,y)$ is the source term. In addition, we introduce following boundary conditions on the river sides:

$$\frac{\partial v}{\partial n} = g_i \text{ on } \Gamma_i^r, \ i = 1,2$$

$$u = 0 \text{ on } \Gamma_i^r, \ i = 1,2$$

$$\frac{\partial S}{\partial n} = h_i \text{ on } \Gamma_i^r, \ i = 1,2$$

where $g_i$ and $h_i$, $i = 1,2$ are assumed to be known functions and, on the boundary part $x = \pm L$ no condition is imposed. Equations of type (6)-(9) are widely studied in the literature. Notably it has been shown that there exists at each time $t$ a shock line where the solution admits a $C^1$ discontinuity [10]. This shock line can be interpreted as a line where silting wrinkles are formed. Along with there exists at time $t$ the solutions which are not continuously differentiable. So, let us assume that we are given of a shock line near a river bank described by the equation

$$y = \eta(t,x)$$

where $\eta$ is a sufficiently smooth function. Then we consider a domain $\Omega^{(\epsilon,t)}$ round the shock line (10) such that $\Omega^{(\epsilon,t)} \rightarrow \Omega^0$ as $\epsilon \searrow 0$, $\Omega^0$ being a subdomain of $\Omega$. Then, in order to describe river silting phenomenon we have to consider the following shock layer in figure 2. For an asymptotic study in the domain $\Omega^{(\epsilon,t)}$

$$\Omega^{\epsilon,t} = \{(x,y) \in \Omega/ - L \leq x \leq L, \eta_1^{\epsilon}(t,x) \leq y \leq \eta_2^{\epsilon}(t,x)\}$$

with $\epsilon \searrow 0$, $\epsilon$ being a parameter considered small compared with the length and with the width of the studied portion and such as $\eta_i^{\epsilon}(t,x) \rightarrow \eta_i^0$ $i = 1,2$ for all $t,x$ fixed. We pose then $\Omega^0 = \lim_{\epsilon \searrow 0}\Omega^{\epsilon,t}$. Next, for the sake of the simplicity, we shall assume that the diffusion term is expressed as

$$\Delta \phi(S^{\epsilon}) = \mu(u^\epsilon,v^\epsilon)\Delta S$$

where $\mu(u^\epsilon,v^\epsilon)$ is the diffusion coefficient. We are looking to develop expressions of $S^\epsilon$ valid in $\Omega^{(\epsilon,t)}$. The asymptotic expansions are given to the neighborhood of the shock line following an internal or external zone.

### 2.2 Inner Asymptotic Expansion

we introduce inner variables by setting

$$x^* = x, \ y^* = \frac{y - \eta(t,x)}{\delta(\epsilon)}, \ \text{and} \ t^* = t$$

Matched asymptotic expansions

Figure 2: Passage from $\Omega^{\varepsilon,t}$ to $\Omega^0$

where $\delta(\varepsilon) \searrow 0$ as $\varepsilon \searrow 0$. We denote by $u^\varepsilon(t, x, y)$, $v^\varepsilon(t, x, y)$ the components of velocity fields and $S^\varepsilon(t, x, y)$ the solution of the system formed by equations (6) and (9) valid in the shock zone $\Omega^{\varepsilon,t}$. Then we look for an asymptotic expansion of these solutions in the form:

$$S^\varepsilon(t, x, y) = \varepsilon^{-\alpha} S^0(t^*, x^*, y^*) + o(\varepsilon^{-\alpha})$$

$$u^\varepsilon(t, x, y) = \varepsilon^{-\beta} u^0(t^*, x^*, y^*) + o(\varepsilon^{-\beta})$$

$$v^\varepsilon(t, x, y) = \varepsilon^{-\gamma} v^0(t^*, x^*, y^*) + o(\varepsilon^{-\gamma})$$

where $\alpha$, $\beta$, et $\gamma$ are appropriate parameters and where we have set

$$\lim_{\varepsilon \searrow 0} \frac{o(\varepsilon^M)}{\varepsilon^M} = 0$$
for some non null real $M$. Using relations (13), We obtain the following derivation formulae:

\[
\begin{align*}
\frac{\partial}{\partial t} &= \frac{\partial}{\partial t^*} - \frac{1}{\delta(\varepsilon)} \left( \frac{\partial \eta}{\partial t} \right) \frac{\partial}{\partial y^*} \\
\frac{\partial}{\partial x} &= \frac{\partial}{\partial x^*} - \frac{1}{\delta(\varepsilon)} \left( \frac{\partial \eta}{\partial x} \right) \frac{\partial}{\partial y^*} \\
\frac{\partial}{\partial y} &= \frac{1}{\delta(\varepsilon)} \frac{\partial}{\partial y^*}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2}{\partial x^2} &= \frac{\partial^2}{\partial x^*^2} - \frac{1}{\delta(\varepsilon)} \left( \frac{\partial^2 \eta}{\partial x^2} \right) \frac{\partial}{\partial y^*} - \frac{2}{\delta(\varepsilon)} \left( \frac{\partial \eta}{\partial x} \right) \frac{\partial^2}{\partial x^* \partial y^*} + \frac{1}{\delta^2(\varepsilon)} \left( \frac{\partial \eta}{\partial x} \right)^2 \frac{\partial^2}{\partial y^*^2} \\
\frac{\partial^2}{\partial y^2} &= \frac{1}{\delta^2(\varepsilon)} \frac{\partial^2}{\partial y^*^2}
\end{align*}
\]  

(18)

We look for the internal asymptotic expansions of the solutions $S^\varepsilon(t, x, y)$, $u^\varepsilon(t, x, y)$ and $v^\varepsilon(t, x, y)$ of equations (3), (4), (5) and (6), on the neighborhood of the shock zone in the form:

\[
\begin{align*}
S^\varepsilon,\text{int}(t, x, y) &= \varepsilon^{-\alpha} S^0,\text{int}(t^*, x^*, y^*) + o(\varepsilon^{-\alpha}) \\
u^\varepsilon,\text{int}(t, x, y) &= \varepsilon^{-\beta} u^0,\text{int}(t^*, x^*, y^*) + o(\varepsilon^{-\beta}) \\
v^\varepsilon,\text{int}(t, x, y) &= \varepsilon^{-\gamma} v^0,\text{int}(t^*, x^*, y^*) + o(\varepsilon^{-\gamma})
\end{align*}
\]  

(19)

(20)

(21)

where:

\[
\lim_{\varepsilon \searrow 0} \frac{o(\varepsilon^{-\alpha})}{\varepsilon^{-\alpha}} = 0; \quad \lim_{\varepsilon \searrow 0} \frac{o(\varepsilon^{-\beta})}{\varepsilon^{-\beta}} = 0; \quad \lim_{\varepsilon \searrow 0} \frac{o(\varepsilon^{-\gamma})}{\varepsilon^{-\gamma}} = 0
\]  

(22)

and $\alpha, \beta, \gamma$ are appropriate parameters.

Considering the hypotheses made higher, the parameters $\alpha \text{ et } \beta$ Have to satisfy:

\[
0 < \beta < \alpha
\]  

(23)

We also suppose that the developments of the perturbed expressions of the terms $k_x(u^\varepsilon, v^\varepsilon)$, $k_y(u^\varepsilon, v^\varepsilon)$ et $\mu(u^\varepsilon, v^\varepsilon)$ Are given by the following relations:

\[
k_x(u^\varepsilon, v^\varepsilon) = k_x^0 u^{0,\text{int}} \varepsilon^{-\beta} + o(\varepsilon^{-\beta})
\]  

(24)

\[
k_y(u^\varepsilon, v^\varepsilon) = k_y^0 v^{0,\text{int}} \frac{\delta(\varepsilon)}{\varepsilon^{-\gamma}} + o(\varepsilon^{-\gamma})
\]  

(25)
\[ \mu(u^\varepsilon, v^\varepsilon) = \mu^0 \varepsilon^{-\gamma} + o(\varepsilon^{-\gamma}) \] (26)

where \( k_x^0, k_y^0, \) and \( \mu^0 \) are constant parameters. Applying formulae (18), it follows:

\[ \begin{aligned}
\frac{\partial S^{\varepsilon, \text{int}}}{\partial t} &= \varepsilon^{-\alpha} \frac{\partial S^0, \text{int}}{\partial t^*} + o(\varepsilon^{-\alpha}) - \frac{\varepsilon^{-\alpha}}{\delta(\varepsilon)} \frac{\partial S^0, \text{int}}{\partial y^*} + o \left( \frac{\varepsilon^{-\alpha}}{\delta(\varepsilon)} \right) \\
\frac{\partial S^{\varepsilon, \text{int}}}{\partial x} &= -\frac{\varepsilon^{-\alpha}}{\delta(\varepsilon)} \frac{\partial \eta}{\partial x} \frac{\partial S^0, \text{int}}{\partial y^*} + o \left( \frac{\varepsilon^{-\alpha}}{\delta(\varepsilon)} \right) \\
\frac{\partial S^{\varepsilon, \text{int}}}{\partial y} &= \frac{\varepsilon^{-\alpha}}{\delta(\varepsilon)} \frac{\partial S^0, \text{int}}{\partial y^*} + o \left( \frac{\varepsilon^{-\alpha}}{\delta(\varepsilon)} \right) \\
\frac{\partial^2 S^{\varepsilon, \text{int}}}{\partial x^2} &= \frac{\varepsilon^{-\alpha}}{\delta^2(\varepsilon)} \frac{\partial \eta}{\partial x^2} + o \left( \frac{\varepsilon^{-\alpha}}{\delta^2(\varepsilon)} \right) \\
\frac{\partial^2 S^{\varepsilon, \text{int}}}{\partial y^2} &= \frac{\varepsilon^{-\alpha}}{\delta^2(\varepsilon)} \frac{\partial^2 S^0, \text{int}}{\partial y^2} + o \left( \frac{\varepsilon^{-\alpha}}{\delta^2(\varepsilon)} \right)
\end{aligned} \] (27)

using the formulae (27), The convective term can be write

\[ k_x(u^\varepsilon, v^\varepsilon) \frac{\partial S^{\varepsilon, \text{int}}}{\partial x} + k_y(u^\varepsilon, v^\varepsilon) \frac{\partial S^{\varepsilon, \text{int}}}{\partial y} = k_x(u^\varepsilon, v^\varepsilon) \left( -\frac{\varepsilon^{-\alpha}}{\delta(\varepsilon)} \frac{\partial \eta}{\partial x} \frac{\partial S^0, \text{int}}{\partial y^*} + o \left( \frac{\varepsilon^{-\alpha}}{\delta(\varepsilon)} \right) \right) \\
+ k_y(u^\varepsilon, v^\varepsilon) \left( \frac{\varepsilon^{-\alpha}}{\delta(\varepsilon)} \frac{\partial S^0, \text{int}}{\partial y^*} + o \left( \frac{\varepsilon^{-\alpha}}{\delta(\varepsilon)} \right) \right) \] (28)

Indeed, considering the first order of development and using developments of (24) and (25) we have:

\[ k_x(u^\varepsilon, v^\varepsilon) \frac{\partial S^{\varepsilon, \text{int}}}{\partial x} + k_y(u^\varepsilon, v^\varepsilon) \frac{\partial S^{\varepsilon, \text{int}}}{\partial y} = k_x^0 \frac{1}{\delta^2(\varepsilon)} \varepsilon^{-\alpha} v^{0, \text{int}} \frac{\partial S^0, \text{int}}{\partial y^*} + o \left( \frac{\varepsilon^{-\alpha}}{\delta^2(\varepsilon)} \right) \] (29)
furthermore,

\[ \mu(u^\varepsilon, v^\varepsilon) \left( \frac{\partial^2 S^{\varepsilon,int}}{\partial x^2} + \frac{\partial^2 S^{\varepsilon,int}}{\partial y^2} \right) = (\mu^{0,int} \varepsilon^{-\gamma} v^{0,int} + 0(\varepsilon^{-\gamma})) \]

\[ \left( \frac{\varepsilon^{-\alpha}}{\delta^2(\varepsilon)} \left( \frac{\partial \eta}{\partial x} \right)^2 \frac{\partial^2 S^{0,int}}{\partial y^*} + o\left( \frac{\varepsilon^{-\alpha}}{\delta^2(\varepsilon)} \right) \right) = \]

\[ \mu^{0,int} v^{0,int} \varepsilon^{-\gamma - \alpha} \frac{1}{\delta^2(\varepsilon)} \left( 1 + \left( \frac{\partial \eta}{\partial x} \right)^2 \right) \frac{\partial^2 S^{0,int}}{\partial y^*} + o\left( \frac{\varepsilon^{-\alpha - \gamma}}{\delta^2(\varepsilon)} \right) = f^{\varepsilon} \]

(30)

Replacing \( S^{\varepsilon,int} \) by the expression of its development (19), And considering the expressions of its derivatives (27), it results from this by multiplying member with member the obtained equation by \( \delta^2(\varepsilon)\varepsilon^{\alpha + \gamma} \),

\[ k_y^0 \frac{1}{\delta^2(\varepsilon)} \varepsilon^{-\alpha - \gamma} v^{0,int} \partial S^{0,int} \frac{\partial}{\partial y^*} - \mu^{0,int} v^{0,int} \varepsilon^{-\gamma - \alpha} \frac{1}{\delta^2(\varepsilon)} \left( 1 + \left( \frac{\partial \eta}{\partial x} \right)^2 \right) \frac{\partial^2 S^{0,int}}{\partial y^*} + o(1) = f^{\varepsilon} \]

(31)

Multiplying by \( \delta^2(\varepsilon)\varepsilon^{\alpha + \gamma} \) This last equation and by passing on the limit when \( \varepsilon \downarrow 0 \), that gives

\[ k_y^0 v^{0,int} \partial S^{0,int} \frac{\partial}{\partial y^*} - \mu v^{0,int} \left( 1 + \left( \frac{\partial \eta}{\partial x} \right)^2 \right) \frac{\partial^2 S^{0,int}}{\partial y^*} = f^{0,int} \]

(32)

where we have posed \( f^{0,int} = \lim_{\varepsilon \downarrow 0} \delta^2(\varepsilon)\varepsilon^{\alpha + \gamma} f^{\varepsilon} \).

We see that (32) is a second order differential equation in \( y^* \) Whose resolution depends on conditions in the limits which will be supplied by the conditions of connecting with the external expansion.

From formulae of derivation (18) and the asymptotic expansions (20) we have
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\[ \begin{align*}
\frac{\partial u^{\varepsilon,\text{int}}}{\partial t} &= -\frac{\varepsilon^{-\beta}}{\delta(\varepsilon)} \left( \frac{\partial \eta}{\partial t} \right) \frac{\partial u^{0,\text{int}}}{\partial y^*} + o\left( \frac{\varepsilon^{-\beta}}{\delta(\varepsilon)} \right) \\
\frac{\partial u^{\varepsilon,\text{int}}}{\partial x} &= -\frac{\varepsilon^{-\beta}}{\delta(\varepsilon)} \left( \frac{\partial \eta}{\partial x} \right) \frac{\partial u^{0,\text{int}}}{\partial y^*} + o\left( \frac{\varepsilon^{-\beta}}{\delta(\varepsilon)} \right) \\
\frac{\partial u^{\varepsilon,\text{int}}}{\partial y} &= \frac{\varepsilon^{-\beta}}{\delta(\varepsilon)} \frac{\partial u^{0,\text{int}}}{\partial y^*} + o\left( \frac{\varepsilon^{-\beta}}{\delta(\varepsilon)} \right) \\
\frac{u^{\varepsilon,\text{int}}}{\partial x} \frac{\partial u^{\varepsilon,\text{int}}}{\partial x} &= -\frac{\varepsilon^{-2\beta}}{\delta(\varepsilon)} u^{0,\text{int}} \left( \frac{\partial \eta}{\partial x} \right) \frac{\partial u^{0,\text{int}}}{\partial y^*} + o\left( \frac{\varepsilon^{-2\beta}}{\delta(\varepsilon)} \right) \\
\frac{v^{\varepsilon,\text{int}}}{\partial y} \frac{\partial u^{\varepsilon,\text{int}}}{\partial y} &= \frac{\varepsilon^{-\beta-\gamma}}{\delta(\varepsilon)} v^{0,\text{int}} \frac{\partial u^{0,\text{int}}}{\partial y^*} + o\left( \frac{\varepsilon^{-\beta-\gamma}}{\delta(\varepsilon)} \right) \\
\frac{\partial^2 u^{\varepsilon,\text{int}}}{\partial x^2} &= \frac{\varepsilon^{-\beta}}{\delta^2(\varepsilon)} \left( \frac{\partial \eta}{\partial x} \right)^2 \frac{\partial^2 u^{0,\text{int}}}{\partial y^*} + o\left( \frac{\varepsilon^{-\beta}}{\delta^2(\varepsilon)} \right) \\
\frac{\partial^2 u^{\varepsilon,\text{int}}}{\partial y^2} &= \frac{\varepsilon^{-\beta}}{\delta^2(\varepsilon)} \frac{\partial^2 u^{0,\text{int}}}{\partial y^*} + o\left( \frac{\varepsilon^{-\beta}}{\delta^2(\varepsilon)} \right) \\
\Delta u^{\varepsilon,\text{int}} &= \frac{\varepsilon^{-\beta}}{\delta^2(\varepsilon)} \left[ 1 + \left( \frac{\partial \eta}{\partial x} \right)^2 \right] \frac{\partial^2 u^{0,\text{int}}}{\partial y^2} + o\left( \frac{\varepsilon^{-\beta}}{\delta^2(\varepsilon)} \right) \\
\frac{\partial p}{\partial x} &= \frac{\partial p}{\partial x^*} - \frac{1}{\delta(\varepsilon)} \left( \frac{\partial \eta}{\partial x} \right) \frac{\partial p}{\partial y^*}
\end{align*} \]

By a similar development (21) We obtain
\[
\frac{\partial \varepsilon^{\text{int}}}{\partial t} = -\frac{\partial \varepsilon^{-\gamma}}{\delta(\varepsilon)} \left( \frac{\partial \eta}{\partial t} \right) \frac{\partial \varepsilon^{\text{int}}}{\partial y^*} + o \left( \frac{\varepsilon^{-\gamma}}{\delta(\varepsilon)} \right)
\]

\[
\frac{\partial \varepsilon^{\text{int}}}{\partial x} = -\frac{\varepsilon^{-\gamma}}{\delta(\varepsilon)} \left( \frac{\partial \eta}{\partial x} \right) \frac{\partial \varepsilon^{\text{int}}}{\partial y^*} + o \left( \frac{\varepsilon^{-\gamma}}{\delta(\varepsilon)} \right)
\]

\[
\frac{\partial \varepsilon^{\text{int}}}{\partial y} = \frac{\varepsilon^{-\gamma}}{\delta(\varepsilon)} \frac{\partial \varepsilon^{\text{int}}}{\partial y^*} + o \left( \frac{\varepsilon^{-\gamma}}{\delta(\varepsilon)} \right)
\]

\[
u^{\text{int}} \frac{\partial \nu^{\text{int}}}{\partial x} = -\frac{\varepsilon^{-\beta-\gamma}}{\delta(\varepsilon)} u^{\text{int}} \left( \frac{\partial \eta}{\partial x} \right) \frac{\partial \nu^{\text{int}}}{\partial y^*} + o \left( \frac{\varepsilon^{-\beta-\gamma}}{\delta(\varepsilon)} \right)
\]

\[
u^{\text{int}} \frac{\partial \nu^{\text{int}}}{\partial y} = \frac{\varepsilon^{-\gamma}}{\delta(\varepsilon)} v^{\text{int}} \frac{\partial \nu^{\text{int}}}{\partial y^*} + o \left( \frac{\varepsilon^{-\gamma}}{\delta(\varepsilon)} \right)
\]

\[
\frac{\partial^2 \nu^{\text{int}}}{\partial x^2} = \frac{\varepsilon^{-\gamma}}{\delta^2(\varepsilon)} \left( \frac{\partial \eta}{\partial x} \right)^2 \frac{\partial^2 \nu^{\text{int}}}{\partial y^*^2} + o \left( \frac{\varepsilon^{-\gamma}}{\delta^2(\varepsilon)} \right)
\]

\[
\frac{\partial^2 \nu^{\text{int}}}{\partial y^2} = \frac{\varepsilon^{-\gamma}}{\delta^2(\varepsilon)} \frac{\partial^2 \nu^{\text{int}}}{\partial y^*^2} + o \left( \frac{\varepsilon^{-\gamma}}{\delta^2(\varepsilon)} \right)
\]

\[
\Delta \nu^{\text{int}} = \frac{\varepsilon^{-\gamma}}{\delta^2(\varepsilon)} \left[ 1 + \left( \frac{\partial \eta}{\partial y} \right)^2 \right] \frac{\partial^2 \nu^{\text{int}}}{\partial y^*^2} + o \left( \frac{\varepsilon^{-\gamma}}{\delta^2(\varepsilon)} \right)
\]

\[
\frac{\partial p}{\partial y} = \frac{1}{\delta(\varepsilon)} \frac{\partial p}{\partial y^*}
\]

We can always set \( \beta = \gamma \) and we consider the hypothesis
\[
\lim_{\varepsilon \to 0} \delta(\varepsilon) \varepsilon^{-\alpha} = \lim_{\varepsilon \to 0} \delta(\varepsilon) \varepsilon^{-\gamma} = 0
\]

Indeed, considering the equations (3) and (4), the asymptotic developments (20) and (21), we obtain a similar development to (31)

\[
-\frac{\varepsilon^{-\beta}}{\delta(\varepsilon)} \left( \frac{\partial \eta}{\partial t} \right) \frac{\partial u^{\text{int}}}{\partial y^*} + o \left( \frac{\varepsilon^{-\beta}}{\delta(\varepsilon)} \right) - \frac{\varepsilon^{-2\beta}}{\delta(\varepsilon)} u^{\text{int}} \left( \frac{\partial \eta}{\partial x} \right) \frac{\partial u^{\text{int}}}{\partial y^*} + o \left( \frac{\varepsilon^{-2\beta}}{\delta(\varepsilon)} \right) +
\]

\[
\frac{\varepsilon^{-\beta-\gamma}}{\delta(\varepsilon)} u^{\text{int}} \frac{\partial u^{\text{int}}}{\partial y^*} + o \left( \frac{\varepsilon^{-\beta-\gamma}}{\delta(\varepsilon)} \right) + \frac{\partial p}{\partial x} - \frac{1}{\delta(\varepsilon)} \left( \frac{\partial \eta}{\partial x} \right) \frac{\partial p}{\partial y^*} -
\]

\[
\frac{1}{Re \delta^2(\varepsilon)} \left[ 1 + \left( \frac{\partial \eta}{\partial x} \right)^2 \right] \frac{\partial^2 u^{\text{int}}}{\partial y^*^2} - o \left( \frac{\varepsilon^{-\beta}}{\delta^2(\varepsilon)} \right) = g_x
\]

(35)
and

\[
- \frac{\varepsilon^{-\gamma}}{\delta(\varepsilon)} \left( \frac{\partial \eta}{\partial t} \right) \frac{\partial v^{0,\text{int}}}{\partial y^*} + o \left( \frac{\varepsilon^{-\gamma}}{\delta(\varepsilon)} \right) - \frac{\varepsilon^{-\beta-\gamma}}{\delta(\varepsilon)} \frac{\partial v^{0,\text{int}}}{\partial x} \frac{\partial v^{0,\text{int}}}{\partial y^*} + o \left( \frac{\varepsilon^{-\beta-\gamma}}{\delta(\varepsilon)} \right) + \\
\frac{\varepsilon^{-2\gamma}}{\delta(\varepsilon)} \frac{\partial v^{0,\text{int}}}{\partial y^*} + o \left( \frac{\varepsilon^{-2\gamma}}{\delta(\varepsilon)} \right) + \frac{1}{\delta(\varepsilon)} \frac{\partial p}{\partial y^*} - \\
\frac{1}{Re \delta^2(\varepsilon)} \left[ 1 + \left( \frac{\partial \eta}{\partial x} \right)^2 \right] \frac{\partial^2 v^{0,\text{int}}}{\partial y^*} - o \left( \frac{\varepsilon^{-\gamma}}{\delta^2(\varepsilon)} \right) = g_y
\]

(36)

(35) is reduced to the relation

\[
- \frac{1}{Re \delta^2(\varepsilon)} \left[ 1 + \left( \frac{\partial \eta}{\partial x} \right)^2 \right] \frac{\partial^2 u^{0,\text{int}}}{\partial y^*} - o \left( 1 + \delta^2(\varepsilon) \varepsilon^\beta \right) \frac{\partial p}{\partial x^*} - \frac{1}{\delta(\varepsilon)} \left( \frac{\partial \eta}{\partial x} \right) \frac{\partial p}{\partial y^*} = g_x
\]

By multiplying both members of (37) by $Re \delta^2(\varepsilon) \varepsilon^\beta$, The relation becomes

\[
- \left[ 1 + \left( \frac{\partial \eta}{\partial x} \right)^2 \right] \frac{\partial^2 u^{0,\text{int}}}{\partial y^*} - o \left( 1 + \delta^2(\varepsilon) \varepsilon^\beta \right) \frac{\partial p}{\partial x^*} - \delta(\varepsilon) \varepsilon^\beta \left( \frac{\partial \eta}{\partial x} \right) \frac{\partial p}{\partial y^*} = Re \delta^2(\varepsilon) \varepsilon^\beta g_x
\]

(38)

By hypothesis, the pressure and $g_x$ are boundary quantities, $g_x$ is a function of low weight, and passing to the limit, these terms aim towards zero. Consequently (38) is reduced to

\[
\left[ 1 + \left( \frac{\partial \eta}{\partial x} \right)^2 \right] \frac{\partial^2 u^{0,\text{int}}}{\partial y^*} = 0
\]

(39)

Which implies

\[
\frac{\partial^2 u^{0,\text{int}}}{\partial y^*} = 0
\]

(40)

$u^{0,\text{int}}$ is an affine function. The analog calculations applied to the relation (36) give

\[
\frac{\partial^2 v^{0,\text{int}}}{\partial y^*} = 0
\]

(41)

The equations (40) and (41) Justify the following expressions

\[
u^{0,\text{int}} = A^0 y^* + B^0
\]

(42a)

\[
v^{0,\text{int}} = A^1 y^* + B^1
\]

(42b)

where $A^0$, $B^0$ $A^1$ et $B^1$ are functions of $(t^*, x^*)$
According to the divergence condition which is described by the equation (5), we obtain

\[
- \frac{\varepsilon^{-\beta}}{\delta(\varepsilon)} \frac{\partial \eta}{\partial x} \frac{\partial u^{0,\text{int}}}{\partial y^*} + o \left( \frac{\varepsilon^{-\gamma}}{\delta(\varepsilon)} \right) \frac{\partial v^{0,\text{int}}}{\partial y^*} + o \left( \frac{\varepsilon^{-\gamma}}{\delta(\varepsilon)} \right) = 0
\] (43)

The exclusive consideration of the expression in the main order gives:

\[
- \left( \frac{\partial \eta}{\partial x} \right) \frac{\partial u^{0,\text{int}}}{\partial y^*} + \frac{\partial v^{0,\text{int}}}{\partial y^*} = 0
\] (44)

this means that

\[
\left( \frac{\partial \eta}{\partial x} \right) A^0 - A^1 = 0
\] (45)

in the main order,

\[
\begin{cases}
  u^{0,\text{int}}(t^*, x^*, y^*) = A^0(t^*, x^*) y^* + B^0(t^*, x^*) \\
  v^{0,\text{int}}(t^*, x^*, y^*) = A^0(t^*, x^*) \left( \frac{\partial \eta}{\partial x} \right) y^* + B^1(t^*, x^*)
\end{cases}
\] (46)

These equations in the main order are coupled with the equation (32), what allows to determine the state of sedimentation to the main order

2.3 Outer asymptotic expansion

We define outer variables by :

\[
\bar{t} = t, \quad \bar{x} = x, \quad \bar{y} = y - \eta(t, x).
\] (47)

And we look for the outer asymptotic development of the solution \( S^e \) of equation (6) in the form :

\[
S^e,\text{ext} = \tau(\varepsilon) S^{0,\text{ext}}(\bar{t}, \bar{x}, \bar{y}) + TAP
\] (48)

where \( \tau(\varepsilon) \) is a chosen function.

From outer variables ensue the following derivation formulae :
Matched asymptotic expansions

\[ \begin{align*}
\frac{\partial}{\partial t} &= \frac{\partial}{\partial t} - \left( \frac{\partial \eta}{\partial t} \right) \frac{\partial}{\partial y}, \\
\frac{\partial}{\partial x} &= \frac{\partial}{\partial x} - \left( \frac{\partial \eta}{\partial x} \right) \frac{\partial}{\partial y}, \\
\frac{\partial}{\partial y} &= \frac{\partial}{\partial y}, \\
\frac{\partial^2}{\partial y^2} &= \frac{\partial^2}{\partial y^2} \\
\frac{\partial^2}{\partial x^2} &= \frac{\partial^2}{\partial x^2} - \left( \frac{\partial^2 \eta}{\partial x^2} \right) \frac{\partial}{\partial y} - 2 \left( \frac{\partial \eta}{\partial x} \right) \frac{\partial^2}{\partial x \partial y} + \left( \frac{\partial \eta}{\partial x} \right)^2 \frac{\partial^2}{\partial y^2}.
\end{align*} \]

These formulae (49) give the following results:

\[ \begin{align*}
\frac{\partial S^{\varepsilon, ext}}{\partial t} &= \tau(\varepsilon) \frac{\partial S^{0, ext}}{\partial t} - \left( \frac{\partial \eta}{\partial t} \right) \tau(\varepsilon) \frac{\partial S^{0, ext}}{\partial y} + o(\varepsilon) \\
\frac{\partial S^{\varepsilon, ext}}{\partial x} &= \tau(\varepsilon) \frac{\partial S^{0, ext}}{\partial x} - \left( \frac{\partial \eta}{\partial x} \right) \tau(\varepsilon) \frac{\partial S^{0, ext}}{\partial y} + o(\varepsilon) \\
\frac{\partial S^{\varepsilon, ext}}{\partial y} &= \tau(\varepsilon) \frac{\partial S^{0, ext}}{\partial y} + o(\varepsilon) \\
\frac{\partial^2 S^{\varepsilon, ext}}{\partial x^2} &= \tau(\varepsilon) \frac{\partial^2 S^{0, ext}}{\partial x^2} - 2\tau(\varepsilon) \left( \frac{\partial \eta}{\partial x} \right) \frac{\partial^2 S^{0, ext}}{\partial x \partial y} + \tau(\varepsilon) \left( \frac{\partial \eta}{\partial x} \right)^2 \frac{\partial^2 S^{0, ext}}{\partial y^2} + o(\varepsilon) \\
\frac{\partial^2 S^{\varepsilon, ext}}{\partial y^2} &= \varepsilon^{-1} \frac{\partial^2 S^{0, ext}}{\partial y^2} + o(\varepsilon^{-1})
\end{align*} \]

By considering the formulae (50), it ensues that the convective term could be written:

\[ \begin{align*}
k_x(u^\varepsilon, v^\varepsilon) \frac{\partial S^{\varepsilon, ext}}{\partial x} + k_y(u^\varepsilon, v^\varepsilon) \frac{\partial S^{\varepsilon, ext}}{\partial y} &= \\
&= k_x(u^\varepsilon, v^\varepsilon) \left( \tau(\varepsilon) \frac{\partial S^{0, ext}}{\partial x} - \tau(\varepsilon) \left( \frac{\partial \eta}{\partial x} \right) \frac{\partial S^{0, ext}}{\partial y} + o(\varepsilon) \right) + k_y(u^\varepsilon, v^\varepsilon) \left( \tau(\varepsilon) \frac{\partial S^{0, ext}}{\partial y} + o(\varepsilon) \right)
\end{align*} \]
The diffusion term could be written
\[
\mu(u^e, v^e) \left( \frac{\partial^2 S_{o,ext}^0}{\partial x^2} + \frac{\partial^2 S_{o,ext}^0}{\partial y^2} \right) = 
\]
\[
\mu(u^e, v^e) \left( \tau(\varepsilon) \frac{\partial^2 S_{o,ext}^0}{\partial x^2} - 2\tau(\varepsilon) \frac{\partial S_{o,ext}^0}{\partial x} \frac{\partial S_{o,ext}^0}{\partial y} + \tau(\varepsilon) \frac{\partial^2 S_{o,ext}^0}{\partial y^2} + \varepsilon^{-\tau} \frac{\partial^2 S_{o,ext}^0}{\partial y^2} + o(\varepsilon^{-\tau}) \right)
\]

By substituting the different terms above, we have:
\[
\tau(\varepsilon) \frac{\partial S_{o,ext}^0}{\partial t} - \left( \frac{\partial S_{o,ext}^0}{\partial y} \right) + k_y(u^e, v^e) \left( \tau(\varepsilon) \frac{\partial S_{o,ext}^0}{\partial x} - \tau(\varepsilon) \frac{\partial S_{o,ext}^0}{\partial y} + o(\tau(\varepsilon)) \right)
\]
\[
+ k_y(u^e, v^e) \left( \tau(\varepsilon) \frac{\partial S_{o,ext}^0}{\partial y} + o(\tau(\varepsilon)) \right)
\]
\[
- \mu(u^e, v^e) \left( \tau(\varepsilon) \frac{\partial^2 S_{o,ext}^0}{\partial x^2} - 2\tau(\varepsilon) \frac{\partial S_{o,ext}^0}{\partial x} \frac{\partial S_{o,ext}^0}{\partial y} + \tau(\varepsilon) \frac{\partial^2 S_{o,ext}^0}{\partial y^2} + \varepsilon^{-\tau} \frac{\partial^2 S_{o,ext}^0}{\partial y^2} + o(\varepsilon^{-\tau}) \right)
\]
\[
= f
\]

By going away slightly from the shock zone, the convection in \( x \) and \( y \) directions are of the same order. The convective effect seems dominated the diffusion effect. These physical analysis allow us to put the The following simplified hypotheses:
\[
\begin{align*}
& k_x(u^e, v^e) = k_x^0 u^0,ext \\
& k_y(u^e, v^e) = k_y^0 v^0,ext \\
& \mu(u^e, v^e) = \tau(\varepsilon) \mu^0 u^0,ext \\
\end{align*}
\]

(54)

The consideration of (54) in (53) and of the main terms, give us the equation of sedimentation in the outer zone:
\[
\tau(\varepsilon) \frac{\partial S_{o,ext}^0}{\partial t} - \tau(\varepsilon) \left( \frac{\partial S_{o,ext}^0}{\partial y} \right) + k_x^0 u^0,ext \left( \tau(\varepsilon) \frac{\partial S_{o,ext}^0}{\partial x} - \tau(\varepsilon) \frac{\partial S_{o,ext}^0}{\partial y} \right)
\]
\[
+ k_y^0 v^0,ext \tau(\varepsilon) \frac{\partial S_{o,ext}^0}{\partial y} + o(\tau(\varepsilon)) + o(\tau^2(\varepsilon)) = f
\]

(55)

By dividing every term of the equation (55) by \( \tau(\varepsilon) \),
\[
\frac{\partial S_{o,ext}^0}{\partial t} + k_x^0 u^0,ext \frac{\partial S_{o,ext}^0}{\partial x}
\]
\[
+ \left( - \frac{\partial \eta}{\partial t} - k_x^0 u^0,ext \left( \frac{\partial \eta}{\partial x} \right) + k_y^0 v^0,ext \right) \frac{\partial S_{o,ext}^0}{\partial \bar{y}} + o(\tau^2(\varepsilon)) = \frac{1}{\tau(\varepsilon)} f
\]

(56)
The outer development of $f$ is given by $f^{ext} = \tau(\varepsilon)f^{0,ext} + \ldots$

And passing to the limit, the equation of sedimentation in the outer zone is:

$$
\frac{\partial S^{0,ext}}{\partial t} + k_{x}^{0,ext}u_{x}^{0,ext} \frac{\partial S^{0,ext}}{\partial x} + \left(- \frac{\partial \eta}{\partial t} - k_{x}^{0,ext}u_{x}^{0,ext} \left( \frac{\partial \eta}{\partial x} \right) + k_{y}^{0,ext}v_{y}^{0,ext} \right) \frac{\partial S^{0,ext}}{\partial y} = f^{0,ext}
$$

(57)

where

$$f^{0,ext} = \lim_{\varepsilon \searrow 0} \frac{1}{\tau(\varepsilon)} f \quad \text{with} \quad \bar{t}, \bar{x}, \bar{y} \quad \text{fixed}.$$

It is a partial differential equation of transport, with variable coefficients. This result is valid in $\Omega^{0} \times ]0, +\infty[$ where $\Omega^{0} = \lim_{\varepsilon \searrow 0} \Omega^{\varepsilon,\bar{t}}$, with $\bar{t}, \bar{x}$ and $\bar{y}$ fixed.

To complete the outer treatment of the problem, it requires boundary conditions. One searches these conditions at the sides.

Taking into account the physical properties [4],[14], [17], it is plausible to suppose that the flow of stranding on the side sections is negligible and that the level of stranding remains low on the edge of $\Omega^{0}$. That is explain by:

$$\frac{\partial S^{0,ext}}{\partial x} = 0 \quad \text{on} \quad x = \pm L \quad (58)$$

and

$$S^{0,ext} = 0 \quad \text{on} \quad \partial \Omega^{0}/x = \pm L \quad (59)$$

**Remark 2.1.** The equation (57) is an equation of type transport. It is easy to see that [18, 11], under conditions of regularity relating to the coefficients of the equation (57), if the initial data $S^{0,ext}(0, \bar{x}, \bar{y})$ is rather regular, it exists a single weak solution of (57) checking the boundary conditions (58) and (59).

### 2.4 Matching condition

According to the theory of the asymptotic expansion [7],[11], the matching of both expansion (inner and outer) obeys the property that there exists a zone where the two developments are equal:

$$\lim_{\varepsilon \searrow 0} (S^{\varepsilon,ext})_{(t^{*}, x^{*}, y^{*}) \text{fixed}} = \lim_{\varepsilon \searrow 0} (S^{\varepsilon,int})_{\bar{t}, \bar{x}, \bar{y} \text{ fixed}}$$
where
\[ S^{\varepsilon,\text{int}} = \varepsilon^{-\alpha} S^{0,\text{int}} \left( t, x, \frac{y - \eta(t, x)}{\delta(\varepsilon)} \right) + \ldots \] (60)
and
\[ S^{\varepsilon,\text{ext}} = \tau(\varepsilon) S^{0,\text{ext}}(\bar{t}, \bar{x}, \bar{y}) + \ldots \] (61)

Taking into account (60) and (61), with the principal order this property results in:
\[
\lim_{\varepsilon \searrow 0} \tau(\varepsilon) \varepsilon^{\alpha} S^{0,\text{ext}}(t^*, x^*, \delta(\varepsilon)y^*) = \lim_{\varepsilon \searrow 0} S^{0,\text{int}}(\bar{t}, \bar{x}, \frac{\bar{y}}{\delta(\varepsilon)})
\]

(62)

Because of property of continuity, one has:
\[
\lim_{\varepsilon \searrow 0} \tau(\varepsilon) \varepsilon^{\alpha} S^{0,\text{ext}}(t^*, x^*, \delta(\varepsilon)y^*) = 0
\]

(63a)

and
\[
\lim_{\varepsilon \searrow 0} S^{0,\text{int}}(\bar{t}, \bar{x}, \frac{\bar{y}}{\delta(\varepsilon)}) = S^{0,\text{int}}(\bar{t}, \bar{x}, \infty)
\]

(63b)

that gives
\[ S^{0,\text{int}}(t^*, x^*, \infty) = 0 \] (64)

The matching condition gives a boundary condition making it possible to deduce that one can find, with the principal order the inner asymptotic solution. One thus has shown the following result:

**Theorem 2.2.** The sedimentation state to the principal order admits an inner asymptotic development in the vicinity of \( y = \eta(t, x) \)
\[ S^\varepsilon(t, x, y) = \varepsilon^{-\alpha} S^0(t^*, x^*, y^*) + o(\varepsilon^{-\alpha}) \] (65)

where \( S^0 \) satisfied the partial derivative equation (32) and the boundary condition which is not other than the matching condition.
\[ S^{0,\text{int}}(t^*, x^*, \infty) = 0 \] (66)

where \( u^0 \) and \( v^0 \), with the principal order are respectively of the form
\[
\begin{align*}
u^{0,\text{int}}(t^*, x^*, y^*) &= A^0(t^*, x^*) y^* + B^0(t^*, x^*) \\
u^{0,\text{int}}(t^*, x^*, y^*) &= A^0(t^*, x^*) \left( \frac{\partial \eta}{\partial x} \right) y^* + B^1(t^*, x^*)
\end{align*}
\] (67)
3 Concluding remarks

In this paper we have presented inner and outer expansion solutions which give rise to a complete description of sediment process near rivers banks. Particularly we established that these solutions can be expressed in terms of inner and outer variables described with moving shock lines. The rivers silting can be explained by an asymptotic behavior near shock curves of the governing equations of sediment deposits. We performed the computation of the leading term of the asymptotic expansions. We give the matching condition under the form of a boundary condition.

References


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