Existence and Uniqueness of Solution for Caginalp Hyperbolic Phase-Field System with a Singular Potential

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Abstract

In this article, we study the existence and the uniqueness of solution for Caginalp hyperbolic phase-field system, with initial conditions, Dirichlet boundary conditions and singular potential in bounded and smooth domain.

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1 Introduction

In this article, we are interested in the study of the following Caginalp hyperbolic phase-field system in a smooth and bounded domain $\Omega \subset \mathbb{R}^n (1 \leq n \leq 3)$.

\begin{align}
\epsilon \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u + f(u) &= \alpha, \\
\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha &= -u - \frac{\partial u}{\partial t},
\end{align}
with homogenous Dirichlet conditions

\[ u|_{\partial\Omega} = \alpha|_{\partial\Omega} = 0, \quad (3) \]

and initial conditions

\[ u|_{t=0} = u_0, \quad \frac{\partial u}{\partial t}|_{t=0} = u_1, \quad \alpha|_{t=0} = \alpha_0, \quad \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1 \quad (4) \]

where \( \epsilon > 0 \) is a relaxation parameter, \( u = u(t,x) \), the order parameter and \( \alpha = \alpha(t,x) \), the relative temperature, \( f \) is a given singular potential function.

Consider the following logarithmic potential function

\[ f(s) = -k_0 s + k_1 \ln\left(\frac{1 + s}{1 - s}\right), \quad s \in ]-1,1[, \quad 0 < k_1 < k_0. \]

The function \( f \) satisfies the following properties:

\[ f \in C^2([-1,1]), \quad \lim_{s \to \pm 1} f(s) = \pm \infty, \quad \lim_{s \to \pm 1} f'(s) = +\infty, \]

\[ -c_0 \leq F(s) \leq f(s)s + c_0 \text{ where } F(s) = \int_0^s f(\tau)d\tau \text{ and } c_0 \geq 0, \quad (5) \]

\[ f(0) = 0, \quad (6) \]

\[ f'(s) \geq -c_1, \quad c_1 \geq 0 \quad \forall s \in ]-1,1[. \quad (7) \]

Such studies had already been realised with \( \epsilon = 0 \) in many works with different types of limit conditions and regular or singular potential functions, see for example [1], [2], [3], [4], [5], [6], [7], [8].

Very recently, such study has been realised with \( \epsilon > 0 \) in [7], [8], [9], in the case of the well defined hyperbolic phase-field Caginalp system with Dirichlet limit conditions and regular or singular logarithmic potential in the aim to establish existence of global attractor.

In the present study, we establish existence and uniqueness theorem of the solution to the considered system different of the one studied in [9].

In this article, we denote by \( \| . \| \) and \( (.,.) \) (or \( \| . \|_{\Phi} \) and \( (.,.)_{\Phi} \)) the norm and the scalar product in \( L^2(\Omega) \) (in \( \Phi \)).
2 A priori estimates

We a priori assume $\|u_0\|_{L^\infty(\Omega)} < 1$ and $\|u\|_{L^\infty((0,T)\times\Omega)} < 1$.

Multiplying (1) by $\frac{\partial u}{\partial \tau}$ and (2) by $\frac{\partial \alpha}{\partial \tau}$, and integrating over $\Omega$, we get the following respective equations:

$$
\frac{d}{dt} \left( \epsilon \| \frac{\partial u}{\partial t} \|^2 + \| \nabla u \|^2 + 2 \int_\Omega F(u) dx \right) + \| \frac{\partial u}{\partial t} \|^2 \leq \| \alpha \|^2, \quad (8)
$$

$$
\frac{d}{dt} \left( \| \frac{\partial \alpha}{\partial t} \|^2 + \| \nabla \alpha \|^2 \right) + \| \frac{\partial \alpha}{\partial t} \|^2 + 2 \| \nabla \frac{\partial \alpha}{\partial t} \|^2 \leq 2 \| u \|^2 + 2 \| \frac{\partial u}{\partial t} \|^2. \quad (9)
$$

Then, adding (8) and $\gamma_1(9)$, with $\gamma_1 > 0$ and $1 - 2 \gamma_1 > 0$, we obtain:

$$
\frac{d}{dt} E_1 + C_1 \| \frac{\partial u}{\partial t} \|^2 + \gamma_1(\| \frac{\partial \alpha}{\partial t} \|^2 + 2 \| \nabla \frac{\partial \alpha}{\partial t} \|^2) \leq C_2 \| \alpha \|^2_{H^1} + C_3 \| u \|^2_{H^1}, C_i > 0, \quad (10)
$$

where

$$
E_1 = \epsilon \| \frac{\partial u}{\partial \tau} \|^2 + \| \nabla u \|^2 + 2 \int_\Omega F(u) dx + \gamma_1(\| \nabla \alpha \|^2 + \| \frac{\partial \alpha}{\partial \tau} \|^2).
$$

Estimate (10) can be written in the form

$$
\frac{d}{dt} E_1 + C_1 \| \frac{\partial u}{\partial t} \|^2 + \gamma_1(\| \frac{\partial \alpha}{\partial t} \|^2 + 2 \| \nabla \frac{\partial \alpha}{\partial t} \|^2) \leq k E_1, \quad (11)
$$

where the strictly positive constant $k$ is independent of $\epsilon$.

Applying the lemma of Gronwall we obtain:

$$
E_1(t) + C_1 \int_0^t \| \frac{\partial u}{\partial \tau} \|^2 d\tau + \gamma_1 \left( \int_0^t \| \frac{\partial \alpha}{\partial \tau} \|^2 d\tau + 2 \int_0^t \| \frac{\partial \alpha}{\partial \tau} \|^2_{H^1} d\tau \right) \leq E_1(0) e^{kt}, \quad (12)
$$

which implies:

$$
u, \alpha \in L^\infty(0, T; H^1_0(\Omega)), \frac{\partial u}{\partial \tau} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \frac{\partial \alpha}{\partial \tau} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)).$$

Multiplying (1) by $- \Delta \frac{\partial u}{\partial \tau}$ and integrating over $\Omega$. We have:

$$
\frac{d}{dt} \left( \epsilon \| \nabla \frac{\partial u}{\partial \tau} \|^2 + \| \Delta u \|^2 \right) + \| \nabla \frac{\partial u}{\partial \tau} \|^2 \leq 2 C_1 \| u \|^2_{H^1} + 2 \| \alpha \|^2_{H^1}. \quad (13)
$$

We get $u \in L^\infty(0, T; H^2(\Omega) \cap H^1_0(\Omega)), \frac{\partial u}{\partial \tau} \in L^\infty(0, T; H^1_0(\Omega)) \cap L^2(0, T; H^1_0(\Omega))$.

Multiplying now (1) by $\frac{\partial^2 u}{\partial \tau^2}$ and integrating over $\Omega$, we obtain:

$$
\frac{d}{dt} \| \frac{\partial u}{\partial \tau} \|^2 + \epsilon \| \frac{\partial^2 u}{\partial t^2} \|^2 \leq C_1(\| \Delta u \|^2 + \| f(u) \|^2 + \| \alpha \|^2). \quad (14)
$$
We deduce that: \( \frac{\partial^2 u}{\partial t^2} \in L^2(0, T; L^2(\Omega)). \)

Multiplying (2) by \(-\Delta \frac{\partial \alpha}{\partial t}\) and integrating over \(\Omega\), we have
\[
\frac{d}{dt} \left( \|\nabla \frac{\partial \alpha}{\partial t}\|^2 + \|\Delta \alpha\|^2 \right) + \|\nabla \frac{\partial \alpha}{\partial t}\|^2 \leq 2\|u\|^2_{H^1} + 2\|\frac{\partial u}{\partial t}\|^2_{H^1}. \tag{15}
\]

Therefore \(\alpha \in L^\infty(0, T; H^2(\Omega) \cap H^1_0(\Omega))\) and \(\frac{\partial \alpha}{\partial t} \in L^\infty(0, T; H^1_0(\Omega)) \cap L^2(0, T; H^1_0(\Omega))\).

Multiplying (2) by \(\frac{\partial^2 \alpha}{\partial t^2}\) and integrating over \(\Omega\), we obtain
\[
\frac{d}{dt} \left( \|\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla \frac{\partial \alpha}{\partial t}\|^2 + 2(\nabla \frac{\partial \alpha}{\partial t}, \nabla \alpha) \right) + \|\frac{\partial^2 \alpha}{\partial t^2}\|^2 \leq 2\|u\|^2 + 2\|\frac{\partial u}{\partial t}\|^2 + 2\|\frac{\partial \alpha}{\partial t}\|^2_{H^1}. \tag{16}
\]

Adding (9) and (16), with \(\gamma_2 > 0, 1 - \gamma_2 > 0\) and \(1 - 2\gamma_2 > 0\) we obtain
\[
\frac{d}{dt} E_2 + \gamma_2 \|\frac{\partial^2 \alpha}{\partial t^2}\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 + C_1\|\nabla \frac{\partial \alpha}{\partial t}\|^2 \leq C_2(\|u\|^2 + \|\frac{\partial u}{\partial t}\|^2), C_i > 0, \tag{17}
\]

where
\[
E_2 = \|\nabla \alpha\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 + \gamma_2(\|\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla \frac{\partial \alpha}{\partial t}\|^2 + 2(\nabla \alpha, \nabla \frac{\partial \alpha}{\partial t})).
\]

\(E_2\) verifies
\[
E_2 \geq \frac{1}{2}\|\nabla \alpha\|^2 + C'\|\nabla \frac{\partial \alpha}{\partial t}\|^2, C' > 0, \tag{18}
\]

which implies \(\frac{\partial \alpha}{\partial t} \in L^2(0, T; L^2(\Omega)).\)

3 Existence and uniqueness of solutions

Theorem 3.1. (Existence) We assume \((u_0, u_1, \alpha_0, \alpha_1) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega) \times (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)\), such that \(\|u_0\|_{L^\infty(\Omega)} < 1\). Then, the system (1)-(2) possesses at least one solution \((u, \alpha)\) such that \(u, \alpha \in L^\infty(0, T; H^2(\Omega) \cap H^1_0(\Omega)),\)
\(\frac{\partial u}{\partial t} \in L^\infty(0, T; H^2_0(\Omega)) \cap L^2(0, T; H^1_0(\Omega)), \frac{\partial \alpha}{\partial t} \in L^\infty(0, T; H^1_0(\Omega)) \cap L^2(0, T; H^1_0(\Omega)),\)
\(\frac{\partial^2 u}{\partial t^2} \in L^2(0, T; L^2(\Omega)),\) and \(\frac{\partial^2 \alpha}{\partial t^2} \in L^2(0, T; L^2(\Omega))\), for all \(T > 0\). Furthermore, \(-1 < u(t, x) < 1\) a.e., \(\forall (t, x) \in [0, T] \times \Omega.\)

Proof. As Miranville & Quintanilla [6], we introduce, for \(N \in \mathbb{N}\), the following function \(f_N \in C^1(\mathbb{R})\) given by
\[
f_N(s) = \begin{cases} 
  f(-1 + \frac{1}{N}) + f'(-1 + \frac{1}{N})(s + 1 - \frac{1}{N}) & \text{if } s < -1 + \frac{1}{N} \\
  f(s) & \text{if } |s| \leq 1 - \frac{1}{N} \\
  f(1 - \frac{1}{N}) + f'(1 - \frac{1}{N})(s - 1 + \frac{1}{N}) & \text{if } s > 1 - \frac{1}{N}, \tag{19}
\end{cases}
\]
where \( f_N \) is a regular potential function which verifies, at least for \( N \) large enough,

\[-c_0 \leq F_N(s) \leq f_N(s)s + c_0 \]

where \( F_N(s) = \int_0^s f_N(\tau) d\tau \) and \( c_0 \geq 0, \forall s \in \mathbb{R} \), (20)

\[ f_N'(s) \geq -c_1, \quad c_1 \geq 0 \quad \forall s \in \mathbb{R}. \] (21)

with the constants independent of \( \epsilon \). Consider the following approximated problem

\[
\frac{\epsilon \partial^2 u^N}{\partial t^2} + \frac{\partial u^N}{\partial t} - \Delta u^N + f_N(u^N) = \alpha^N, \quad (t, x) \in [0, T] \times \Omega, \tag{22}
\]

\[
\frac{\partial^2 \alpha^N}{\partial t^2} + \frac{\partial \alpha^N}{\partial t} - \Delta \frac{\partial \alpha^N}{\partial t} - \Delta \alpha^N = -u^N - \frac{\partial u^N}{\partial t}, \quad (t, x) \in [0, T] \times \Omega, \tag{23}
\]

\[
u^N|_{\partial \Omega} = \alpha^N|_{\partial \Omega} = 0, \quad \tag{24}
\]

\[
u^N|_{t=0} = u_0, \quad \frac{\partial u^N}{\partial t}|_{t=0} = u_1, \quad \alpha^N|_{t=0} = \alpha_0, \quad \frac{\partial \alpha^N}{\partial t}|_{t=0} = \alpha_1, \quad \forall x \in \Omega. \tag{25}
\]

Proceeding as in [8], the system (22)-(23) possesses a unique solution \((u^N, \alpha^N)\) such that \( u^N, \alpha^N \in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)), \frac{\partial u^N}{\partial t}, \frac{\partial \alpha^N}{\partial t} \in L^\infty(0, T; H^1_0(\Omega)) \cap L^2(0, T; H^1_0(\Omega)), \frac{\partial^2 u^N}{\partial t^2}, \frac{\partial^2 \alpha^N}{\partial t^2} \in L^2(0, T; L^2(\Omega)) \) and \( \frac{\partial u^N}{\partial t}, \frac{\partial \alpha^N}{\partial t} \in L^2(0, T; L^2(\Omega)), \) for all \( T > 0. \) Since the constants in (20)-(21) are independent of \( N, \) from the sequence \((u^N, \frac{\partial u^N}{\partial t}, \alpha^N, \frac{\partial \alpha^N}{\partial t})\), there exists at least a subsequence which we do not relabel and \((u, \frac{\partial u}{\partial t}, \alpha, \frac{\partial \alpha}{\partial t})\) such that, for every \( T > 0, \)

\[
u^N \rightharpoonup u \text{ in } L^\infty(0, T; H^2(\Omega)) \text{ weak}-* \text{ a.e.} (t, x) \in \Omega, \quad \frac{\partial u^N}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ in } L^\infty(0, T; H^1_0(\Omega)) \text{ weak}-* \text{ and in } L^2(0, T; H^1_0(\Omega)) \text{ weak},
\]

\[
\frac{\partial^2 u^N}{\partial t^2} \rightharpoonup \frac{\partial^2 u}{\partial t^2} \text{ in } L^2(0, T; L^2(\Omega)) \text{ weak},
\]

\[
\alpha^N \rightharpoonup \alpha \text{ in } L^\infty(0, T; H^2(\Omega)) \text{ weak}-*, \quad \frac{\partial \alpha^N}{\partial t} \rightharpoonup \frac{\partial \alpha}{\partial t} \text{ in } L^\infty(0, T; H^1_0(\Omega)) \text{ weak}-* \text{ and in } L^2(0, T; H^1_0(\Omega)) \text{ weak},
\]

\[
\frac{\partial^2 \alpha^N}{\partial t^2} \rightharpoonup \frac{\partial^2 \alpha}{\partial t^2} \text{ in } L^2(0, T; L^2(\Omega)) \text{ weak}.
\]

The only difficulty, to pass to the limit in the approximated problems, is to prove that \( f_N(u^N) \) tends to \( f(u) \) in a proper sense. In order to prove this, we need a uniform (with respect \( N \)) estimate on \( f_N(u^N) \) in \( L^2([0, T] \times \Omega). \) From equation (22), we have

\[
f_N(u^N) = -\epsilon \frac{\partial^2 u^N}{\partial t^2} - \frac{\partial u^N}{\partial t} + \Delta u^N + \alpha^N,
\]
which implies,

$$\| f_N(u^N) \| \leq C(\varepsilon) \frac{\partial^2 u_N}{\partial t^2} + \| \frac{\partial u_N}{\partial t} \| + \| u_N \|_{H^2} + \| \alpha^N \|,$$

where the constants $C$ is independent of $N$. Hence, $f_N(u^N)$ is bounded in $L^2([0,T] \times \Omega)$. We easily deduce that $f_N(u^N)$ is bounded, uniformly with respect to $N$, in $L^1([0,T] \times \Omega)$. Arguing as in Miranville & Quintanilla [6], we can conclude

$$\text{meas}\{ (t, x) \in [0, T] \times \Omega, |u(t, x)| \geq 1 \} = 0,$$

so that

$$-1 < u(t, x) < 1 \text{ a.e., } \forall (t, x) \in [0, T] \times \Omega,$$

and $f_N(u^N) \rightarrow f(u)$ in $L^2([0, T] \times \Omega)$ weak. Finally, we conclude that $(u, \alpha)$ is the solution of the system (1)-(2). \hfill \square

**Theorem 3.2. (Uniqueness)** Assume the hypothesis of Theorem 3.1 verified, then the system (1)-(2) possesses a unique solution $(u, \alpha)$ such that $u, \alpha \in L^\infty(0, T; \tilde{H}^2(\Omega) \cap H^1_0(\Omega)), \frac{\partial u}{\partial t} \in L^\infty(0, T; H^1_0(\Omega)) \cap L^2(0, T; H^1_0(\Omega)), \frac{\partial \alpha}{\partial t} \in L^\infty(0, T; H^1_0(\Omega)) \cap L^2(0, T; H^1_0(\Omega)), \frac{\partial^2 \alpha}{\partial t^2} \in L^2(0, T; L^2(\Omega))$ and $\frac{\partial^2 u}{\partial t^2} \in L^2(0, T; L^2(\Omega))$, $\forall T > 0$.

**Proof.** Let $(u^{(1)}, \alpha^{(1)})$ and $(u^{(2)}, \alpha^{(2)})$ two solutions of the system (1)-(2) with initial conditions $(u_0^{(1)}, u_1^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)})$ and $(u_0^{(2)}, u_1^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)})$; respectively. Let $u = u^{(1)} - u^{(2)}$ and $\alpha = \alpha^{(1)} - \alpha^{(2)}$. Then $(u, \alpha)$ satisfies

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u + l.u = \alpha,$$

$$\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -u - \frac{\partial u}{\partial t},$$

where $l(t, x) = \int_0^1 f'(su^{(1)}(t, x) + (1-s)u^{(2)}(t, x))ds$. Following theorem 3.1, $u^{(1)}$ and $u^{(2)}$ belong to $L^\infty(0, T; \tilde{H}^2(\Omega))$ and are such that $-1 < u^{(1)}(t, x) < 1$ and $-1 < u^{(2)}(t, x) < 1$ a.e., $\forall (t, x) \in [0, T] \times \Omega$. Then $su^{(1)} + (1-s)u^{(2)} \in L^\infty(0, T; \tilde{H}^2(\Omega)), \forall s \in (0, 1)$ and $-1 < su^{(1)}(t, x) + (1-s)u^{(2)}(t, x) < 1$ a.e., $\forall (s, t, x) \in (0, 1) \times [0, T] \times \Omega$. Since $f'$ is continuous and $\lim_{s \to \pm 1} f'(s) = +\infty$, then there exists $c \in (0, 1)$ independent of $t$ and $x$ such that $f'(c) > 0$ and $|su^{(1)}(t, x) + (1-s)u^{(2)}(t, x)| \leq c$. Moreover $f'$ convex and even, therefore

$$|l(t, x)| \leq \int_0^1 |f'(su^{(1)}(t, x) + (1-s)u^{(2)}(t, x))|ds \leq f'(c), \forall (t, x) \in [0, T] \times \Omega.$$
Multiplying (26) by $\frac{\partial u}{\partial t}$ and (27) by $\frac{\partial \alpha}{\partial t}$ and integrating over $\Omega$, we obtain

$$\frac{d}{dt} \left( \epsilon \| \frac{\partial u}{\partial t} \|^2 + \| \nabla u \|^2 \right) + \| \frac{\partial u}{\partial t} \|^2 \leq 2f'(c)\| u \|^2 + 2\| \alpha \|^2. \tag{28}$$

$$\frac{d}{dt} \left( \| \frac{\partial \alpha}{\partial t} \|^2 + \| \nabla \alpha \|^2 \right) + \| \frac{\partial \alpha}{\partial t} \|^2 + 2\| \nabla \frac{\partial \alpha}{\partial t} \|^2 \leq 2\| u \|^2 + 2\| \frac{\partial u}{\partial t} \|^2. \tag{29}$$

Adding (28) and (29), we have

$$\frac{d}{dt} E_3 + \| \frac{\partial \alpha}{\partial t} \|^2 + 2\| \nabla \frac{\partial \alpha}{\partial t} \|^2 \leq C\| u \|^2 + 2\| \alpha \|^2 + \| \frac{\partial u}{\partial t} \|^2, \tag{30}$$

where

$$E_3 = \| \nabla u \|^2 + \epsilon \| \frac{\partial u}{\partial t} \|^2 + \| \nabla \alpha \|^2 + \| \frac{\partial \alpha}{\partial t} \|^2,$$

satisfies

$$E_3 \geq c(\| \nabla u \|^2 + \| \nabla \alpha \|^2 + \epsilon \| \frac{\partial u}{\partial t} \|^2), \tag{32}$$

Gronwall’s lemma, together with (32), then yields the continuous dependence of the solution relative to the initial conditions, hence the uniqueness of the solution. \(\square\)

**Theorem 3.3. (Existence)** Assume $(u_0, u_1, \alpha_0, \alpha_1) \in (H^3(\Omega) \cap H^1_0(\Omega)) \times (H^2(\Omega) \cap H^1_0(\Omega)) \times (H^2(\Omega) \cap H^1_0(\Omega))$ and $\|u_0\|_{L^\infty(\Omega)} < 1$. Then the system (1)-(2) possesses a unique solution $(u, \alpha)$ such that $u, \alpha \in L^\infty(0,T; H^3(\Omega) \cap H^1_0(\Omega))$, $\frac{\partial u}{\partial t} \in L^\infty(0,T; H^2(\Omega) \cap H^1_0(\Omega)) \cap L^2(0,T; (H^2(\Omega) \cap H^1_0(\Omega)))$, $\frac{\partial \alpha}{\partial t} \in L^\infty(0,T; H^2(\Omega) \cap H^1_0(\Omega))$, $\frac{\partial^2 u}{\partial t^2} \in L^2(0,T; H^1_0(\Omega)) \cap L^2(0,T; H^1_0(\Omega))$, $\frac{\partial^2 \alpha}{\partial t^2} \in L^2(0,T; H^1_0(\Omega))$. Moreover, $-1 < u(t, x) < 1$, a.e., $\forall (t, x) \in [0,T] \times \Omega$.

**Proof.** Following theorem 3.1, system (1)-(2) has a unique solution $(u, \alpha)$ such that $u, \alpha \in L^\infty(0,T; H^2(\Omega) \cap H^1_0(\Omega)) \cap L^2(0,T; H^1_0(\Omega))$, $\frac{\partial u}{\partial t} \in L^\infty(0,T; H^1_0(\Omega)) \cap L^2(0,T; H^1_0(\Omega))$, $\frac{\partial \alpha}{\partial t} \in L^2(0,T; H^1_0(\Omega))$, $\forall T > 0$, $\frac{\partial^2 u}{\partial t^2} \in L^2(0,T; H^1_0(\Omega))$, $\forall T > 0$.

Multiplying (1) by $\Delta^2 \frac{\partial u}{\partial t}$ and integrating over $\Omega$, we obtain

$$\frac{d}{dt} \left( \epsilon \| \frac{\partial u}{\partial t} \|^2 + \| \nabla u \|^2 \right) + \| \frac{\partial u}{\partial t} \|^2 \leq 2\| \Delta f(u) \|^2 + 2\| \Delta \alpha \|^2. \tag{33}$$
Which implies \( u \in L^\infty(0, T; H^3(\Omega) \cap H_0^1(\Omega)), \frac{\partial u}{\partial t} \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \).

Multiplying (1) by \(-\Delta \frac{\partial^2 u}{\partial t^2}\) and integrating over \( \Omega \), we have

\[
\frac{d}{dt} \| \frac{\partial u}{\partial t} \|^2_{H^1} + \epsilon \| \frac{\partial^2 u}{\partial t^2} \|^2_{H^1} \leq C(\| u \|^2_{H^3} + \| \nabla f(u) \|^2 + \| \alpha \|^2_{H^1}), C > 0. \tag{34}
\]

We deduce that \( \frac{\partial^2 u}{\partial t^2} \in L^2(0, T; H_0^1(\Omega)) \).

Multiplying (2) by \(-\Delta \frac{\partial \alpha}{\partial t}\) and integrating over \( \Omega \), we have

\[
\frac{d}{dt} \left( \| \frac{\partial \alpha}{\partial t} \|^2_{H^1} + \| \Delta \alpha \|^2 \right) + \| \Delta \frac{\partial \alpha}{\partial t} \|^2 + 2\| \nabla \frac{\partial \alpha}{\partial t} \|^2 \leq C(\| u \|^2_{H^3} + \| u \|^2_{H^1} + \| \partial u \|^2_{H^1}), C > 0. \tag{35}
\]

Therefore \( \frac{\partial \alpha}{\partial t} \in L^\infty(0, T; H_0^1(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)). \)

Multiplying (2) by \(-\Delta \frac{\partial^2 \alpha}{\partial t^2}\) and integrating over \( \Omega \), we have

\[
\frac{d}{dt} \left( \| \nabla \frac{\partial \alpha}{\partial t} \|^2 + \| \Delta \frac{\partial \alpha}{\partial t} \|^2 + 2(\Delta \alpha, \Delta \frac{\partial \alpha}{\partial t}) \right) + \| \nabla \frac{\partial^2 \alpha}{\partial t^2} \|^2 \leq 2\| u \|^2_{H^1} + 2\| \partial u \|^2_{H^1} + 2\| \frac{\partial u}{\partial t} \|^2_{H^2}, \tag{36}
\]

Adding (35) and \( \gamma_3(36) \), where \( \gamma_3 > 0, 1 - 2\gamma_3 > 0 \), we have

\[
\frac{d}{dt} E_4 + 2\| \nabla \frac{\partial \alpha}{\partial t} \|^2 + C_1 \| \frac{\partial \alpha}{\partial t} \|^2_{H^2} + \gamma_3 \| \nabla \frac{\partial^2 \alpha}{\partial t^2} \|^2 \leq C' \left( \| u \|^2_{H^1} + \| \partial u \|^2_{H^1} \right), C_1, C' > 0, \tag{37}
\]

where

\[
E_4 = C_2 \| \nabla \frac{\partial \alpha}{\partial t} \|^2 + \| \Delta \alpha \|^2 + \gamma_3(\| \Delta \frac{\partial \alpha}{\partial t} \|^2 + 2(\Delta \alpha, \Delta \frac{\partial \alpha}{\partial t})), C_2 > 0.
\]

Therefore (37) implies

\( \frac{\partial \alpha}{\partial t} \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)), \frac{\partial^2 \alpha}{\partial t^2} \in L^2(0, T; H_0^1(\Omega)). \)

Multiplying (2) by \( \Delta^2 \frac{\partial \alpha}{\partial t}\) and integrating over \( \Omega \), we have

\[
\frac{d}{dt} \left( \| \nabla \Delta \alpha \|^2 + \| \Delta \frac{\partial \alpha}{\partial t} \|^2 \right) + 2\| \nabla \Delta \frac{\partial \alpha}{\partial t} \|^2 + \| \Delta \frac{\partial \alpha}{\partial t} \|^2 \leq 2\| \Delta u \|^2 + 2\| \frac{\partial u}{\partial t} \|^2. \tag{38}
\]

Estimate (38) implies \( \alpha \in L^\infty(0, T; H^3(\Omega) \cap H_0^1(\Omega)), \frac{\partial \alpha}{\partial t} \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^2(0, T; H^3(\Omega) \cap H_0^1(\Omega)). \)

Finally we have \( u, \alpha \in L^\infty(0, T; H^3(\Omega) \cap H_0^1(\Omega)), \frac{\partial u}{\partial t} \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^2(0, T; H^3(\Omega) \cap H_0^1(\Omega)), \frac{\partial \alpha}{\partial t} \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^2(0, T; H^3(\Omega) \cap H_0^1(\Omega)), \frac{\partial^2 u}{\partial t^2} \in L^2(0, T; H_0^1(\Omega)) \) and \( \frac{\partial^2 \alpha}{\partial t^2} \in L^2(0, T; H_0^1(\Omega)). \) \( \square \)
4 Conclusion

We have just shown the theorems of existence and uniqueness of solution for Caginalp hyperbolic phase-field system with singular potential. The principal results obtained in this study confirm those already obtained with a regular potential.

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References


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