Asymptotic Growth Bounds for the Vlasov-Poisson System in Convex Bounded Domains

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Abstract

We consider smooth compactly supported solutions to the classical three-dimensional Vlasov-Poisson system in convex bounded domains. In the plasma physics case, we show that the size of the velocity support of the distribution function grows at most $t^{\frac{10}{21} + \epsilon}$ for any $\epsilon > 0$.

Keywords: Vlasov-Poisson system, bounded convex domains, specular reflection, asymptotic behavior

1 Introduction and main result

We study the Vlasov Poisson system in a smooth, convex and bounded domain $\Omega$ with specular reflection on the boundary:

\begin{align}
\partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f &= 0, \quad x \in \Omega \subset \mathbb{R}^3, \quad v \in \mathbb{R}^3, \quad t > 0, \quad (1.1) \\
\Delta \phi &= \int_{\mathbb{R}^3} f dv = \rho(t,x), \quad x \in \Omega, \quad t > 0, \quad (1.2) \\
\frac{\partial \phi}{\partial n_x}(t,x) &= h(x), \quad x \in \partial \Omega, \quad t > 0, \quad (1.3) \\
f(0,x,v) &= f_0(x,v), \quad x \in \Omega, \quad v \in \mathbb{R}^3, \quad (1.4) \\
f(t,x,v) &= f(t,x,v^*), \quad x \in \Omega, \quad v \in \mathbb{R}^3, \quad t > 0, \quad (1.5)
\end{align}
where $\Omega$ is a convex bounded domain with $C^5$ boundary, $n_x$ denotes the outer normal to $\partial\Omega$ and

$$f_0(x, v) \geq 0,$$

is the prescribed initial datum. Here $f(t, x, v)$ denotes the distribution density of electrons, $\phi(t, x)$ is the electric potential and $\rho(t, x)$ denotes the macroscopic charge density. The function $h$ in (1.3) will be assumed to be positive and satisfy the following compatibility condition:

$$\int_\Omega f_0(x, v) dx dv = \int_{\partial\Omega} h(x) dS_x.$$

We will also assume throughout the paper that the distribution function $f$ satisfies the specular reflection boundary condition (1.5) where the reflected velocity $v^*$ is defined: Given $(x, v) \in \partial\Omega \times \mathbb{R}^3$

$$v^* \equiv v - 2n_x.v$$

where from now on $n_x$ is the outer normal vector to $\partial\Omega$ at the point $x$.

For the whole space case $\Omega = \mathbb{R}^3$, the system (1.1) reduces to the well-known Vlasov-Poisson equation, in two dimensions, a smooth solution is known to exist globally in time [20]. For the three dimensional case Batt [3], Horst [10], and Bardos and Degond [2] proved the global existence of classical solutions for spherical, cylindrically symmetric, and general but small initial data respectively. The global existence of smooth solutions for general initial data was first shown for the whole space case in [16] by Pfaffelmoser and simple versions were followed by Schaeffer [18] and Horst [9]. A different method was developed to show global existence of a smooth solution in [14] by Lions and Perthame. It was proved in [7] that classical solutions for the Vlasov Poisson system may not exist in general without the nonnegativity assumption if $\Omega$ is the half space $\mathbb{R}^3_+$. On other hand, it was also proved in [7] that even with the nonnegativity assumption the derivatives of the solutions of (1.1)-(1.5) can not be uniformly bounded near the boundary of $\Omega$ due to the fact that a Lipschitz estimate for the characteristics in terms of the initial data is not possible.

One of the main technical difficulties that must be considered in order to solve (1.1)-(1.5), even for short times, is a careful study of the evolution of the characteristic curves (1.1) which remain close during their evolution to the so-called singular set, defined as follows

$$\Gamma = \{(x, v) \in \Omega \times \mathbb{R}^3 : x \in \partial\Omega, v \in T_x\partial\Omega\},$$

where $T_x\partial\Omega \subset \mathbb{R}^3$ is the tangent plane to $\partial\Omega$ at the point $x$.

Notice that the projection of such characteristic curves in the domain $\Omega$ bounces repeatedly at the boundary $\partial\Omega$. 
In the case of half space $\Omega = \mathbb{R}^3_+$ the global existence result was first shown in [6]. A new proof of global modifying Pfaffelmoser’s idea [16] was recently proved in [13]. [1] improved Hwang’s estimates for the growth of the solutions. For general convex bounded domains but with the Neumann boundary condition for $\phi$, the global well-posedness was shown in [11]. With specular reflection boundary condition for $f$ and satisfying Dirichlet boundary condition for $\phi$ was recently proved in [12]. The main contribution of this paper is to introduce geometric methods to the problem of general bounded convex domains with curvatures in order to establish a better upper bound for (1.1) by combining the methods in [12], [13], [17].

By assumption $\partial \Omega$ is a $C^5$ and we will parametrized it locally using a set of coordinate $(\mu_1, \mu_2)$. Let us denote $x_{\parallel}(\mu_1, \mu_2)$ the point of $\partial \Omega$ characterized by the values of the parameters $(\mu_1, \mu_2)$. We will denote as $n(\mu_1, \mu_2)$ the outer normal to $\partial \Omega$ at the point $x_{\parallel}(\mu_1, \mu_2)$.

The Implicit Function Theorem shows that for $\delta > 0$ sufficiently small we can parametrize uniquely the set of points $\partial \Omega + B_\delta(0) \subset \mathbb{R}^3$ by means of the unique values $(\mu_1, \mu_2, x_\perp)$. solving the equation:

$$x = x_{\parallel}(\mu_1, \mu_2) - x_\perp n(\mu_1, \mu_2).$$ (1.10)

Given $x \in \partial \Omega + B_\delta(0)$ we will represent any vector $v \in \mathbb{R}^3$ as:

$$v = v_{\parallel}(\mu_1, \mu_2) - v_\perp n(\mu_1, \mu_2),$$ (1.11)

where $v_{\parallel}(\mu_1, \mu_2) \in T_{x_{\parallel}(\mu_1, \mu_2)}(\partial \Omega), v_\perp \in \mathbb{R}$ and $(\mu_1, \mu_2)$ are as in (1.10). Moreover, we will represent $v_{\parallel} = v_{\parallel}(\mu_1, \mu_2)$ using the two coordinates $(w_1, w_2)$ defined by means of:

$$v_{\parallel} = w_1 u_1 + w_2 u_2,$$ (1.12)

where $\{u_1, u_2\}$ are the basis of $T_{x_{\parallel}(\mu_1, \mu_2)}(\partial \Omega)$ given by:

$$u_i = \frac{\partial x_{\parallel}(\mu_1, \mu_2)}{\partial \mu_i}, \quad i = 1, 2.$$ (1.13)

The system of coordinates $(\mu_1, \mu_2, x_\perp, w_1, w_2, v_\perp)$ will provide a more convenient representation of the set of points in the phase space $\Omega \times \mathbb{R}^3$ that are close to singular set $\Gamma$ defined in (1.9).

By using the set of coordinates $(\mu_1, \mu_2, x_\perp, w_1, w_2, v_\perp)$, we can rewrite the equation (1.1) for $(x, v) \in [\partial \Omega + B_\delta(0)] \times \mathbb{R}^3$ in the form:

$$\frac{\partial f}{\partial t} + \sum_{i=1}^2 \frac{w_i}{1 + k_i x_\perp} \frac{\partial f}{\partial \mu_i} + v_\perp \frac{\partial f}{\partial x_\perp} + \sum_{i=1}^2 \sigma_i \frac{\partial f}{\partial w_i} + F \frac{\partial f}{\partial v_\perp} = 0,$$ (1.14)
where

\[
\sigma_i \equiv E_i - \frac{v_i w_i k_i}{1 + k_i x_\perp} - \sum_{j,l=1}^{2} \frac{\Gamma_{j,l}^i w_j w_l}{1 + k_j x_\perp}, \quad F \equiv E_\perp + \sum_{j=1}^{2} \frac{w_j^2 b_j}{1 + k_j x_\perp},
\]

(1.15)

where \(k_j\) are the principal curvatures, \(b_j\) are the coefficients \(e\) and \(g\) from the second fundamental form according to the notation in [19] and \(\Gamma_{j,l}^i\) are the Christoffel symbols of the surface \(\partial \Omega\). The vector \(E = \nabla_x \phi\) has been written in the form

\[
E = E_1 u_1 + E_2 u_2 - E_\perp n(\mu_1, \mu_2),
\]

(1.16)

where \(u_1, u_2\) are as in (1.13). We note that the proof of the formulas (1.14), (1.15) and (1.16) are just standard lengthy change of variables that makes use of the classical Gauss-Weingarten equations (cf. [19] page 124), and since the domain \(\Omega\) is convex with \(f_0 \geq 0\) we have \(F < 0\).

Now, we need to impose some compatibility conditions on the initial data \(f_0(x, v)\) at the reflection points of \(\partial \Omega \times \mathbb{R}^3\) see [12, 7]. These conditions are the following:

\[
f_0(x, v) = f_0(x, v^*),
\]

\[
v_\perp[\nabla_x^\perp f_0(x, v^*) + \nabla_v^\perp f_0(x, v)] + 2E_\perp(0, x)\nabla_v^\perp f_0(x, v) = 0,
\]

where \(E_\perp(0, x)\) is the decomposition of the field \(E(0, x)\) given by (1.16) and \(\nabla_x^\perp, \nabla_v^\perp\) are the normal component to \(\partial \Omega\) of the gradients \(\nabla_x, \nabla_v\) respectively.

A key point in the proof of global existence of classical solutions to (1.1), (1.2) and (1.4) is to control the growth of velocity support for the phase space density

\[
Q(t) = \sup\{|v| : (x, v) \in \text{supp} f(s), \ 0 \leq s \leq t\}.
\]

(1.17)

As a matter of fact, upper bounds of \(Q(t)\) are closely related to large behavior of these systems and have received a great deal of discussion. For classical Vlasov-Poisson system if the \(\Omega = \mathbb{R}^3\), different upper bounds of \(Q(t)\) were established by many authors see, e.g.: [15, 17, 4], and if \(\Omega = \mathbb{R}^3_+\), recently sub-linear estimate was obtained in [1]. For the system (1.1)-(1.5), so far only the following upper bound was obtained :for any \(\epsilon > 0\)

\[
Q(t) \leq C(1 + t)^{\frac{33}{32} + \epsilon}, \quad t \geq 0,
\]

where \(C\) is a generic positive constant depending only upon initial data \(f_0\). In this paper, we shall establish a better upper bound for (1.1), the main result can be described as follows:
Theorem 1.1 Let $f_0 \in C^{1,\mu}(\Omega \times \mathbb{R}^3)$ with $0 < \mu < 1$. Suppose that $f \in C^{1,\lambda,\mu}([0,T] \times \Omega \times \mathbb{R}^3)$ is a solution of (1.1)-(1.6) with $\lambda \in (0,1), 0 < T < \infty$. Then

$$Q(t) \leq C(1 + t)^{16/21} \ln^{3/7}(2 + t),$$

(1.18)

for some constant $C > 0$.

In order to keep our notation simple we write $f \lesssim g$ as a short for $f \leq Cg$ where $C > 0$ is a constant that may change from line to line, but that may depend on the only upon the initial data and $T$.

2 Definitions and Notations

1. Define $(X(s; t, x, v), V(s; t, x, v))$ by

$$\frac{dX}{ds} = V, \frac{dV}{ds} = E = \nabla \phi, \quad X(t; t, x, v) = x, V(t; t, x, v) = v.$$  (2.1)

as long as $X \in \Omega$. The reflection boundary conditions says that if $X(s_1; t, x, v) \in \partial \Omega$ for some $s_1 \in [0, T]$, then

$$V(s_1^+; t, x, v) = \lim_{s \to s_1^+}\ V(s_1^-; t, x, v) = (V(s_1^-; t, x, v))^* = (\lim_{s \to s_1^+}\ V(s_1^-; t, x, v))^*,$$

where $(.)^*$ is as in (1.8).

2. Define

$$I(t, \delta) = \sup\{\delta \in [0, t] : (x, v) \in \text{supp} f(t) \Rightarrow U(t, \delta, x, v) = \int_{t-\delta}^{t} |E(X(s; t, x, v), s)|ds\},$$

$$\Delta(t, I) = \sup\{\delta \in (0, t) : I(t, \delta) \leq I\}.$$  (2.2)

The analysis centers on estimating

$$U(t, \delta, x, v) = \int_{t-\delta(t)}^{t} |E(Y(s), s)|ds \leq \int_{t-\delta(t)}^{t} \int_{\Omega \times \mathbb{R}^3} \frac{f(s, y, w)}{|y - Y(s)|^2} dydwd\delta + \delta(||h||_{\infty})$$

$$= \int_{\Omega \times \mathbb{R}^3} f(t, x, v) \int_{t-\delta(t)}^{t} |X(s, t, x, v) - Y(s)|^{-2} dsdxdv + \delta(||h||_{\infty}),$$

(2.2)

where we are using the following change of variables: $(X(s), V(s)) = (y, w) \rightarrow (x, v) = (X(t), V(t))$. We also note the measure preserving property (dydw=dxdv) that is due to the fact that the evolution of the characteristics is Hamiltonian away from the boundary and that the measure dxdv is also preserved by reflection on the boundary.

We will introduce a new coordinate system that will be convenient to study the dynamics of the characteristic curves for bouncing trajectories. Suppose
that \((μ₁, μ₂, x_\perp, w₁, w₂, v_\perp)\) are as in (1.10)-(1.13). We then define two new coordinates \((α(t, μ₁, μ₂, x_\perp, w₁, w₂, v_\perp), β(t, μ₁, μ₂, x_\perp, w₁, w₂, v_\perp))\) as follows:

\[
α(t, μ₁, μ₂, x_\perp, w₁, w₂, v_\perp) = \frac{v_\perp^2}{2} - F(t, μ₁, μ₂, 0, w₁, w₂)x_\perp,
\]

\[
β(t, μ₁, μ₂, x_\perp, w₁, w₂, v_\perp) = -2πH(t, μ₁, μ₂, x_\perp, w₁, w₂, v_\perp) = π(1 - \frac{v_\perp}{\sqrt{2α}}),
\]

where the function \(H\) will increase by one at each bounce and \(F\) is as in (1.15). Since \(v_\perp\) change from \(-\sqrt{2α}\) to \(\sqrt{2α}\) in each bounce, it follows that \(β\) is continuous along characteristic. Notice that \(β\) is just a coordinate that indicates the specific point in the surface \(\{α = \text{constant}\}\) where the trajectory lies. In all the following, we will write for simplicity \(F(t, 0)\) instead of \(F(t, μ₁, μ₂, 0, w₁, w₂)\) and we will drop the dependence of \(H\) on the variables \(μ₁, μ₂, x_\perp, w₁, w₂, v_\perp\) if there is no risk of confusion. We can then write:

\[
x_\perp = -\frac{α}{F(t, 0)} \left[1 - \left(1 - \frac{β - 2πH(t)}{π}\right)^2\right],
\]

\[
v_\perp = \sqrt{2α}(1 - \frac{β - 2πH(t)}{π}).
\]

Making the change of variables \((t, μ₁, μ₂, x_\perp, w₁, w₂, v_\perp) → (t, μ₁, μ₂, α, w₁, w₂, β)\), we transform the system (1.14) by the following form in lemma below, which is in [11] and we skip its proof.

**Lemma 2.1** Under the assumptions of Theorem 1.1 we have that \(F(t, μ₁, μ₂, 0, w₁, w₂) \leq -Λ < 0\) in \(Ω\) in any time interval \(0 ≤ t ≤ t^*\), in which the function \(Λ\) will be defined in the next section. Moreover, in the coordinate system \((t, μ₁, μ₂, α, w₁, w₂, β)\) the problem (1.14) can be reformulated in \([Ω + B_δ(0)] × \mathbb{R}^3\) as:

\[
\frac{∂f}{∂t} + \sum_{i=1}^{2} \frac{w_i}{1 + k_i x_\perp} \frac{∂f}{∂μ_i} + \sum_{i=1}^{2} σ_i \frac{∂f}{∂w_i} + \left[v_\perp(F(t, x_\perp) - F(t, 0))\right]
\]

\[
- x_\perp \left\{\sum_{i=1}^{2} \left(\frac{w_i}{1 + k_i x_\perp} \frac{∂F(t, o)}{∂μ_i} + σ_i \frac{∂F(t, 0)}{∂w_i}\right)\right\} \frac{∂f}{∂α} + \left[ - \frac{πv_\perp^2}{(2α)^{\frac{3}{2}}} F(t, 0)\right]
\]

\[
+ \frac{2πF(t, 0)F(t, x_\perp)x_\perp}{(2α)^{\frac{3}{2}}} - \frac{πx_\perp v_\perp}{(2α)^{\frac{3}{2}}} \left\{\sum_{i=1}^{2} \left(\frac{w_i}{1 + k_i x_\perp} \frac{∂F(t, o)}{∂μ_i}\right)\right\} \frac{∂f}{∂β} = 0.
\]

### 3 First Integral Estimate

Consider a characteristic, \((\hat{X}(s), \hat{V}(s))\), along which \(f(s, \hat{X}(s), \hat{V}(s)) \neq 0\). For \(t > 0\) and \(δ ∈ (0, t)\) note that

\[
\int_{t-δ(t)}^{t} |E(Y(s), s)|ds ≤ \int_{t-δ(t)}^{t} \int_{Ω × \mathbb{R}^3} \frac{f(s, y, w)}{|y - Y(s)|} dydwds + δ(∥h∥_∞).
\]
We shall consider the following standard partition of the integration domain with take into account the bounces at the boundary:

\[ S_1 = \{(s, y, w) \in (t - \delta, t) \times \Omega \times \mathbb{R}^3 : |y| \leq 200P, |w - \hat{V}(t)| \leq 200P, |w - \hat{V}^+(t)| \leq 200P\}, \]

\[ S_2 = D(\Lambda) \backslash (S_1), \]

\[ S_3 = (t - \delta, t) \times \Omega \times \mathbb{R}^3 \backslash (S_1 \cup S_2). \]

Here \( D(\Lambda) = \{(s, y, w) : |y - \hat{X}(s) \leq \Lambda\} \) where \( \Lambda \equiv \frac{R}{|w|^2} \left( \frac{1}{|w - \hat{V}(s)|} + \frac{1}{|w - \hat{V}^+(s)|} \right) \), if the characteristic curve \( \hat{X}(s) \) does not intersect \( \partial \Omega \) on the interval \( s \in [t - \delta, t] \), and otherwise \( \Lambda \equiv \frac{R}{|w|^2} \left( \frac{1}{|w - \hat{V}(s)|} + \frac{1}{|w - \hat{V}^+(s)|} \right) \). The parameter \( R \) will be made precise later. Here \( \hat{V}^+(t) = \hat{V}^*(s_0) \), where \( s_0 = \sup \{s \in [t - \delta, t] : \hat{X}(s) \in \partial \Omega \} \).

We now give the following Lemma which is in [11], and we omit the proof, which utilizing to drive a rough estimate for the right-hand side of (3.1).

**Lemma 3.1** Let \( f_0, f \) be as defined in Theorem 1.1 and set \( E(t, x) = \nabla_x \phi(t, x) \) in (1.1): we have the following estimate

\[
(i) \sup_{0 \leq t \leq T} \|\rho(t, \cdot)\|_{L^2(\Omega)} \leq C,
\]

\[
(ii) \sup_{0 \leq t \leq T} \int_{\Omega} v^2 f(t, x, v) dx dv \leq C,
\]

\[
(iii) \|f(t)\|_{L^p(\Omega \times \mathbb{R}^3)} = \|f_0\|_{L^p(\Omega \times \mathbb{R}^3)}, \quad \text{for all} \quad 1 \leq p \leq \infty,
\]

\[
(iv) \frac{|w|}{2} \leq |v| \leq 2|w|,
\]

\[
\frac{|w - \hat{V}(t)|}{2} + \frac{|w - \hat{V}^+(t)|}{2} \leq |v - \hat{V}(t)| + |v - \hat{V}^+(t)| \leq 2|w - \hat{V}(t)| + 2|w - \hat{V}^+(t)|,
\]

\[
(v) ||X(s) - \hat{X}(s)|| \geq C(\Lambda + \min\{|V(t) - \hat{V}(t)|s - s_0|, |V(t) - \hat{V}^+(t)||s - s_1|\}).
\]

To derive an upper bound, we choose \( \delta \) satisfying: \( \delta Q^\frac{1}{2}(t) = c_0 P \), where \( c_0 \) is small, and in order to estimate the contribution of the set \( S_1 \), we define:

\[ \rho_{S_1}(y, s) \equiv \int_{S_1} f(s, y, w) dw \]

standard estimates yields \( \|\rho_{S_1}\| \leq CP^3 \). So that by Lemma 3.1 (i), we have

\[
\int_{S_1} \frac{f(s, y, w)}{|y - \hat{X}(s)|^2} ds dy dw \lesssim \delta P^\frac{3}{2} \tag{3.2}
\]

Now by Lemma 3.1 (iv), we have \( \Lambda \leq \frac{sR}{|w|^2} \left( \frac{1}{|w - \hat{V}(s)|} + \frac{1}{|w - \hat{V}^+(s)|} \right) \) it follows that

\[
\int_{S_2} \frac{f(s, y, w)}{|y - \hat{X}(s)|^2} ds dy dw \lesssim \int_{t - \delta}^t ds \int_{|w| \leq Q} \Lambda dw \lesssim R \int_{t - \delta}^t \ln \frac{Q^2(s)}{|\hat{V}(s)||\hat{V}^+(s)|} ds \tag{3.3}
\]

where \( |\hat{V}(t)| \geq 1, \) since otherwise the corresponding characteristics have an effect of order one in the variation of \( Q \).

Finally, to estimate the contribution of \( S_3 \), the following Lemma is the main tool for handling the time integration.
Lemma 3.2 Let \((x, v) \in \text{supp} f(t)\) and \(\delta \in [0, t]\) verifying
\[
\delta \leq \triangle(t, \frac{1}{20} \min \{|V(t) - \hat{V}(t)|, |V(t) - \hat{V}^+(t)|\}), \tag{3.4}
\]
suppose there exists \(\alpha > 0\) such that for any \(s \in [t - \delta, t]\) we have
\[
\Lambda(s, V(s)) \geq \alpha \Lambda(t, V(t)), \tag{3.5}
\]
then
\[
\int_{t-\delta}^{t} \frac{1_{D(\Lambda)^c}(s, X(s), V(s))}{|X(s) - \hat{X}(s)|^2} ds \lesssim \frac{1}{\alpha \Lambda} \left( \frac{1}{|v - \hat{V}(t)|} + \frac{1}{|v - \hat{V}^+(t)|} \right). \tag{3.6}
\]

Proof. For all \(s \in [t - \delta, t]\) and Lemma 3.1(v). We have
\[
\int_{t-\delta}^{t} |(X(s) - \hat{X}(s)|^{-2} 1_{D(\Lambda)^c}(s, X(s), V(s)) ds \\
\leq \int_{t-\delta}^{t} C(\Lambda + \min \{|v - \hat{V}(t)||s - s_0|, |v - \hat{V}^+(t)||s - s_1|\})^2 ds \\
\leq \frac{C}{\Lambda} \left( \frac{1}{|v - \hat{V}(s)|} + \frac{1}{|v - \hat{V}^+(s)|} \right). \tag{3.7}
\]
If \(s^* \in [t - \delta, t]\), for some \(s^* \in [t - \delta, t]\) then \(\min \{|V(s^*)|, |V(s^*) - \hat{V}(s^*)|, |V(s^*) - \hat{V}^+(s^*)|\} \geq 200P\). But then
\[
\delta \leq \triangle(t, P) \leq \frac{1}{200} \min \{|V(s^*) - \hat{V}(s^*)|, |V(s^*) - \hat{V}^+(s^*)|\} \\
\leq \triangle(t, \frac{99}{2} \min \{|v - \hat{V}(s)|, |v - \hat{V}^+(s)|\}).
\]
and (3.4) holds true. By the same way we find \(\delta \leq \triangle(t, \frac{|v|}{199})\), so
\[
\delta \leq \min \{\triangle(t, \frac{|v|}{199}), \triangle(t, \frac{99}{2} |v - \hat{V}(t)|), \triangle(t, \frac{99}{2} |v - \hat{V}^+(t)|)\}.
\]
In view of the definition of \(\Lambda(s, V(s))\) (3.5) is satisfied, and obtain from the above Lemma
\[
f_{t-\delta}^{t} |(X(s) - \hat{X}(s)|^{-2} 1_{S_3}(s, X(s), V(s)) ds \leq \frac{Cv^2}{R}. \tag{3.8}
\]
Thus we may use the Lemma and (3.8) eventually follows from the fact that \(S_3 \subset D(\Lambda)^c\). Integrating yields
\[
\int_{S_3} \frac{f(s, y, w)}{|y - X(s)|^2} ds dy dw \lesssim \frac{1}{R} \int_{S_3} v^2 f(t, x, v) dx dv \lesssim R^{-1}. \tag{3.9}
\]
Collecting (3.2), (3.3) and (3.9), we get for any $P > 0$ and $\delta \leq \triangle(t,P)$

$$U(t,\delta,x,v) \lesssim (\delta P^\frac{4}{3} + R \int_{t-\delta}^t \ln \frac{Q^2(s)}{|\hat{V}(s)||\hat{V}^+(s)|} + R^{-1}).$$

(3.10)

Now, we give two useful corollaries, the first one provides a proof of the following super linear growth rate, similar to the one obtained in [11] except that an additional logarithmic disappears here. The second corollary allows us to derive an upper bound for $I(t,\tau)$.

**Corollary 3.3** For any $t \geq 0$ we have

$$Q(t) \lesssim (1 + t)^\frac{33}{17}. \quad (3.11)$$

**Corollary 3.4** For any $t \geq \tau \geq 0$, we have

(i) $I(t,\tau) \lesssim \max(\tau^\frac{3}{2}\ln^\frac{1}{2}(2 + t), \tau \ln^\frac{3}{4}(2 + t))$.

(ii) $\triangle(t,I) \gtrsim \min(t, I^2 \ln^{-1}(2 + t), I \ln^{-\frac{3}{4}}(2 + t)).$

In the sequel we denote by $\triangle_-(t,I)$ the lower bound obtained in the last inequality. **Proof of Corollary 3.3.** Firstly we show

$$Q(t) - Q(t - \delta) \lesssim \delta Q^{\frac{16}{17}}(t). \quad (3.12)$$

Let $\delta \in (0,1)$ and $t \geq 1$ verifies $Q(t) = \max_{[0,t]} Q \geq 3$, choose $|v| = Q(t)$ and if $\delta \leq \triangle(t,\frac{Q(t)}{2})$ then we have $|\hat{V}(s)| \gtrsim |v|$, $|\hat{V}^+(s)| \gtrsim |v|$ and $Q(s) \lesssim Q(t)$ for any $s \in (t-\delta,t)$, therefore we infer from (3.10) that

$$Q(t) - Q(t - \delta) \lesssim \delta Q^\frac{4}{3} + \delta^\frac{1}{2}, \quad (3.13)$$

because $Q(t) \geq Q(s)$ for any $s \leq t$, it yields $\triangle(t,P) \geq \min(t, c_0 Q^{\frac{16}{17}}(t)P)$, with $c_0$ sufficiently small. Suppose $P = Q(t)^\frac{4}{17}$ and $\delta = c_0 Q(t)^\frac{16}{17}$, we get $\triangle(t,P) \geq \min(t, c_0 Q(t)^\frac{16}{17}) = \delta$, as a result the two assumptions $2P \leq Q(t)$ and $\delta \leq \triangle(t,P)$ hold true and (3.13) gives (3.12), but since $Q(t - \delta) \leq Q(t)$, so

$$Q^{\frac{16}{17}}(t) - Q^{\frac{16}{17}}(t - \delta) \lesssim \delta. \quad (3.14)$$

Define $\phi(t) = M(1 + t)^\frac{33}{17}$ with $M = 1 + \max_{[0,2]} Q + C^{-\frac{33}{17}}$ where $C > 1$ is the implied constant in (3.14). Assume there exists $t_0 = \min\{t \geq 0 : Q(t) = \phi(t)\}$. Through this assumption, we have necessarily $t_0 \geq 2$, $Q(t_0) \geq 3$ and $Q(t_0) \geq Q(t)$ for any $t \in [0,t_0]$, using (3.14) and the equality $Q(t_0) = \phi(t_0)$ we find

$$M^{\frac{17}{34}}(1 + t_0) - Q^{\frac{17}{34}}(t_0 - \delta) \leq C \delta.$$

Since $C \leq M^{\frac{17}{34}}$ implies

$$\phi^{\frac{17}{34}}(t_0 - \delta) = M^{\frac{17}{34}}(1 + t_0 - \delta) \leq Q^{\frac{17}{34}}(t_0 - \delta),$$

but this is a contradiction and the proof of the corollary is complete. \qed

Corollary 3.4 can be proved by applying Schaeffer’s method [17] in the case of the whole space and the proof is same here.
4 Final Growth Estimate

Let us pick \( t > 0, \delta > 0 \) and choose \((x, v)\) such that
\[
|v| = Q(t) \geq 200, \tag{4.1}
\]
\[
\delta = \min\left(\frac{t}{20}, \triangle_-(t, \frac{Q(t)}{5})\right). \tag{4.2}
\]

Below, we aim at finding an upper bound on \( U(t, \delta, x, v) \) using the same method as above, with the same definition of the sets \( S_1, S_2 \) and \( S_3 \) in the previous section. Once again \( D(\Lambda) = \{(s, y, w) : |X(s) - \hat{X}(s)| \leq \Lambda(s, v)\} \) but the definition of \( \Lambda \) differs from the one given in the previous section. In particular, an additional term appears. We set
\[
\Lambda \equiv R \left(\frac{1}{|v|^2 m(s, v)} \left(\frac{1}{|v - \hat{V}(s)|} + \frac{1}{|v - \hat{V}+(s)|}\right)\right)
\]
if the characteristic curve \( \hat{X}(s) \) does not intersect \( \partial \Omega \) on the interval \( s \in [t - \delta, t] \), and
\[
\Lambda \equiv R \left(\frac{1}{|v|^2 m(s, v)} \left(\frac{1}{|v - \hat{V}(s)|} + \frac{1}{|v - \hat{V}+(s)|}\right)\right)
\]
Here \( m(s, v) \) is defined using the lower bound \( \triangle_- \) in Corollary 3.4
\[
m(s, v) = \min\left\{\frac{s}{20}, \triangle_-(s, \frac{|v|}{20}), \triangle_-(s, \frac{|v - \hat{V}(s)|}{20}), \triangle_-(s, \frac{|v - \hat{V}+(s)|}{20})\right\}. \tag{4.3}
\]

By the definition of \( m \), we deduce from (3.4) that \( m(s, V(s)) \leq \triangle(s, \frac{|V(s)|}{20}) \). Thus, (3.4) holds true with \( t = s \) and \( \delta = m(s, V(s)) \). Also, by the definition of \( m \), we note that it follows (3.5) holds true as well. By the definition of \( \Lambda \) and using the fact that \( m(s, V(s)) \leq \triangle(s, \frac{|V(s)|}{20}) \), we conclude from Lemma 4.1 in [1] that
\[
\int \int_{S_3} f \int_{s-m(s, V(s))}^{s} \frac{|V(s)|^2}{R} ds dx dv \lesssim R^{-1} \delta. \tag{4.4}
\]

Let \( \psi = \{v : p \leq |v| \leq Q(s), p \leq |v - \hat{V}(s)| \text{ and } p \leq |v - \hat{V}+(s)|\} \) and write
\[
\int \int \int_{S_2} \int_{t-\delta}^{t} \int_{\mu} \frac{R}{|v|^2 m(s, v)} \left(\frac{1}{|v - \hat{V}(s)|} + \frac{1}{|v - \hat{V}+(s)|}\right) dv ds ds dt.
\]

Using similar method in [1], several cases which may arise from definition of \( m(s, v) \) where \( \psi = \bigcup_{i=1}^{7} M_i \). Consequently, we obtain
\[
\int \int \int_{S_2} \int_{t-\delta}^{t} \left(\frac{1}{\nu} \ln(2 + t) + \frac{1}{\nu} \ln(2 + t) \right) (\frac{1}{|V(t)|} + \frac{1}{|V+(t)|})
\]
Asymptotic growth bounds for the Vlasov-Poisson system

\[ + \ln(2 + t) \ln(1 + \frac{\ln^\frac{3}{2}(2 + t)}{p}) \left( \frac{1}{|V(t)|^2} + \frac{1}{|V^+(t)|^2} \right). \]

taking \( p = \left( \frac{\ln(2+t)}{|Q(t)|} \right)^\frac{3}{8} \) and Corollary 3.4, yield

\[
\int \int \int S \lesssim \delta R \frac{\ln^\frac{3}{2}(2 + t)}{|Q(t)|^{\frac{3}{8}}}.
\]

(4.5)

Adding the preceding estimates (3.2), (4.4) and (4.5), and optimizing \( R \), we get for every \( t \geq 0 \)

\[ Q(t) - Q(t - \delta) \lesssim \delta |Q(t)|^{-\frac{3}{8}} \ln^{\frac{3}{8}} (2 + t). \]

Using the same processing of proof Corollary 3.3 yield the expected upper bound (1.18) and the proof of Theorem 1.1 is complete.

\[ \square \]

Acknowledgments. The author would like to thank professor X. Zhang (School of Mathematics and Statistics, Huazhong University of Science and Technology) for giving advice on this work and improving this paper.

References


Received: October 5, 2015; Published: November 21, 2015