Analysis of a Ratio-Dependent
Predator-Prey Interaction with
Epidemic in the Predator

Huaying Wang

LMIB-School of Mathematics and Systems Science
Beihang University, Beijing 100191, China

Xia Li

School of Mathematics and Science
Shijiazhuang University of Economics, Shijiazhuang 050031, China

Luona Zhang

Department of Electrical and Computer Engineering
University of Alberta, Edmonton, Alberta T6G2V4, Canada

Yulei Pang

Department of Mathematics
Southern Connecticut State University, New Haven, CT 06515, USA

Abstract

This paper studies a kind of ratio-dependent predator-prey model. Assuming that predator populations suffered from epidemics, and the infected individuals also have some foraging capability and the predation rate is a monotonic function of the general. By analyzing the boundedness in the solutions and the existence of equilibrium point in the system, we gradually reaching local asymptotic stability of disease free
equilibrium and epidemic equilibrium, and global asymptotic stability of equilibrium of predator extinction.

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## 1 Introduction

From the construction of Lotka-Voltera equation [8] till now, people have put a lot of effort to study the population dynamics, which include the cases while species spreading the disease, see [4,6]. Sun and Yuan [3] discussed the model with infected predator and Holling I type functional response, and assuming that infected predator cannot hunt, analyzed the stability of equilibrium point. Zhang [7] discussed a more general model with functional response, and studied on persistence of the system solutions, to promote the conclusion of the previous literature. The work [2] and [10] discussed the spread of epidemics in the prey, finding epidemic threshold conditions. However, in many circumstances, ratio-dependent predator-prey model, which can be roughly stated as that the per capita predator growth rate should be a function of the ratio of prey to predator abundance, tend to be a better fit supported by numerous field and laboratory experiments [1] and observations and getting more and more attention in the predator-prey system. The paper [5] discussed eco-epidemic model of ratio dependent, proposed control parameters to avoid the extinction of population. This paper builds the following eco-epidemic model based on the SI compartment model of Kermack-Mckendrick (1927),

\[
\begin{align*}
\frac{dX}{dt} &= X(a - bX) - \frac{XS}{mS+X} - \varphi(X)I \\
\frac{dS}{dt} &= \frac{kXS}{mS+X} - d_1S - \beta SI \\
\frac{dI}{dt} &= \beta SI - d_2I,
\end{align*}
\]

we assume infected predator also have foraging capability, and using a more general function to represent predation conversion rate. Among these parameters, \(X\) represent the population density of prey, \(S\) and \(I\) represent the density of susceptible and infected predator group respectively. All the coefficients are positive constants. \(a\) is the intrinsic increasing rate of prey, \(b\) is capacity related coefficient, \(\frac{XS}{mS+X}\) is the ratio-dependent response function and \(k\) is the conversion rate. Taking into account that infected predator also have the ability of predation, we suppose \(\varphi(X)\) is positive, incremental, bounded continuous function, and \(\varphi(0) = 0\). \(d_1\) is the death rate for predators, \(d_2\) is the rate of epidemic mortality for predators, \(\beta\) is the contact rate, and once infected, on longer recover.
2 Main Results

Theorem 2.1. Any solution of system (1) is eventually bounded in $R^3_+$.

Proof. From the first equation in (1), we get $X' \leq X(a-bX)$, it implies that there exists $T > 0$, when $t > T$, we have $X(t) \leq \frac{a}{b}$. Let $W(t) = X(t) + \frac{1}{k}S(t) + \frac{1}{k}I(t)$, and take derivative, $W'(t) = X(a-bX) - \frac{d_1}{k}S - \frac{d_2}{k}I - \varphi(X)I$. Let $\tau = \min(d_1, d_2)$, when $t > T$, we get, $W'(t) + \tau W(t) \leq X(a+\tau-bX) - \frac{d_1-\tau}{k}S - \frac{d_2-\tau}{k}I \leq \frac{a}{b}(a+\tau) = L$.

According to Comparison theorem, it follows that, $0 < W(t) < \frac{L}{\tau}(1-e^{-\tau(t-T)}) + W(T)e^{-\tau(t-T)}$. So it deduces $0 < W(t) < \frac{L}{\tau}$, as $t \to +\infty$. Thus, all solutions of system (1) within $R^3_+$ will eventually enter the domain $K$,

$$K = \{(X, S, I) \in R^3_+: X \leq \frac{a}{b}, W = X + \frac{1}{k}S + \frac{1}{k}I \leq \frac{L}{\tau}\}.$$

In the following we analysis the stability of equilibriums in system (1).

(i) When $I = 0$, from the equations as follows,

\[
\begin{align*}
X(a-bX) - \frac{XS}{mS+X} &= 0, \\
\frac{kX}{mS+X} - d_1S &= 0,
\end{align*}
\]

we obtain equilibriums as $E_1 = (\frac{a}{b}, 0, 0)$, $E_2 = (X_2, S_2, 0)$, here $S_2 = \frac{kX_2(a-bX_2)}{d_1}$ and $X_2$ satisfies the equation $k - d_1 = km(a-bX_2)$. Due to $0 < X_2 < \frac{a}{b}$, that is $k - d_1 > 0$, $E_2$ is so-called disease-free equilibrium.

(ii) When $I \neq 0$, the positive equilibrium exists if the following equations have positive solutions,

\[
\begin{align*}
X(a-bX) - \frac{XS}{mS+X} - \varphi(X)I &= 0, \\
\frac{kX}{mS+X} - d_1 - \beta I &= 0, \\
\beta S - d_2 &= 0.
\end{align*}
\]

from the second and the third equations of (2), we have,

$$S = \frac{d_2}{\beta} = S_3, \quad I = \frac{1}{\beta} \left( \frac{kX}{mS+X} - d_1 \right).$$

Substitute (3) into the first equation of (2), it follows,

$$\beta X(a-bX)(md_2 + \beta X) - \beta d_2 X - \varphi(X)[\beta kX - d_1(md_2 + \beta X)] = 0.$$  (4)

Suppose (4) has some root in $(0, \frac{a}{b})$ (denoted $X_3$) and substitute $X_3$ into (3), we get $I = \frac{1}{\beta} \left( \frac{kX_3}{mS_3+X_3} - d_1 \right) = I_3$, in this case there exists the positive equilibrium $E_3 = (X_3, S_3, I_3)$ in system (1).
Theorem 2.2. When \( 0 < X_2 < \frac{a}{b} \), disease-free equilibrium \( E_2(X_2, S_2, 0) \) exists. When the equation (4) has root in \( (0, \frac{a}{b}) \), the positive equilibrium \( E_3 \) (that means coexistence) exists.

In the following, we will study the stability of equilibria of system (1). The Jacobian matrix of the system about any equilibrium point \((x, y)\) is given by,

\[
J(x, y) = \begin{pmatrix}
a - 2bX - \frac{mS^2}{(mS + X)^2} & \frac{X^2}{(mS + X)^2} - \varphi'(X) & -\varphi(X) \\
\frac{kmS^2}{(mS + X)^2} & \frac{kmS^2}{(mS + X)^2} - d_1 - \beta I & -\beta S \\
0 & \beta I & \beta S - d_2
\end{pmatrix}.
\]

At the axial equilibrium point \( E_1 \),

\[
J(E_1) = \begin{pmatrix}
-a & -1 & -\varphi(0) \\
0 & k - d_1 & 0 \\
0 & 0 & -d_2
\end{pmatrix},
\]

so \( E_1 \) is locally asymptotically stable if \( k - d_1 < 0 \). Due to (i), the disease-free equilibrium \( E_2 \) is nonexistent.

For \( E_2 \), the Jacobian matrix takes the form,

\[
J(E_2) = \begin{pmatrix}
a - 2bX - \frac{mS^2}{(mS + X)^2} & \frac{X^2}{(mS + X)^2} - \varphi'(X) & -\varphi(X) \\
\frac{kmS^2}{(mS + X)^2} & \frac{kmS^2}{(mS + X)^2} - \frac{kmS^2}{(mS + X)^2} & -\beta S \\
0 & \beta I & \beta S - d_2
\end{pmatrix},
\]

its characteristic equation is,

\[
(\beta S - d_2 - \lambda)[(\frac{kmS^2}{(mS + X)^2} - \lambda)(a - 2bX - \frac{mS^2}{(mS + X)^2} - \lambda) + \frac{kmS^2}{(mS + X)^2}] = 0.
\]

It is clearly that \( \lambda_1 = \beta S - d_2 \) and \( \lambda_2 = \frac{kmS^2}{(mS + X)^2} > 0 \). If \( a - 2bX < 0 \) and \( \lambda_1 = \beta S - d_2 < 0 \), then \( \lambda_2 + 3 < 0 \), that is, \( E_2 \) is locally asymptotically stable.

Theorem 2.3. When \( k - d_1 < 0 \), \( E_1 \) is locally asymptotically stable, and there is no disease-free equilibrium exists. When \( \frac{a}{b} < X_2 < \frac{a}{b} \) and \( \beta S_d - d_2 < 0 \), the disease-free equilibrium is locally asymptotically stable.

For the positive equilibrium point \( E_3 \), its Jacobian matrix is as follows,

\[
J(E_3) = \begin{pmatrix}
a - 2bX - \frac{mS^2}{(mS + X)^2} & -\varphi'(X_3) & -\varphi(X_3) \\
\frac{kmS^2}{(mS + X)^2} & \frac{kmS^2}{(mS + X)^2} - \frac{kmS^2}{(mS + X)^2} & -\beta S_3 \\
0 & \beta I_3 & 0
\end{pmatrix}.
\]

Let \( J(E_3) = (a_{ij})_{3\times3} \), its characteristic equation is,

\[
\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0,
\] (5)
where,
\[
A_1 = -(a_{11} + a_{22}) - \left( a - 2bX_3 - \frac{mS_3^2}{(mS_3 + X_3)^2} - \varphi'(X_3)I_3 - \frac{kmS_3 X_3}{(mS_3 + X_3)^2} \right),
\]
and \(A_1 > 0\) if \(a - 2bX_2 < 0\). Next, we consider \(A_2\) and \(A_3\).
\[
A_2 = a_{11} a_{22} - a_{21} a_{12} - a_{32} a_{23},
A_3 = a_{32}(a_{11} a_{23} - a_{13} a_{21}),
\]
it is clearly that when \(a - 2bX_2 < 0\), we get \(a_{11} a_{22} > 0\) and \(a_{11} a_{23} - a_{13} a_{21} > 0\), hence \(A_2, A_3 > 0\).
Combining \(A_1, A_2\) and \(A_3\), we have,
\[
\Delta = A_1 A_2 - A_3
= -(a_{11} + a_{22})(a_{11} a_{22} - a_{21} a_{12} - a_{23} a_{32}) - a_{32}(a_{11} a_{23} - a_{13} a_{21})
= -(a_{11} + a_{22})(a_{11} a_{22} - a_{21} a_{12}) + a_{32}(a_{22} a_{23} + a_{13} a_{21}),
\]
if \(a - 2bX_2 < 0\), then \(-(a_{11} + a_{22})(a_{11} a_{22} - a_{21} a_{12}) > 0\), as well as if \(a_{22} a_{23} + a_{13} a_{21} > 0\) (that is \(\varphi(X_3) < \frac{\alpha_{22} \alpha_{23}}{\alpha_{32}} = \beta X_3\)), then we ultimately obtain \(\Delta > 0\).

According to Hurwitz’s criterion, the real part of each root for characteristic equation (5) is negative, it implies that \(E_3\) is locally asymptotically stable.
We conclude the above analysis as follows,

**Theorem 2.4.** When \(\max\left( \frac{a_{11}}{2b}, \frac{\varphi'(X_3)}{\beta} \right) < X_3 < \frac{a}{b}\), the positive equilibrium \(E_1\) is locally asymptotically stable.

With respect to global asymptotically stability of equilibrium of predator extinction \(E_3\), we have the following conclusion,

**Theorem 2.5.** If \(k - d_1 < \frac{-\tau}{mk(a + \tau)}\), then equilibrium of predator extinction \(E_1\) is global asymptotically stable.

**Proof.** Due to Theorem 2.3, if \(k - d_1 < 0\), then \(E_1\) is locally asymptotically stable. In addition, from Theorem 2.1, all solutions of system (1) within \(R^3_+\) will eventually enter the domain \(K\), so it is enough to show that \(E_1\) is asymptotically stable in the domain \(K\).

Choose Lyapunov function, \(V(t) = V(X, S, I) = X - \frac{a}{b} - \frac{a}{b} \ln \frac{kX}{mS + X} + S + I\) and take derivative along system (1), we get,
\[
V'(t) = \frac{X - \frac{a}{b}}{X} \frac{dX}{dt} + \frac{dS}{dt} + \frac{dI}{dt}
\leq (X - \frac{a}{b}) \left( a - bX - \frac{S}{mS + X} \right) + S \left( \frac{kX}{mS + X} - d_1 \right)
\leq (X - \frac{a}{b}) \left( -b(X - \frac{a}{b}) - \frac{S}{mS + X} \right) + S(k - d_1)
= -b(X - \frac{a}{b})^2 + (\frac{a}{b} - X) \frac{S}{mS + X} + S(k - d_1),
\]
from Theorem 2.1, when $t$ is large enough, we have $0 < X < \frac{a}{b}$ and $0 < S < \frac{kL}{\tau}$, then $V'(t) \leq \frac{a}{bm} + \frac{kL}{\tau}(k - d_1)$. Because $k - d_1 < \frac{-\tau}{mk(a+\tau)}$, hence $V'(t) < 0$, that is, $E_1$ is locally asymptotically stable.

3 Discussion

The predator-epidemics model this paper studies assumes infected predators have some predation, and to represent predation rate as a more general monotonic function, which just met some basic requirements. In nature, the predation rate of infected individuals in some populations is lower than individuals without the disease, while others may exhibit an extremely efficient aggressiveness, so the predation rate function can also be constructed due to the specific circumferences. In addition, some study on new modeling method for predator-prey interaction has been appeared, see [9]. That may support some new ideas for mathematical modeling of eco-epidemiological phenomenons.

This paper has discussed the global stability of predator extinction equilibrium point. Unfortunately, however, we did not find the requirements for global stability of disease-free equilibrium (predator exist). In addition, only sufficient condition for distinction is not enough. After all, bio-diversity of species ensure macro stability in nature. These deficiencies are all we need to improve in the future place.

References


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