Numerical Solution of Third Order BVPs
by Using Non-polynomial Spline with FDM

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Abstract

In this paper, a numerical scheme is proposed for the numerical solutions of the third order two point boundary value problems (BVPs) using non-polynomial spline method with finite difference method (FDM). The numerical results are obtained for different values of \( n \). Two test problems have been considered to test the accuracy of the proposed method, and to compare the compute results with exact solutions and other known methods.

Keywords: non-polynomial spline, finite difference, tow-point boundary value problem, exact solution

1. Introduction

The boundary value problems of ordinary differential equation play a significant role in wide variety of problems such as electrostatic potential between two concentric metal, chemical reaction, heat transfer and deflection of a beam. These problems can be presented by using boundary value problem with two boundary conditions.

Solutions of BVPs can sufficiently closely be approximated by simple and efficient numerical method. Among these numerical methods are finite difference method, standard 5-point formula, iteration method, relaxation method and standard analytic method. But here the non-polynomial spline method [1-2,4-9] with finite difference will be considered. BVPs arise in several branches of differ-
ential equations of mathematical physics. Problems involving the wave equation, such as the determination of normal modes, are often stated as BVPs. The analysis of these problems involves the eigen functions of a differential operator. Third-order boundary-value problems for differential equations play an important role in a variety of different areas of applied mathematics and physics ([3],[10]).

In this paper, we will show the numerical solutions for two-point boundary value problem of linear third order ordinary differential equation:

\[
y'' = p_2(x)y'' + p_1(x)y' + p_0(x)y + r(x) \quad x \in [a,b]
\]  

(1)

With the boundary conditions:

\[
y(a) = \gamma_1, \quad y'(a) = \gamma_2, \quad y(b) = \gamma_3
\]  

(2)

where \( \gamma_1, \gamma_2, \) and \( \gamma_3 \) are finite real constants and \( p_2(x), p_1(x), p_0(x) \) and \( r(x) \) are continuous functions defined on the interval \([a,b]\).

2. Derivation of the Non-polynomial Spline Method

We introduce a finite set of grid point \( x_i \) by dividing the interval \([a,b]\) into \( n \) equal subintervals where

\[
x_i = a + ih, \quad i = 0, 1, 2 \ldots n, \quad x_0 = a, \quad x_n = b \quad \text{and} \quad h = \frac{b-a}{n}.
\]

Let \( y(x) \) be the exact solution of the problem (1)-(2) and \( y_i \) be the approximation to \( y(x_i) \) obtained by the spline function \( P_i(x) \) passing through the point \((x_i, y_i)\) and each quartic non-polynomial spline segment has the form

\[
P_i(x) = a_i \cos k(x-x_i) + b_i \sin k(x-x_i) + c_i (x-x_i)^2 + d_i (x-x_i)^3 + e_i,
\]

for \( i = 0, 1, 2 \ldots n-1 \),

\[
(3)
\]

Where \( a_i, b_i, c_i, d_i, \) and \( e_i \) are constants and \( k \) is a free parameter to be determined later, consider the non-polynomial function \( P_i(x) \) of the class \( C^3[a,b] \), dependent on free parameter \( k \), which interpolates \( y(x) \) at the mesh points \( x_i, i = 0, 1, \ldots, n \), and reduces to an ordinary spline \( P_i(x) \) in \([a,b]\) as \( k \to 0 \).

First, we develop expressions for the coefficients of equation (3) in the terms of \( y_i, y_{i+1}, D_i, S_i, \) and \( S_{i+1} \). We need the following notations.
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\[ P_i(x_i) = y_i, \quad P_i(x_{i+1}) = y_{i+1}, \quad P_i'(x_i) = D_i, \quad P_i''(x_i) = S_i, \quad P_i'''(x_{i+1}) = S_{i+1}. \]  

From algebraic manipulation of equation (3) and using the notation of (4) is following that:

\[ a_i = h^3 \left( \frac{S_{i+1} - S_i \cos(\theta)}{\theta^3 \sin(\theta)} \right), \quad b_i = -h^3 \frac{S_i}{\theta^3}, \]
\[ c_i = \frac{(y_{i+1} - y_i)}{h^2} + \frac{h(S_{i+1} + S_i)(1 - \cos(\theta)) - D_i - hS_i}{h^2 \theta^3}, \]
\[ d_i = D_i + h^2 \frac{S_i}{\theta^3}, \quad e_i = y_i - h^3 \left( \frac{S_{i+1} - S_i \cos(\theta)}{\theta^3 \sin(\theta)} \right), \]

Where \( \theta = kh, \ i = 0,1,...,n - 1. \)

Using the continuity conditions of the first and second derivatives at the point \( (x_i, y_i) \), that is \( P_{(\mu)}(x_i) = P_{(\mu)}(x_{i+1}) \), \( \mu = 1,2 \) we get the following consistency relations for \( i = 0,1,...,n, \)

\[ D_i + D_{i-1} = \frac{2}{h} (y_i - y_{i-1}) + h^2 \left( -\frac{1}{\theta^2} + \frac{2(1 - \cos(\theta))}{\theta^3 \sin(\theta)} \right) (S_{i-1} + S_i) \]  
(5)

\[ D_i - D_{i-1} = \frac{1}{h} (y_{i-1} - 2y_i + y_{i+1}) - h^2 \left( \frac{1}{2\theta \sin(\theta)} + \frac{(1 - \cos(\theta))}{\theta^3 \sin(\theta)} \frac{1}{\theta} \right) S_{i-1} + h^2 \left( \frac{\cos(\theta)}{\theta \sin(\theta)} - \frac{1}{\theta} \right) S_i \]  
(6)

Adding equations (5) and (6) we get

\[ D_i = \frac{1}{2h} (y_{i+1} - y_{i-1}) + h^2 \left( \frac{1 - \cos(\theta)}{2\theta^3 \sin(\theta)} - \frac{1}{4\theta \sin(\theta)} \right) (S_{i-1} + S_{i+1}) \]
\[ + h^2 \left( -\frac{1}{\theta^2} + \frac{2(1 - \cos(\theta))}{2\theta^3 \sin(\theta)} + \frac{\cos(\theta)}{2\theta \sin(\theta)} \right) S_i \]  
(7)

We replace \( i \) by \( i - 1 \) in equation (7) we get the following


\[ D_{i-1} = \frac{1}{2h} (y_i - y_{i-2}) + h^2 \left( \frac{1 - \cos(\theta)}{2\theta^3 \sin(\theta)} - \frac{1}{4\theta \sin(\theta)} \right) (S_{i-2} + S_i) + h^2 \left( \frac{-1}{\theta^2} + \frac{2(1 - \cos(\theta))}{2\theta^3 \sin(\theta)} + \frac{\cos(\theta)}{2\theta \sin(\theta)} \right) S_{i-1} \]  

(8)

substituting equations (7) and (8) into equation (5) we get the following

\[-y_{i-2} + 3y_{i-1} - 3y_i + y_{i+1} = h^3 \left( \frac{\cos(\theta) - 1}{\theta^3 \sin(\theta)} + \frac{1}{2\theta \sin(\theta)} \right) S_{i-2} + h^3 \left( \frac{1 - \cos(\theta)}{\theta^3 \sin(\theta)} - \frac{\cos(\theta)}{2\theta \sin(\theta)} + \frac{1}{2\theta \sin(\theta)} \right) S_{i-1} + h^3 \left( \frac{\cos(\theta) - 1}{\theta^3 \sin(\theta)} + \frac{1}{2\theta \sin(\theta)} \right) S_{i+1} \]

for \( i = 2, 3, \ldots, n - 1, \)  

(9)

For simplicity, equation (9) can be re-written in the following form:

\[-y_{i-2} + 3y_{i-1} - 3y_i + y_{i+1} = h^3 \left[ \alpha (S_{i-2} + S_{i+1}) + \beta (S_{i-1} + S_i) \right] \]

(10)

where

\[ \alpha = \left( \frac{\cos(\theta) - 1}{\theta^3 \sin(\theta)} + \frac{1}{2\theta \sin(\theta)} \right), \quad \beta = \left( \frac{1 - \cos(\theta)}{\theta^3 \sin(\theta)} - \frac{\cos(\theta)}{2\theta \sin(\theta)} + \frac{1}{2\theta \sin(\theta)} \right). \]

Subtracted equation (5) from (6), we obtain that

\[ 2hD_{i-1} = -3y_{i-1} + 4y_i - y_{i+1} + h^3 \left[ -\frac{2}{\theta^2} + \frac{3(1 - \cos(\theta))}{\theta^3 \sin(\theta)} + \frac{1}{2\theta \sin(\theta)} \right] S_{i-1} + h^3 \left[ \frac{2(1 - \cos(\theta))}{\theta^3 \sin(\theta)} - \frac{\cos(\theta)}{\theta \sin(\theta)} \right] S_i + h^3 \left[ \frac{-1 - \cos(\theta)}{\theta^3 \sin(\theta)} - \frac{1}{2\theta \sin(\theta)} \right] S_{i+1} \]

(11)
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If \( i = 1 \) we can be re-written the equation (11) in the following form:

\[
3y_0 - 4y_1 + y_2 = -2hD_0 + h^3 \left[ -\frac{2}{\theta^2} + \frac{3(1 - \cos(\theta))}{\theta^3 \sin(\theta)} + \frac{1}{2\theta \sin(\theta)} \right] S_0 \\
+ h^3 \left[ \frac{2(1 - \cos(\theta))}{\theta^3 \sin(\theta)} - \frac{\cos(\theta)}{\theta \sin(\theta)} \right] S_1 + h^3 \left[ \frac{(1 - \cos(\theta))}{\theta^3 \sin(\theta)} - \frac{1}{2\theta \sin(\theta)} \right] S_2
\]

for simplicity, equation (12) can be re-written in the following form:

\[
3y_0 - 4y_1 + y_2 = -2hD_0 + h^3 \left[ w_1 S_0 + w_2 S_1 + w_3 S_2 \right]
\]

where,

\[
w_1 = \left( -\frac{2}{\theta^2} + \frac{3(1 - \cos(\theta))}{\theta^3 \sin(\theta)} + \frac{1}{2\theta \sin(\theta)} \right), \quad w_2 = \left( \frac{2(1 - \cos(\theta))}{\theta^3 \sin(\theta)} - \frac{\cos(\theta)}{\theta \sin(\theta)} \right),
\]

and \( w_3 = \left( \frac{(1 - \cos(\theta))}{\theta^3 \sin(\theta)} - \frac{1}{2\theta \sin(\theta)} \right) \).

If \( k \to 0 \) in equation (13) which is given by,

\[
3y_0 - 4y_1 + y_2 = -2hD_0 - \frac{h^3}{24} (5S_0 + 10S_1 + S_2).
\]

The local truncation errors \( t_i \), \( i = 1, 2, ..., n \), in equation (10) and (13) can be obtained as follows:

First we re-write the equation (10) and (13) in the form,

\[
-y_{i-2} + 3y_{i-1} - 3y_i + y_{i+1} = h^3 \left[ \alpha (y_{i-2}^{(3)} + y_{i+1}^{(3)}) + \beta (y_{i-1}^{(3)} + y_i^{(3)}) \right]
\]

\[
3y_0 - 4y_1 + y_2 = -2hD_0 + h^3 \left[ w_1 y_0^{(3)} + w_2 y_1^{(3)} + w_3 y_2^{(3)} \right]
\]

The terms \( y_{i-2}^{(3)} \), \( y_{i-1}^{(3)} \), \( y_i^{(3)} \) and \( y_{i+1}^{(3)} \) in equation (15) are expanded around the point \( x_i \) using Taylor series and the expressions for \( t_i \), \( i = 2, ..., n \) can be obtained,
\[ t_i = \left[ 1 - \left( 2\alpha - 2\beta \right) \right] h^3 y_i^{(3)} + \left[ \left( \alpha + \beta \right) - \frac{1}{2} \right] h^4 y_i^{(4)} + \left[ \frac{1}{4} - \frac{5\alpha + \beta}{2} \right] h^5 y_i^{(5)} + \left[ -\frac{1}{12} + \left( \frac{7\alpha + \beta}{6} \right) \right] h^6 y_i^{(6)} + \left[ \frac{1}{40} - \left( \frac{17\alpha + \beta}{24} \right) \right] h^7 y_i^{(7)} + O(h^8) \]  

(17)

the local truncation error corresponding to (16) is given by

\[ t_i = \left[ -\frac{2}{3} + \left( w_1 + w_2 + w_3 \right) \right] h^3 y_i^{(3)} + \left[ \frac{1}{6} \left( w_1 - w_3 \right) \right] h^4 y_i^{(4)} + \left[ -\frac{1}{15} + \frac{1}{2} \left( w_1 + w_3 \right) \right] h^5 y_i^{(5)} + \left[ \frac{1}{90} - \frac{1}{6} \left( w_1 - w_3 \right) \right] h^6 y_i^{(6)} + \left[ -\frac{1}{420} + \frac{1}{24} \left( w_1 + w_3 \right) \right] h^7 y_i^{(7)} + O(h^8) \]  

(18)

Second order method:

For arbitrary value of \( \alpha \) and \( \beta \) along with \( \alpha + \beta = \frac{1}{2}, \alpha \neq 0 \) and, then the local truncation error given by equation (17) and (18) are

\[ t_i = \begin{cases} 
-2\alpha h^3 y_i^{(3)} + O(h^6) & i = 2, \ldots, n, \\
-\frac{7}{120} h^3 y_i^{(3)} + O(h^6) & i = 1.
\end{cases} \]  

(19)

Third order method:

For \( \alpha = 0 \), \( \beta = \frac{1}{2} \) and \( (w_1, w_2, w_3) = \left( \frac{3}{20}, \frac{8}{15}, \frac{1}{60} \right) \),

then the local truncation error given by equation(17) and (18) are

\[ t_i = \begin{cases} 
\frac{1}{240} h^7 y_i^{(7)} + O(h^8), & i = 2, \ldots, n, \\
\frac{1}{60} h^6 y_i^{(6)} + O(h^7), & i = 1.
\end{cases} \]  

(20)
3. Analysis of the Method

To illustrate the application of the non-polynomial spline method developed in the previous section we consider the linear boundary value problem that is given in equation (1) at grid point \((x_i, y_i)\).

We choose an integer \(n > 0\) and divide the interval \([a, b]\) into \(n\) equal subintervals whose end point are the mesh points \(x_i = a + ih\) for \(i = 1, 2, ..., n - 1\), where the step size \(h = \frac{b - a}{n}\), also approximate the third derivative that occurs in equation (1) by using non-polynomial spline i.e. substituting \(S_i = y''\) in equation (1), we get the following equations.

\[
S_{i-2} = p_2(x_{i-2})y'''_{i-2} + p_1(x_{i-2})y''_{i-2} + p_0(x_{i-2})y'_{i-2} + r(x_{i-2}),
\]

\[
S_{i-1} = p_2(x_{i-1})y'''_{i-1} + p_1(x_{i-1})y''_{i-1} + p_0(x_{i-1})y'_{i-1} + r(x_{i-1}),
\]

\[
S_i = p_2(x_i)y'''_i + p_1(x_i)y''_i + p_0(x_i)y'_i + r(x_i),
\]

\[
S_{i+1} = p_2(x_{i+1})y'''_{i+1} + p_1(x_{i+1})y''_{i+1} + p_0(x_{i+1})y'_{i+1} + r(x_{i+1})
\]

the first and second-order derivative will be approximated by using the FDM thus we have,

\[
\begin{align*}
y'_{i+1} = & \frac{y_{i+1} - y_i}{2h}, \\
y''_{i+1} = & \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, \\
y'''_{i+1} = & \frac{y_{i+1} - 6y_i + 12y_{i-1} - 6y_{i-2} + y_{i-3}}{2h^3}, \\
y'_{i-1} = & \frac{y_{i-1} - 4y_i + 6y_{i+1} - 4y_{i+2} + y_{i+3}}{2h}, \\
y''_{i-1} = & \frac{-3y_{i-1} + 4y_i - y_{i+1}}{2h}, \\
y'''_{i-1} = & \frac{-5y_{i-1} + 8y_i - 3y_{i+1}}{2h}, \\
y'_{i-2} = & \frac{y_{i-2} - 6y_{i-1} + 14y_i - 6y_{i+1} + y_{i+2}}{2h^2}, \\
y''_{i-2} = & \frac{-y_{i-2} + 6y_{i-1} - 11y_i + 6y_{i+1} - y_{i+2}}{2h^2}, \\
y'''_{i-2} = & \frac{-y_{i-2} + 6y_{i-1} - 11y_i + 8y_{i+1} - 2y_{i+2}}{h^2},
\end{align*}
\]
so equation (21)–(24) and using (25) we get

\[ S_{i-2} = \frac{p_2(x_{i-2})(-y_{i-3} + 6y_{i-2} - 11y_{i-1} + 8y_i - 2y_{i+1})}{h^2} + \frac{p_1(x_{i-2})(-5y_{i-3} + 8y_{i-2} - 3y_{i-1})}{2h} + p_0(x_{i-2})y_{i-2} + r(x_{i-2}), \]  

(26)

\[ S_{i-1} = \frac{p_2(x_{i-1})(-y_{i-3} + 6y_{i-2} - 10y_{i-1} + 6y_i - y_{i+1})}{2h^2} + \frac{p_1(x_{i-1})(-3y_{i-3} + 4y_{i-2} - y_{i-1})}{2h} + p_0(x_{i-1})y_{i-1} + r(x_{i-1}), \]  

(27)

\[ S_i = \frac{p_2(x_i)(y_{i-2} - 2y_i + y_{i+1}) + p_1(x_i)(y_{i+1} - y_{i-1})}{h^2} + \frac{p_0(x_i)y_i + r(x_i)}{2h}, \]  

(28)

\[ S_{i+1} = \frac{p_2(x_{i+1})(y_{i-3} - 6y_{i-2} + 14y_{i-1} - 14y_i + 5y_{i+1})}{2h^2} + \frac{p_1(x_{i+1})(y_{i+1} - 4y_i + 3y_{i+1})}{2h} + p_0(x_{i+1})y_{i+1} + r(x_{i+1}), \]  

(29)

now substituting the equations (26)–(29) in equation (10) we get the difference equation,

\[
\begin{bmatrix}
\alpha h \left( \frac{p_1(x_{i+1})}{2} - p_2(x_{i+2}) \right) - \beta h p_2(x_{i+1}) \\
+ [1 + \alpha h (-3p_1(x_{i+1}) + 6p_2(x_{i+2})) + 3h \beta p_1(x_{i+1}) + \alpha h' p_0(x_{i+2})] y_{i+2} \\
-3 + \alpha h (-1p_1(x_{i+2}) + 7p_2(x_{i+1})) + \beta h (-5p_2(x_{i+1}) + p_1(x_{i+2})) \\
+ \alpha h^2 \left( -\frac{5}{2} p_1(x_{i+2}) + \frac{p_2(x_{i+1})}{2} \right) + \beta h^2 \left( -\frac{3p_1(x_{i+2})}{2} - \frac{p_2(x_{i+1})}{2} \right) y_{i+1} \\
+ \beta h p_0(x_{i+2}) \\
+ 3 + \alpha h (8p_1(x_{i+2}) - 7p_2(x_{i+1})) + \beta h (3p_2(x_{i+1}) - 2p_2(x_{i+2})) \\
+ \alpha h^2 (4p_1(x_{i+2}) - 2p_1(x_{i+1})) + 2\beta h^2 p_1(x_{i+1}) + \beta h^2 p_0(x_{i+2}) y_i \\
-1 + \alpha h (-2p_1(x_{i+2}) + 5p_2(x_{i+1})) + \beta h \left( -\frac{p_2(x_{i+1})}{2} + p_1(x_{i+2}) \right) \\
+ \alpha h^2 \left( -\frac{3p_1(x_{i+2})}{2} + \frac{3p_2(x_{i+1})}{2} \right) + \beta h^2 \left( -\frac{p_1(x_{i+2})}{2} + \frac{p_2(x_{i+1})}{2} \right) y_{i+1} \\
+ \alpha h' p_0(x_{i+2}) \\
\end{bmatrix} \\
= -\alpha h' (r(x_{i+2}) + r(x_{i+1})) - \beta h' (r(x_{i+1}) + r(x_i))
\]

for \( i = 3, 4, \ldots, n - 1, \)  

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However, this formula is suitable for $n - 3$ mesh points, so we need to derive special formula when $i = 1, 2$ using the boundary conditions.

First, if $i = 1$ we must use the following approximation for the first and second-order derivative,

$$
\begin{align*}
&y'_0 = \frac{y_1 - y_{-1}}{2h}, \quad y''_0 = \frac{y_1 - 2y_0 + y_{-1}}{h^2}, \quad y'_1 = \frac{y_2 - y_0}{2h}, \\
&y'_i = \frac{y_{i+2} - 2y_{i+1} + y_i}{h^2}, \quad y''_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{2h}, \quad y''_i = \frac{y_{i+2} - 2y_{i+1} + y_i}{h^2},
\end{align*}
$$

(31)

We can find $y_{-1}$ by using the first condition, then becomes $y_{-1} = y_1 - 2hy'_0$. Therefore from equation (31) and using equations (22), (23) and (24) we get the following equations,

$$
\begin{align*}
S_0 &= p_2(x_0)(2y_1 - 2y_0 - 2hy'_0) + p_1(x_0)y'_0 + p_0(x_0)y_0 + r(x_0), \\
S_1 &= p_2(x_1)(y_{i+2} - 2y_{i+1} + y_i) + p_1(x_1)(y_{i+1} - 2y_i + y_{i-1}) + p_0(x_1)y_i + r(x_1), \\
S_2 &= p_2(x_2)(y_3 - 2y_2 + y_1) + p_1(x_2)(y_3 - y_1) + p_0(x_2)y_2 + r(x_2),
\end{align*}
$$

(32)

Now substituting (32) in equation (13) we get the following linear algebraic equation.

$$
\begin{align*}
&\left[ h \left( 2w_1p_2(x_0) - 2w_2p_2(x_1) + w_3p_2(x_2) \right) - \frac{w_1h^2p_1(x_2)}{2} + \frac{w_3h^2p_0(x_1)}{1} + 4 \right] y_1 \\
+ &\left[ h \left( w_3p_2(x_1) - 2w_3p_2(x_2) + h^2w_2p_1(x_1) - h^2w_3p_0(x_2) - 1 \right) \right] y_2 \\
+ &\left[ hw_3p_2(x_2) + \frac{h^2w_3p_1(x_2)}{2} \right] y_3
\end{align*}
$$

(33)

Second, to find the formula at $i = 2$ from equations (26)-(29) and using first condition ($y'_0 = y'_2$) we get the following equation.
The scheme (30) along with the equations (33) and (34) gives rise to a linear system of order \((n - 1) \times (n - 1)\) and may be written in the matrix form as

\[ AY + h^3DR = G \]  

Where \(A = N + hBp_x + h^2Bp + h^3Bp_o\), and matrix \(N\) has the following form,

\[
N = \begin{bmatrix}
4 & -1 & 0 & & & & \\
-3 & 3 & -1 & 0 & & & \\
1 & -3 & 3 & -1 & 0 & & \\
0 & 1 & -3 & 3 & -1 & 0 & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
0 & 1 & -3 & 3 & -1 & 0 & \\
0 & 1 & -3 & 3 & -1 & 0 & \\
0 & 1 & -3 & 3 & & & \\
\end{bmatrix}
\]

\[ Bp_x = Z, \quad Bp = U, \quad Bp_o = V \quad \text{where} \]

\[
\begin{align*}
-3 + ah\left(\frac{15p_1(x_3)}{2} - 12p_2(x_0)\right) + \beta h \left(\frac{11p_2(x_1)}{2} - p_2(x_3)\right) & + ah^2 \left(-\frac{5p_1(x_3)}{2} + p_1(x_1)\right) + \beta h^2 \left(-\frac{3p_1(x_1)}{2} - p_1(x_3)\right) + \beta h^3 p_o(x_1) \\
3 + ah\left(8p_2(x_0) - 7p_2(x_1)\right) + \beta h \left(3p_2(x_1) - 2p_2(x_2)\right) & + ah^2 \left(4p_1(x_0) - 2p_1(x_1)\right) + 2\beta h^2 p_1(x_1) + \beta h^3 p_o(x_2) \\
-1 + ah\left(-2p_2(x_0) + 5p_2(x_1)\right) + \beta h \left(-\frac{p_2(x_1)}{2} + p_2(x_2)\right) & + ah^2 \left(-\frac{3p_1(x_0)}{2} + 3p_1(x_1)\right) + \beta h^2 \left(-\frac{p_1(x_1)}{2} + p_1(x_2)\right) \\
+ ah^3 p_o(x_3) & = \left[-\beta h \left(6\alpha p_2(x_0) + 3\beta p_2(x_1) - 3\alpha p_2(x_3)\right) - \alpha h^3 p_o(x_0) - 1\right] y_0 \\
- h^2 \left[2\alpha p_2(x_0) - \alpha p_2(x_1) + \beta p_2(x_1)\right] y'_0 - \alpha h^3 \left(r(x_0) + r(x_3)\right) - \beta h^3 \left(r(x_1) + r(x_2)\right)
\end{align*}
\]  

(34)
Numerical solution of third order BVPs

\[
\begin{align*}
2w_i p_2(x_i) - 2w_j p_2(x_j) + w_i p_2(x_j) &= \quad i = j = 1, \\
0.5 w_i p_2(x_i) - 2w_j p_2(x_j) &= \quad i = 1, j = 2, \\
w_i p_2(x_j) &= \quad i = 1, j = 3, \\
\alpha \left( \frac{15}{2} p_2(x_j) - 12 p_2(x_i) \right) + \beta \left( p_2(x_j) - \frac{11}{2} p_2(x_i) \right) &= \quad i = 2, j = 1, \\
Z_{ij} = \begin{cases} 
\alpha \left( -p_2(x_{i-2}) + \frac{1}{2} p_2(x_{i-1}) \right) - \frac{1}{2} \beta p_2(x_{i-1}) & j = i - 3, i = 4, \ldots, n - 1, \\
\alpha \left( -5p_2(x_{i-1}) + 6p_2(x_{i-2}) \right) - 3\beta p_2(x_{i-1}) & j = i - 2, i = 3, \ldots, n - 1, \\
\alpha \left( -11p_2(x_{i-2}) + 7p_2(x_{i-1}) \right) + \beta \left( p_2(x_i) - 5p_2(x_{i-1}) \right) & j = i - 1, i = 3, \ldots, n - 1, \\
\alpha \left( -7p_2(x_{i-2}) + 8p_2(x_{i-1}) \right) + \beta \left( -2p_2(x_i) + 3p_2(x_{i-1}) \right) & j = i, i = 2, \ldots, n - 1, \\
\alpha \left( \frac{5}{2} p_2(x_{i-2}) - 2p_2(x_{i-1}) \right) + \beta \left( -\frac{1}{2} p_2(x_{i-1}) + p_2(x_i) \right) & j = i + 1, i = 2, \ldots, n - 2. 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
-\frac{1}{2} w_i p_1(x_i) &= \quad i = 1, j = 1, \\
\frac{1}{2} w_i p_1(x_i) &= \quad i = 1, j = 2, \\
\frac{1}{2} w_i p_1(x_i) &= \quad i = 1, j = 3, \\
U_{ij} = \begin{cases} 
\alpha \left( \frac{1}{2} p_1(x_{i-1}) - \frac{5}{2} p_1(x_{i-2}) \right) + \beta \left( -\frac{3}{2} p_1(x_{i-2}) - \frac{1}{2} p_1(x_i) \right) & j = i - 1, i = 2, \ldots, n - 1, \\
\alpha \left( 4p_1(x_{i-2}) - 2p_1(x_{i-1}) \right) + 2\beta p_1(x_{i-1}) & j = i, i = 2, \ldots, n - 1, \\
\alpha \left( \frac{3}{2} p_1(x_{i-1}) - \frac{3}{2} p_1(x_{i-2}) \right) + \beta \left( \frac{1}{2} p_1(x_i) - \frac{1}{2} p_1(x_{i-1}) \right) & j = i + 1, i = 2, \ldots, n - 2, 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
w_2 p_0(x_1) &= \quad i = j = 1, \\
w_2 p_0(x_2) &= \quad i = 1, j = 2, \\
\alpha p_0(x_{i-2}) &= \quad j = i - 2, i = 3, \ldots, n - 1, \\
\beta p_0(x_{i-1}) &= \quad j = i - 1, i = 2, \ldots, n - 1, \\
\beta p_0(x_i) &= \quad i = j, i = 2, \ldots, n - 1, \\
\alpha p_0(x_{i+1}) &= \quad i = j + 1, i = 2, \ldots, n - 2, 
\end{align*}
\]

\[
V_{ij} = \begin{cases} 
\end{cases}
\]
\[
D = \begin{bmatrix}
    w_2 & w_3 & & \\
    \beta & \beta & \alpha & \\
    \alpha & \beta & \alpha & \\
    \alpha & \beta & \alpha & \\
    \alpha & \beta & \alpha & \\
    \alpha & \beta & \beta & \alpha \\
    \alpha & \beta & \beta & \beta & \\
    \alpha & \beta & \beta & \beta & \\
    \alpha & \beta & \beta & \beta & \\
    \alpha & \beta & \beta & \beta & \\
    \alpha & \beta & \beta & \beta & \\
\end{bmatrix},
\]

\[ R = (r(x_1), r(x_2), ..., r(x_{n-1})) \]
\[ Y = (y_1, y_2, ..., y_{n-2}, y_{n-1})^T \]

and

\[ G = (g_1, g_2, g_3, 0, 0, ..., g_{n-1})^T \]

Where

\[ g_1 = \left[ h\left(2w_1p_2(x_0) - w_2p_2(x_1)\right) + h^2w_2\frac{p_1(x_1)}{2} - h^3w_1p_0(x_0) + 3\right]y_0 \]
\[ + \left[-2h + 2h^2w_1p_2(x_0) + h^3w_1p_1(x_o)\right]y_0' - w_1r(x_0), \]

\[ g_2 = \left[-h(6\alpha p_2(x_0) + 3\beta p_2(x_1) - 3\alpha p_2(x_1)) - \alpha h^3p_0(x_0) - 1\right]y_0 \]
\[ - h^2\left[\alpha p_2(x_0) - \alpha p_2(x_1) + \beta p_2(x_1)\right]y_0' - \alpha h^3r(x_0), \]

\[ g_3 = \left[-\alpha h\left(p_2(x_0)\frac{2}{2} - p_2(x_1)\frac{2}{2}\right) + \beta h p_2(x_0)\frac{2}{2}\right]y_0, \]

\[ g_i = 0, \text{ for } i = 4, ..., n - 2, \]

\[ g_{n-1} = \left[1 + ah\left(-2p_2(x_{n-3}) + \frac{5p_2(x_0)}{2}\right) - \beta h \left(-p_2(x_{n-2}) + p_2(x_{n-1})\right)\right]y_n \]
\[ - ah^2 \left(-\frac{3p_3(x_{n-3})}{2} + \frac{3p_3(x_1)}{2}\right) - \beta h^2 \left(-\frac{p_1(x_{n-3})}{2} + \frac{p_1(x_{n-1})}{2}\right) \]
\[ - ah^3p_n(x_n), \]

we assume that

\[ \bar{Y} = (y(x_1), y(x_2), ..., y(x_{n-1}))^T, \]

be the exact solution of the given BVP (1) at the points \( x_i = 0, 1, 2, ..., n - 1 \), and then we have
\[
A\bar{Y} + h^3DR = G + T \tag{36}
\]
If we subtract equation (35) from equation (36) we get the following
\[
A(\bar{Y} - Y) = AE = T \tag{37}
\]

5. Convergence Analysis

Our main purpose now is to derive a bound \(\|E\|_\infty\). We now turn back to the error equation in (37) and re-write it in the form,

\[
E = A^{-1}T = \left[N + hBp_2 + h^2Bp_1 + h^3Bp_0\right]^{-1}T
\]

Since \(N\) is nonsingular ([17]) we can re-write the matrix \(E\) in the form,

\[
E = \left[I + N^{-1}\left(hBp_2 + h^2Bp_1 + h^3Bp_0\right)\right]^{-1}N^{-1}T
\]

And using Cauchy Schwarz inequality we get

\[
\|E\|_\infty \leq \left\|I + N^{-1}\left(hBp_2 + h^2Bp_1 + h^3Bp_0\right)\right\|_\infty \|N^{-1}\|_\infty \|T\|_\infty, \tag{38}
\]

In order to derive the bound on \(\|E\|_\infty\), the following lemma is needed.

**Lemma 1**: The matrix \(A\) is nonsingular if

\[
\|p_2\|_\infty \leq \frac{h^2\lambda_1}{\omega(48\alpha + 16\beta)},
\]

\[
\|p_1\|_\infty \leq \frac{h\lambda_2}{\omega(12\alpha + 5\beta)} \quad \text{and} \quad \|p_0\|_\infty \leq \frac{\lambda_3}{\omega} \quad \text{where} \quad \omega = \frac{2}{81}(b-a)^3\left[1 + \frac{3h^2}{2(b-a)^2}\right],
\]

\(\lambda_1, \lambda_2, \lambda_3 < 1\) and \(\lambda_1 + \lambda_2 + \lambda_3 = 1\).

**Proof:**

Since

\[
A = \left(N + hBp_2 + h^2Bp_1 + h^3Bp_0\right) = \left[I + N^{-1}\left(hBp_2 + h^2Bp_1 + h^3Bp_0\right)\right]N
\]

and the matrix \(N\) is nonsingular, so to prove \(A\) nonsingular it is sufficient to show that \(\left[I + N^{-1}\left(hBp_2 + h^2Bp_1 + h^3Bp_0\right)\right]\) is nonsingular, by using Cauchy
Schwarz and triangle inequalities we get the following,

\[
\left\| N^{-1} \left( hBP_2 + h^2BP_1 + h^3BP_0 \right) \right\|_\infty \leq \left\| N^{-1} \left( hBP_2 + h^2BP_1 + h^3BP_0 \right) \right\|_\infty \\
\leq \left\| N^{-1} \right\|_\infty \left[ h \left\| BP_2 \right\|_\infty + h^2 \left\| BP_1 \right\|_\infty + h^3 \left\| BP_0 \right\|_\infty \right] \quad (39)
\]

Moreover,

\[
\left\| N^{-1} \right\|_\infty \leq \frac{1}{81}(n) \left[ 2(n-1)^2 + 4(n-1) + 5 \right] = \frac{2}{81} \left[ n + \frac{3}{2n^2} \right] = \frac{2}{81} \left[ b - a \right] \left[ 1 + \frac{3h^2}{2(b-a)^2} \right] h^3 \quad , [7]
\]

\[
\left\| BP_2 \right\|_\infty = (48\alpha + 16\beta) \left\| p_2 \right\|_\infty , \left\| BP_1 \right\|_\infty = (12\alpha + 5\beta) \left\| p_1 \right\|_\infty \\
\left\| BP_0 \right\|_\infty = 2(\alpha + \beta) \left\| p_0 \right\|_\infty 
\]

since \( \alpha + \beta = \frac{1}{2} \) we get \( \left\| BP_0 \right\|_\infty = \left\| p_0 \right\|_\infty \).

Where

\[
\left\| p_2 \right\|_\infty = \max_{a \leq x_i \leq b} \left| p_2(x_i) \right| , \left\| p_1 \right\|_\infty = \max_{a \leq x_i \leq b} \left| p_1(x_i) \right| \quad \text{and} \quad \left\| p_0 \right\|_\infty = \max_{a \leq x_i \leq b} \left| p_0(x_i) \right|.
\]

Therefore, substituting \( \left\| N \right\|_\infty , \left\| BP_2 \right\|_\infty , \left\| BP_1 \right\|_\infty \) and \( \left\| BP_0 \right\|_\infty \) in equation (39) we get the following equation,

\[
\left\| N^{-1} \left( hBP_2 + h^2BP_1 + h^3BP_0 \right) \right\|_\infty \leq \frac{2}{81h^3 \left( b - a \right)} \left[ 1 + \frac{3h^2}{2(b-a)^2} \right] \left[ h \left( 48\alpha + 16\beta \right) \left\| p_2 \right\|_\infty \\
+ h^2 \left( 12\alpha + 5\beta \right) \left\| p_1 \right\|_\infty + h^3 \left\| p_0 \right\|_\infty \right] , \quad (40)
\]

since,

\[
\left\{ \left\| p_2 \right\|_\infty \leq \frac{h^2\lambda_1}{\omega (48\alpha + 16\beta)} , \left\| p_1 \right\|_\infty \leq \frac{h\lambda_2}{\omega (12\alpha + 5\beta)} , \left\| p_0 \right\|_\infty \leq \frac{\lambda_3}{\omega} \right. \right. \quad (41)
\]

Therefore equations (40)-(41) lead to \( \left\| N^{-1} \left( hBP_2 + h^2BP_1 + h^3BP_0 \right) \right\|_\infty \leq 1, \)
so that the matrix $A$ is nonsingular, since,

$$\left\| N^{-1} \left( hBp_2 + h^2Bp_1 + h^3Bp_0 \right) \right\|_\infty \leq 1$$

so by using equation (38) follow that,

$$\| E \|_\infty \leq \frac{\| N^{-1} \|_\infty \| F \|_\infty}{1 - \| N^{-1} \|_\infty \| (hBp_2 + h^2Bp_1 + h^3Bp_0) \|_\infty}$$

from equation (19) we have

$$\| F \| = 2ch^5M_5, \; M_5 = \max_{a \leq x \leq b} \left| y^{(5)}(x) \right|, \; \text{then},$$

$$\| E \|_\infty \leq \frac{\| N^{-1} \|_\infty \| F \|_\infty}{1 - \| N^{-1} \|_\infty \| (hBp_2 + h^2Bp_1 + h^3Bp_0) \|_\infty} \approx O(h^2), \; (42)$$

Also from equation (20) we have

$$\| F \| = \frac{1}{60}M_6, \; M_6 = \max_{a \leq x \leq b} \left| y^{(6)}(x) \right|, \; \text{then},$$

$$\| E \|_\infty \leq \frac{\| N^{-1} \|_\infty \| F \|_\infty}{1 - \| N^{-1} \|_\infty \| (hBp_2 + h^2Bp_1 + h^3Bp_0) \|_\infty} \approx O(h^3). \; (43)$$

**Theorem 1:**

Let $y(x)$ be the exact solution of the continuous BVP (1) with the boundary condition (2) and let $y(x_i), \; i = 1, 2, ..., n - 1$, satisfies the discrete BVP (35). Further, if $e_i = y(x_i) - y_i$ then

1. $\| E \|_\infty \approx O(h^2)$ for second order convergent method.
2. $\| E \|_\infty \approx O(h^3)$ for third order convergent method.

**6. Numerical Examples**

In this section we will illustrate the numerical techniques discussed in the previous sections by the following two BVPs of equation (1) and the boundary conditions (2), in order to illustrate the comparative performance of the proposed method (12) over other existing methods. All calculations are implemented by using Maple 13.
Example 1: Consider the linear third-order BVP:[3]

\[ y''' = 2x^2y'' - 3xy' - 5x^2y + e^{2x} \left( 3x^3 - x^2 - 5x - 4 \right) \]

With the boundary conditions

\[ y(0) = 1, \quad y'(0) = 1, \quad y(1) = 0 \]

where the exact solution is given by, \( y(x) = (1 - x) e^{2x} \),

the numerical results of the example(1) are presented in the Table (1) for different values of \( n \). Table(2) comparison of the error proposed method with FDM [3]. Figure(1) shows the comparison of the exact and numerical solutions for choosing \( h = 0.05 \).

Example 2: Consider the linear third-order BVP:[10].

\[ y''' - xy = \left( x^3 - 2x^2 - 5x - 3 \right) e^x \quad x \in [0, 1] \]

With the boundary conditions

\[ y(0) = 0, \quad y'(0) = 1, \quad y(1) = 0 \]

where the exact solution is given by, \( y(x) = (x - x^2) e^x \),

the numerical result of the example(2) are present in the Table (3) for different values of \( n \). Table(4) comparison of the error proposed method with FDM[10]. Figure (2) shows the comparison of the exact and numerical solutions for choosing \( h = \frac{1}{32} \).

The numerical result for our second and third order methods are summarized in Tables (5)-(6).
Table 1: The numerical solutions and exact solution of example (1) at different values of \( n \).

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>( n = 10 )</th>
<th>( n = 20 )</th>
<th>( n = 40 )</th>
<th>( n = 80 )</th>
<th>Exact solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.099248322</td>
<td>1.099257993</td>
<td>1.099261195</td>
<td>1.099262286</td>
<td>1.099262482</td>
</tr>
<tr>
<td>0.2</td>
<td>1.193407021</td>
<td>1.193441938</td>
<td>1.193454591</td>
<td>1.193458880</td>
<td>1.193459758</td>
</tr>
<tr>
<td>0.3</td>
<td>1.275368665</td>
<td>1.275443150</td>
<td>1.275471500</td>
<td>1.275481056</td>
<td>1.275483160</td>
</tr>
<tr>
<td>0.4</td>
<td>1.335125541</td>
<td>1.335253651</td>
<td>1.335303829</td>
<td>1.335320609</td>
<td>1.335324557</td>
</tr>
<tr>
<td>0.5</td>
<td>1.358836867</td>
<td>1.359031508</td>
<td>1.359108892</td>
<td>1.359134492</td>
<td>1.359140914</td>
</tr>
<tr>
<td>0.6</td>
<td>1.327625366</td>
<td>1.327894728</td>
<td>1.328002288</td>
<td>1.328037353</td>
<td>1.328046769</td>
</tr>
<tr>
<td>0.7</td>
<td>1.216030761</td>
<td>1.216369627</td>
<td>1.216504405</td>
<td>1.216547698</td>
<td>1.216559990</td>
</tr>
<tr>
<td>0.8</td>
<td>0.9900284107</td>
<td>0.9904000744</td>
<td>0.9905464005</td>
<td>0.9905927146</td>
<td>0.9906064848</td>
</tr>
<tr>
<td>0.9</td>
<td>0.6044981074</td>
<td>0.6047998414</td>
<td>0.6049169321</td>
<td>0.6049534588</td>
<td>0.6049647464</td>
</tr>
</tbody>
</table>

Table 2: Comparison of the error proposed method with FDM [9] for example(1).

<table>
<thead>
<tr>
<th>( n )</th>
<th>The proposed method</th>
<th>FDM [4]</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5.780741 \times 10^{-4}</td>
<td>7.7 \times 10^{-3}</td>
</tr>
<tr>
<td>20</td>
<td>2.064104 \times 10^{-5}</td>
<td>2.4 \times 10^{-3}</td>
</tr>
<tr>
<td>40</td>
<td>6.00843 \times 10^{-5}</td>
<td>6.4415 \times 10^{-4}</td>
</tr>
<tr>
<td>80</td>
<td>1.37702 \times 10^{-5}</td>
<td>1.6704 \times 10^{-4}</td>
</tr>
</tbody>
</table>
Table 3: The numerical solutions and exact solution of example (2) at different values of \( n \).

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>( n = 8 )</th>
<th>( n = 16 )</th>
<th>( n = 32 )</th>
<th>( n = 64 )</th>
<th>Exact solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1/8 )</td>
<td>0.1239372396</td>
<td>0.1239380823</td>
<td>0.1239381109</td>
<td>0.1239381116</td>
<td>0.1239381120</td>
</tr>
<tr>
<td>( 2/8 )</td>
<td>0.2407530455</td>
<td>0.2407547006</td>
<td>0.2407547628</td>
<td>0.2407547640</td>
<td>0.2407547657</td>
</tr>
<tr>
<td>( 3/8 )</td>
<td>0.3410111596</td>
<td>0.3410135124</td>
<td>0.3410136077</td>
<td>0.3410136125</td>
<td>0.3410136129</td>
</tr>
<tr>
<td>( 4/8 )</td>
<td>0.4121773507</td>
<td>0.4121801886</td>
<td>0.4121803112</td>
<td>0.4121803159</td>
<td>0.4121803178</td>
</tr>
<tr>
<td>( 5/8 )</td>
<td>0.4378670084</td>
<td>0.4378700027</td>
<td>0.4378701387</td>
<td>0.4378701455</td>
<td>0.4378701462</td>
</tr>
<tr>
<td>( 6/8 )</td>
<td>0.3969346831</td>
<td>0.3969373688</td>
<td>0.3969374960</td>
<td>0.3969375008</td>
<td>0.3969375032</td>
</tr>
<tr>
<td>( 7/8 )</td>
<td>0.2623751433</td>
<td>0.2623768944</td>
<td>0.2623769803</td>
<td>0.2623769854</td>
<td>0.2623769853</td>
</tr>
</tbody>
</table>

Table 4: Comparison of the error proposed method with FDM [10] for example(2).

<table>
<thead>
<tr>
<th>( n )</th>
<th>The proposed method</th>
<th>FDM [28]</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>3.1378 ( \times 10^6 )</td>
<td>8.5 ( \times 10^5 )</td>
</tr>
<tr>
<td>16</td>
<td>1.435 ( \times 10^{-7} )</td>
<td>2.14 ( \times 10^{-2} )</td>
</tr>
<tr>
<td>32</td>
<td>7.7 ( \times 10^{-9} )</td>
<td>5.35 ( \times 10^{-3} )</td>
</tr>
<tr>
<td>64</td>
<td>2.4 ( \times 10^{-9} )</td>
<td>1.34 ( \times 10^{-4} )</td>
</tr>
</tbody>
</table>

Table 5: The observed maximum absolute error for example(1).

<table>
<thead>
<tr>
<th>( n )</th>
<th>Second order ( \alpha = \frac{1}{24}, \beta = \frac{11}{24} )</th>
<th>Third order ( \alpha = 0, \beta = \frac{1}{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>3.322303 ( \times 10^{-3} )</td>
<td>5.780741 ( \times 10^{-4} )</td>
</tr>
<tr>
<td>20</td>
<td>7.68942 ( \times 10^{-4} )</td>
<td>2.064104 ( \times 10^{-5} )</td>
</tr>
<tr>
<td>40</td>
<td>1.82619 ( \times 10^{-4} )</td>
<td>6.00843 ( \times 10^{-5} )</td>
</tr>
<tr>
<td>80</td>
<td>6.1183 ( \times 10^{-5} )</td>
<td>1.37702 ( \times 10^{-5} )</td>
</tr>
</tbody>
</table>
Table 6: The observed maximum absolute error for example(2).

<table>
<thead>
<tr>
<th>n</th>
<th>Second order $\alpha = \frac{1}{24}, \beta = \frac{11}{24}$</th>
<th>Third order $\alpha = 0, \beta = \frac{1}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>$9.382133 \times 10^{-4}$</td>
<td>$3.1378 \times 10^{-6}$</td>
</tr>
<tr>
<td>16</td>
<td>$2.398831 \times 10^{-4}$</td>
<td>$1.435 \times 10^{-7}$</td>
</tr>
<tr>
<td>32</td>
<td>$5.99442 \times 10^{-5}$</td>
<td>$7.7 \times 10^{-9}$</td>
</tr>
<tr>
<td>64</td>
<td>$1.50204 \times 10^{-5}$</td>
<td>$2.4 \times 10^{-9}$</td>
</tr>
</tbody>
</table>

Figure(1): Comparison of the exact and the proposed method of example(1) for $h = 0.05$.

Figure (2): Comparison of exact and the proposed method of example(2) for $h = \frac{1}{32}$. 
7. Conclusion

The non-polynomial spline method with finite difference is developed for the approximate solution of third order two–point BVPs in this paper. Two examples are considered for numerical illustration of the method. This method is shown to be second and third ordered convergent methods which are better than other methods. Numerical result are presented in tables (1) and (3) and compared with the exact solutions.

The obtained numerical results show that the proposed methods maintain a high accuracy which make them are very encouraging for dealing with the solution of this type of two point boundary value problems.

References


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