On a Boundary Value Problem for a Quasi-linear Elliptic Equation Degenerating into a Parabolic Equation in an Infinite Strip

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Abstract

In this paper we construct complete asymptotics on the small parameter of the solution of a singularly perturbed boundary value problem for a quasi-linear elliptic equation degenerating in an infinite strip into the parabolic equation, and the remainder term is estimated.

Keywords: Asymptotics, Boundary layer function, Remainder term

1. Introduction

While studying numerous real phenomena with non-uniform transitions from one physical characteristics to another ones, we have to investigate singularly perturbed boundary value problems. A lot of papers have been devoted to the asymptotics of the solution of different boundary value problems for nonlinear elliptic equations with a small parameter at higher derivatives. In a great number of papers on nonlinear singularly perturbed elliptic equations, the input equations degenerate for a zero value of the small parameter into functional equations (see. [1]-[4], [6]). Besides, in all these papers with the exception of the paper [4], the derivatives of the desired function enter linearly to the equation, only the desired function itself enters into the equation nonlinearly. All these and other problems known to us are considered only in finite domains.

In the present paper, in an infinite strip $\Pi = \{(x, y) | 0 \leq x \leq 1, -\infty < y < +\infty\}$ we consider the following boundary value problem

$$L_\varepsilon U \equiv -\varepsilon^p \left[\frac{\partial}{\partial x} \left(\frac{\partial U}{\partial x} \right)^p + \frac{\partial}{\partial y} \left(\frac{\partial U}{\partial y} \right)^p \right] - \varepsilon \Delta U + \frac{\partial U}{\partial x} - \frac{\partial^2 U}{\partial y^2} + aU - f(x, y) = 0, \quad (1)$$

$$U|_{x=0} = U|_{x=1} = 0, \quad (-\infty < y < +\infty); \lim_{|y| \rightarrow +\infty} U = 0, \quad (0 \leq x \leq 1), \quad (2)$$

where $\varepsilon > 0$ is a small parameter, $p = 2k + 1$, k is an arbitrary natural number, Δ is a Laplace operator, $a > 0$ is a constant, $f(x, y)$ is a given smooth function.

The goal of the paper is to construct the asymptotic expansion of the generalized solution of problem (1), (2) from the class $\overset{0}{W}_{p+1}(\Pi)$.

2. The first iterative process

In the first iterative process, we'll look for the approximate solution of equation (1) in the form

$$W = W_0 + \varepsilon W_1 + \dots + \varepsilon^n W_n, \quad (3)$$

and the functions $W_i(x, y)$ will be chosen so that

$$L_\varepsilon W = 0(\varepsilon^{n+1}). \quad (4)$$

Substituting (3) in (4), expanding the nonlinear terms in powers of ε and equating the terms with the same powers of ε , for determining the function $W_i; i = 0, 1, \dots, n$ we get the following recurrently connected equations:

$$\frac{\partial W_i}{\partial x} - \frac{\partial^2 W_i}{\partial y^2} + aW_i = f_i(x, y), \quad i = 0, 1, \dots, n, \quad (5)$$

where $f_0 = f(x, y)$, $f_j(x, y) = \Delta W_{j-1}$ for $j = 1, 2, \dots, 2k$; $f_s(x, y) = \Delta W_{s-1} + g_s(W_0, W_1, \dots, W_{s-2k-1})$ for $s = 2k + 1, 2k + 2, \dots, n$, and the functions g_s are dependent polynomially on the first and second derivatives of $W_0, W_1, \dots, W_{s-2k-1}$.

We'll solve equations (5) under the following boundary conditions:

$$W_i|_{x=0} = 0, \quad (-\infty < y < +\infty); \lim_{|y| \rightarrow +\infty} W_i = 0, \quad (0 \leq x \leq 1), i = 0, 1, \dots, n. \quad (6)$$

For $i = 0$, problem (5), (6) is said to be a degenerated problem corresponding to problem (1), (2).

The following lemma is valid

Lemma 1. Let $f(x, y)$ be a function given in Π , having continuous derivatives with respect to x to the $(n+1)$ -th order inclusively, be infinitely differentiable with respect to y and satisfy the condition

$$\sup_y \left(1 + |y|^l \right) \left| \frac{\partial^k f(x, y)}{\partial x^{k_1} \partial y^{k_2}} \right| = C_{lk_1 k_2}^{(1)} < +\infty, \quad (7)$$

where l is a nonnegative number, $k = k_1 + k_2$, $k_1 \leq n+1$, k_2 is arbitrary, $C_{lk_1k_2}^{(1)} > 0$. Then the function $W_0(x, y)$ being the solution of problem (5), (6) for $i = 0$, in Π has continuous derivatives with respect to x to the $(n+2)$ -th order inclusively, is infinitely differentiable with respect to y and satisfies the condition

$$\sup_y \left(1 + |y|^l\right) \left| \frac{\partial^k W_0(x, y)}{\partial x^{k_1} \partial y^{k_2}} \right| = C_{lk_1k_2}^{(2)} < +\infty, \quad (8)$$

where $k_1 \leq n+2$, $C_{lk_1k_2}^{(2)} > 0$.

Proof. Applying the Fourier transformation with respect to y , problem (5), (6) for $i = 0$ is reduced to the problem

$$\frac{d\tilde{W}_0}{dx} + (a + \lambda^2)\tilde{W}_0 = \tilde{f}(x, \lambda), \quad \tilde{W}_0|_{x=0} = 0. \quad (9)$$

Here $\tilde{W}_0(x, \lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} W_0(x, y) e^{-i\lambda y} dy$, $\tilde{f}(x, \lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x, y) e^{-i\lambda y} dy$. The solution of problem (9) is of the form

$$\tilde{W}_0(x, \lambda) = \int_0^x e^{(a+\lambda^2)(x-\tau)} \tilde{f}(\tau, \lambda) d\tau. \quad (10)$$

$W_0(x, y)$ is found as the inverse Fourier transformation of the function $\tilde{W}_0(x, \lambda)$ from the following formula:

$$W_0(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{W}_0(x, \lambda) e^{i\lambda y} d\lambda. \quad (11)$$

From condition (7) it follows that the function $\tilde{f}(x, \lambda)$ and all its derivatives with respect to x to the $(n+1)$ -th order inclusively, with respect to the variable λ belong to the $S.L.$ Schwarts space (in the sequel we'll denote it by S_λ). Obviously, for proving lemma 1 it suffices to show that the function $\tilde{W}_0(x, \lambda)$ that is the solution of problem (9), and all its derivatives with respect to x to the $(n+2)$ -th order inclusively belong to S_λ .

By the mathematical induction method we can prove the validity of the following formula:

$$\frac{\partial^k \tilde{W}_0}{\partial \lambda^k} = \int_0^x \left\{ \sum_{j=0}^k a_j(\lambda, x-\tau) \frac{\partial^{k-j} \tilde{f}(\tau, \lambda)}{\partial \lambda^{k-j}} \right\} e^{-(a+\lambda^2)(x-\tau)} d\tau. \quad (12)$$

Here $a_j(\lambda, x-\tau)$ is a polynomial with respect to λ and $x-\tau$, more exactly,

$$a_j(\lambda, x-\tau) = \sum_{r=0}^j C_r \lambda^r (x-\tau)^r, \quad (C_j \neq 0),$$

moreover the coefficients C_r are real numbers and some of them may equal zero.

From belongness of the function $\tilde{f}(x, \lambda)$ to the space S_λ it follows that

$$\sup_{\lambda} (1 + |\lambda|^l) \left| \frac{\partial^k \tilde{f}(x, \lambda)}{\partial \lambda^k} \right| = C_{lk}^{(3)} < +\infty. \quad (13)$$

From (12) and (13), we get

$$\begin{aligned} \sup_{\lambda} (1 + |\lambda|^l) \left| \frac{\partial^k \tilde{W}_0(x, \lambda)}{\partial \lambda^k} \right| &\leq \sup_{\lambda} (1 + |\lambda|^l) \int_0^x \left[\sum_{j=0}^k |a_j(\lambda, x - \tau)| e^{-\lambda^2(x-\tau)} \left| \frac{\partial^{k-j} \tilde{f}(\tau, \lambda)}{\partial \lambda^{k-j}} \right| \right] d\tau \leq \\ &\leq \sup_{\lambda} (1 + |\lambda|^l) \int_0^x \left[\sum_{j=0}^k b_j \left| \frac{\partial^{k-j} \tilde{f}(\tau, \lambda)}{\partial \lambda^{k-j}} \right| \right] d\tau = \sum_{j=0}^k b_j \int_0^x \sup_{\lambda} (1 + |\lambda|^l) \left| \frac{\partial^{k-j} \tilde{f}(\tau, \lambda)}{\partial \lambda^{k-j}} \right| d\tau \leq \sum_{j=0}^k b_j C_{lk-j}^{(3)}. \end{aligned}$$

Denoting in the last inequality $\sum_{j=0}^k b_j C_{lk-j}^{(3)} = C_{lk}^{(4)}$, we get

$$\sup_{\lambda} (1 + |\lambda|^l) \left| \frac{\partial^k \tilde{W}_0(x, \lambda)}{\partial \lambda^k} \right| = C_{lk}^{(4)}, \quad (14)$$

i.e. $\tilde{W}_0(x, \lambda) \in S_{\lambda}$. While obtaining (14) we used $|a_j(\lambda, \sigma) e^{-\sigma \lambda^2}| \leq b_j; (\sigma \geq 0)$, where $j = 0, 1, \dots, k; b_j > 0$ are some numbers.

Now prove $\frac{\partial^{k_1} \tilde{W}_0(x, \lambda)}{\partial x^{k_1}} \in S_{\lambda}; k = 1, 2, \dots, n+2$. It is easy to show that the derivatives of the function $\tilde{W}_0(x, \lambda)$ with respect to x of any order are expressed by the formula

$$\frac{\partial^{k_1} \tilde{W}_0(x, \lambda)}{\partial x^{k_1}} = \left[-(a + \lambda^2) \right]^{k_1} \tilde{W}_0 + \sum_{j=0}^{k_1-1} \left[-(a + \lambda^2) \right]^{k_1-j-1} \frac{\partial^j \tilde{f}(x, \lambda)}{\partial x^j}. \quad (15)$$

The functions $\varphi_m(\lambda) = \left[-(a + \lambda^2) \right]^m$ with respect to λ have a polynomial growth. Above we proved the belongness of the function $\tilde{W}_0(x, \lambda)$ to the space S_{λ} . Thus each summand contained in the right hand side of (15) is the product of two functions, one of them has a polynomial growth, the another one enters into the space S_{λ} . Therefore, the relation $\frac{\partial^{k_1} \tilde{W}_0(x, \lambda)}{\partial x^{k_1}} \in S_{\lambda}$ is valid, whence it follows

that $\frac{\partial^{k_1} W_0(x, y)}{\partial x^{k_1}} \in S_y; k_1 = 0, 1, \dots, n+2$.

Lemma 1 is proved.

The remaining functions $W_1, W_2, \dots, W_n$ contained in expansion (3) will be sequentially determined from boundary value problems (5), (6) for $i = 1, 2, \dots, n$. From lemma 1 it follows that the functions W_i being the solutions of problems (5), (6) for $i = 1, 2, \dots, n$, will have continuous derivatives with respect to x to the $(n+2-i)$ -th order and condition (8) for the function W_i will be satisfied for $k_1 \leq n+2-i; i = 1, 2, \dots, n$.

From (3) and (6) we get that the constructed function W satisfies the following boundary conditions:

$$W|_{x=0} = 0, \quad (-\infty < y < +\infty); \quad \lim_{|y| \rightarrow +\infty} W = 0, \quad (0 \leq x \leq 1). \quad (16)$$

The function W doesn't satisfy, generally speaking, boundary condition (2) for $x=1$. For compensating the missed boundary condition it is necessary to construct a boundary layer type function near the boundary $x=1$.

3. The Second Iterative Process-Construction of Boundary Layer Functions

Let's construct a boundary layer type function near the boundary $x=1$. The first iterative process is conducted on the base of decomposition (1) of the operator L_ε . For conducting the second iterative process by means of which we'll construct a boundary layer function near the boundary $x=1$, it is necessary to write a new decomposition of the operator L_ε near this boundary. We make change of variables: $1-x = \varepsilon\tau$, $y = y$. Let's consider the auxiliary function $r = \sum_{j=0}^{n+1} \varepsilon^j r_j(\tau, y)$, where $r_j(\tau, y)$ are some smooth functions determined near $x=1$. Expansion of $L_\varepsilon(r)$ in powers of ε in the coordinates (τ, y) has the form:

$$L_{\varepsilon,1}r \equiv -\varepsilon^{-1} \left\{ \frac{\partial}{\partial \tau} \left(\frac{\partial r_0}{\partial \tau} \right)^{2k+1} + \frac{\partial^2 r_0}{\partial \tau^2} + \frac{\partial r_0}{\partial \tau} + \sum_{j=1}^{n+1} \varepsilon^j \left[(2k+1) \frac{\partial}{\partial \tau} \left(\left(\frac{\partial r_0}{\partial \tau} \right)^{2k} \frac{\partial r_j}{\partial \tau} \right) + \frac{\partial^2 r_j}{\partial \tau^2} + \frac{\partial r_j}{\partial \tau} + \Phi_j(r_0, r_1, \dots, r_{j-1}) \right] + o(\varepsilon^{n+2}) \right\}, \quad (17)$$

where Φ_j are the known functions dependent on $\tau, y, r_0, r_1, \dots, r_{j-1}$ and their first, second derivatives.

We look for a boundary layer type function near the boundary $x=1$ in the form:

$$V = V_0 + \varepsilon V_1 + \dots + \varepsilon^{n+1} V_{n+1}, \quad (18)$$

as the solution of the equation

$$L_{\varepsilon,1}(W + V) - L_{\varepsilon,1}W = 0(\varepsilon^{n+1}). \quad (19)$$

Expanding each function $W_i(1-\varepsilon\tau, y)$, $i=0,1,\dots,n$ in Taylor formula at the point $(1, y)$, we get a new expansion of the function W in powers of ε in the coordinates (τ, y) in the following form:

$$W = \sum_{j=0}^{n+1} \varepsilon^j \omega_j(\tau, y) + o(\varepsilon^{n+2}), \quad (20)$$

where $\omega_0 = W_0(1, y)$ are independent of τ , and the remaining functions ω_k are determined from the formula

$$\omega_k = \sum_{i+j=k} (-1)^i \frac{\partial^i W_j(1, y)}{\partial x^i} \tau^i; \quad k = 1, 2, \dots, n+1. \quad (21)$$

Substituting expressions (18), (21) for the functions V, W to (19) and taking into account (17), for determining $V_0, V_1, \dots, V_{n+1}$ we get the following equations:

$$\frac{\partial}{\partial \tau} \left(\frac{\partial V_0}{\partial \tau} \right)^{2k+1} + \frac{\partial^2 V_0}{\partial \tau^2} + \frac{\partial V_0}{\partial \tau} = 0, \quad (22)$$

$$\frac{\partial}{\partial \tau} \left\{ \left[(2k+1) \left(\frac{\partial V_0}{\partial \tau} \right)^{2k} + 1 \right] \frac{\partial V_j}{\partial \tau} \right\} + \frac{\partial V_j}{\partial \tau} = Q_j, \quad (23)$$

where Q_j are the known functions dependent on $V_0, V_1, \dots, V_{j-1}, \omega_0, \omega_1, \dots, \omega_j$, $j = 1, 2, \dots, n+1$ their first and second derivatives. We can write the formulae for Q_j obviously, but they are of bulky form. Here we give formulae only for Q_1 and Q_2 :

$$\begin{aligned} Q_1(\tau, y) &= -\frac{\partial^2 V_0}{\partial y^2} + aV_0 - (2k+1) \frac{\partial}{\partial \tau} \left[\left(\frac{\partial V_0}{\partial \tau} \right)^{2k} \frac{\partial \omega_1}{\partial \tau} \right], \\ Q_2(\tau, y) &= -\frac{\partial^2 V_1}{\partial y^2} + aV_1 - (2k+1) \frac{\partial}{\partial \tau} \left[\left(\frac{\partial V_0}{\partial \tau} \right)^{2k} \frac{\partial \omega_2}{\partial \tau} \right] - \\ &\quad - \frac{(2k+1)(2k)}{2!} \frac{\partial}{\partial \tau} \left[\left(\frac{\partial V_0}{\partial \tau} \right)^{2k-1} \left(\frac{\partial \omega_1}{\partial \tau} + \frac{\partial V_1}{\partial \tau} \right)^2 \right]. \end{aligned}$$

The boundary conditions for equations (22), (23) are obtained from the requirement that the sum $W + V$ should satisfy the boundary condition

$$(W + V)|_{x=1} = 0. \quad (24)$$

Substituting the expressions for W and V , respectively from (3) and (18) into (24), taking into account that we look for V_j ; $j = 0, 1, \dots, n+1$ as a boundary layer type function, we have

$$V_j|_{\tau=0} = \varphi_j(y), \quad \lim_{\tau \rightarrow +\infty} V_j = 0; \quad j = 0, 1, \dots, n+1, \quad (25)$$

where $\varphi_i(y) = -W_i(1, y)$ for $i = 0, 1, \dots, n$; $\varphi_{n+1} \equiv 0$.

The following lemma is valid

Lemma 2. For each $y \in (-\infty, +\infty)$ problem (22), (25) (for $j = 0$) has a unique solution that is infinitely differentiable with respect to both variables τ and y . And the following estimation is valid

$$\left| \frac{\partial^i V_0(\tau, y)}{\partial \tau^{i_1} \partial y^{i_2}} \right| \leq G_{i_1 i_2} \left(|\varphi_0(y)|, |\varphi'_0(y)|, \dots, |\varphi_0^{(i_2)}(y)| \right) e^{-\tau}; \quad i = i_1 + i_2, \quad (26)$$

where $G_{i_1 i_2}(t_1, t_2, \dots, t_{i_2+1})$ are some known polynomials of their own arguments with non-negative coefficients, the free members of these polynomials equal zero, and even one of other coefficients is non-zero.

Proof. Existence and uniqueness of the solution of problem (22), (25) for $j = 0$ were proved in [5] (see theorem 2). The solution of problem (22), (25) for $j = 0$ in the parametric form is as follows

$$\tau = -\frac{2k+1}{2k} (t^{2k} - t_0^{2k}) + \ln \left| \frac{t}{t_0} \right|, V_0 = -(t^{2k+1} + t), \quad (27)$$

where t is a parameter, $t_0(y)$ is a real root of the algebraic equation

$$t_0^{2k+1} + t_0 + \varphi_0(y) = 0. \quad (28)$$

Note that if $\varphi_0(y_0) = 0$ for some $y_0 \in (-\infty, +\infty)$, then the corresponding real root $t_0(y_0)$ of algebraic equation (28) also vanishes and the expression for τ in (27) loses its sense. For $\varphi_0(y_0) = 0$ as the solution $V_0(\tau, y_0)$ of problem (22), (25) for $j = 0$ we can take $V_0(\tau, y_0) \equiv 0$.

Thus, the desired solution of problem (22), (25) for $j = 0$ is given in the parametric form (27) if $\varphi_0(y) \neq 0$ and is predetermined by an identity zero, if $\varphi_0(y) = 0$.

The infinite differentiability of $V_0(\tau, y)$ with respect to τ was also proved in [5]. But there $\varphi_0(y)$ has continuous derivatives with respect to y to definite finite order. In connection with the fact that here $\varphi_0(y) \in S_y$, is infinitely differentiable, $t_0(y)$ also will be an infinitely differentiable function. Hence it follows an infinite differentiability of $V_0(\tau, y)$ with respect to y .

Prove the validity of estimation (26). From the first equality of (27), we can get an estimation of the form

$$|t| \leq |t_0(y)| \exp \left[\frac{2k+1}{2k} t_0^{2k}(y) \right] \exp(-\tau). \quad (29)$$

Having transformed equation (28) we have: $t_0(y) = -[t_0^{2k}(y) + 1]^{-1} \varphi_0(y)$, whence it follows that $|t_0(y)| \leq |\varphi_0(y)|$. Hence it is seen that the function $\exp \left[\frac{2k+1}{2k} t_0^{2k}(y) \right]$ is bounded, i.e. $\exp \left[\frac{2k+1}{2k} t_0^{2k}(y) \right] \leq C_0$. Consequently, from (29) we get the following estimation

$$|t| \leq C_0 |\varphi_0(y)| \exp(-\tau). \quad (30)$$

Taking into account (30), in the second equality of (27) we have

$$|V_0| \leq C|\varphi_0(y)|\exp(-\tau), \quad C > 0. \quad (31)$$

Recalling that the parametric form of the solution of problem (22), (25) for $j = 0$ was obtained by means of substitution $\frac{\partial V_0}{\partial \tau} = q$, from (30) we get an estimation for $\frac{\partial V_0}{\partial \tau}$

$$\left| \frac{\partial V_0}{\partial \tau} \right| \leq C_0|\varphi_0(y)|\exp(-\tau), \quad C_0 > 0 \quad (32)$$

We can represent the function $\frac{\partial^2 V_0}{\partial \tau^2}$ in the form

$$\frac{\partial^2 V_0}{\partial \tau^2} = -B^{-1}(\tau, y) \frac{\partial V_0}{\partial \tau}, \quad (33)$$

where $B(\tau, y)$ denotes the following function:

$$B(\tau, y) = (2k+1) \left(\frac{\partial V_0}{\partial \tau} \right)^{2k} + 1. \quad (34)$$

$0 < B^{-1}(\tau, y) \leq 1$, from (32), (33) we get an estimation for $\frac{\partial^2 V_0}{\partial \tau^2}$. For obtaining estimations for the derivatives $V_0(\tau, y)$ with respect to τ , we differentiate sequentially the both parts of (33) with respect to τ , and each time take into account the estimations of previous derivatives. These estimations will be of the form (32), i.e. $\left| \frac{\partial^i V_0}{\partial \tau^i} \right| \leq C|\varphi_0(y)|\exp(-\tau)$, $i = 2, 3, \dots$; $C > 0$.

Now prove the estimations for the derivatives $V_0(\tau, y)$ with respect to y and for mixed derivatives. The function $\frac{\partial V_0}{\partial y} = \psi$ satisfies the equation in variations that is obtained from equation (22) by differentiating with respect to y :

$$\frac{\partial}{\partial \tau} \left[B(\tau, y) \frac{\partial \psi}{\partial \tau} \right] + \frac{\partial \psi}{\partial \tau} = 0. \quad (35)$$

For $j = 0$, from (25) we get that the function ψ should satisfy the boundary conditions

$$\psi|_{\tau=0} = \varphi'_0(y), \quad \lim_{\tau \rightarrow +\infty} \psi = 0. \quad (36)$$

The solution of problem (35), (36) is of the form

$$\psi = \varphi'_0(y) \exp \left[- \int_0^\tau B^{-1}(\xi, y) d\xi \right]. \quad (37)$$

Using (34) and estimation (32), we estimate $\exp\left[-\int_0^\tau B^{-1}(\xi, y)d\xi\right]$ in the following way:

$$\begin{aligned} \exp\left[-\int_0^\tau B^{-1}(\xi, y)d\xi\right] &\leq \exp\left[-\int_0^\tau \frac{d\xi}{(2k+1)C_1^{2k} \exp(-2k\xi)+1}\right] = \\ &= \exp\left[-\int_0^\tau \frac{\exp(2k\xi)}{(2k+1)C_1^{2k} + \exp(2k\xi)}d\xi\right] = \exp\left[-\frac{1}{2k} \ln|(2k+1)C_1^{2k} + \exp(2k\xi)|_{\xi=0}^{\xi=\tau}\right] = \\ &= \frac{[(2k+1)C_1^{2k} + 1]^{\frac{1}{2k}}}{[(2k+1)C_1^{2k} + \exp(2k\tau)]^{\frac{1}{2k}}} \leq C_2 e^{-\tau}, \end{aligned}$$

where $C_0|\varphi_0(y)| \leq C_1$, $C_2 = [(2k+1)C_1^{2k} + 1]^{\frac{1}{2k}}$. Hence and from (37) we get the estimation

$$|\psi| = \left| \frac{\partial V_0}{\partial y} \right| \leq C|\varphi'_0(y)| \exp(-\tau), \quad C > 0 \quad (38)$$

From (37) it follows that $\frac{\partial \psi}{\partial \tau} = -B^{-1}(\tau, y)\psi$. Taking into account (38), hence we get an estimation for the mixed derivative

$$\left| \frac{\partial \psi}{\partial \tau} \right| = \left| \frac{\partial^2 V_0}{\partial y \partial \tau} \right| \leq C|\varphi'_0(y)| \exp(-\tau). \quad (39)$$

Now we can get an estimation also for $\frac{\partial^2 V_0}{\partial y^2}$. Differentiating the both sides of (37) with respect to y , we have

$$\frac{\partial \psi}{\partial y} = -\left\{ \int_0^\tau [B^{-1}(\xi, y)]_y d\xi \right\} \psi + \varphi''_0(y) \exp\left[-\int_0^\tau B^{-1}(\xi, y)d\xi\right]. \quad (40)$$

From (34) it follows that $[B^{-1}(\tau, y)]_y = -(2k+1)(2k)B^{-2}(\tau, y)\left(\frac{\partial V_0}{\partial \tau}\right)^{2k-1} \frac{\partial^2 V_0}{\partial y \partial \tau}$.

Obviously, $0 < B^{-i} \leq 1$ for any natural number i . Knowing estimation (32) for $\frac{\partial V_0}{\partial \tau}$ and estimation (39) for $\frac{\partial^2 V_0}{\partial y \partial \tau}$, we estimate $[B^{-1}(\tau, y)]_y$:

$$\left| [B^{-1}(\tau, y)]_y \right| \leq C|\varphi_0(y)| |\varphi'_0(y)| \exp(-\tau). \quad (41)$$

Taking into account (38) and (41) in (40), we have

$$\left| \frac{\partial \psi}{\partial y} \right| = \left| \frac{\partial^2 V_0}{\partial y^2} \right| \leq \left[C_1 |\varphi_0(y)| |\varphi'_0(y)|^2 + C_2 |\varphi''_0(y)| \right] \exp(-\tau).$$

The validity of estimation (26) for subsequent derivatives is proved in the same way.

Lemma 2 is proved.

By theorem whose proof is given in [5], (see. theorem 3), there exists a unique solution of each problem (22), (25) for $j = 1, 2, \dots, n+1$, and these solutions are represented by the following formula:

$$V_j(\tau, y) = \left\{ \varphi_j(y) - \int_0^\tau \left[B^{-1}(z, y) e^{\nu(z, y)} \int_z^{+\infty} Q_j(\xi, y) d\xi \right] dz \right\} \exp[-\nu(\tau, y)]. \quad (42)$$

Here $\nu(\tau, y)$ denotes the function

$$\nu(\tau, y) = \int_0^\tau B^{-1}(\xi, y) d\xi. \quad (43)$$

Unlike the estimations in the paper [5], here it is necessary to get such estimations for $V_1, V_2, \dots, V_{n+1}$, that could enable to study the behavior of functions not only as $\tau \rightarrow +\infty$, and also as $|y| \rightarrow +\infty$.

Substituting $j = 1$ in (42), we get a formula for $V_1(\tau, y)$. Using the obvious expressions for the functions $Q_1(\tau, y)$ and (21), at $k = 1$ for ω_k , and taking into account the known estimations for V_0 , $\frac{\partial V_0}{\partial \tau}$, $\frac{\partial V_0}{\partial y}$, $\frac{\partial^2 V_0}{\partial \tau^2}$, also the belongness of the functions $W_1(1, y)$ and $\frac{\partial W_0(1, y)}{\partial x}$ to the space S_y , we get

$$|Q_1(\tau, y)| \leq |q_1(y)| \exp(-\tau), \quad (44)$$

where $q_1(y)$ is a known function from the space S_y . Following (44), from (42) for $j = 1$ we can get the estimation

$$|V_1(\tau, y)| \leq C(|\varphi_1(y)| + \tau |q_1(y)|) \exp(-\tau), \quad C > 0. \quad (45)$$

Differentiating the both sides of (42), for $j = 1$ with respect to τ we have

$$\frac{\partial V_1}{\partial \tau} = -B^{-1}(\tau, y) \left[V_1 + \int_\tau^{+\infty} Q_1(\xi, y) d\xi \right]. \quad (46)$$

Using estimates (44), (45) in (46) we get an estimation for $\frac{\partial V_1}{\partial \tau}$. The estimates for higher derivatives with respect to τ are obtained from formulae obtained by sequential differentiation of both sides of (46) and from the estimations for previous derivatives of $V_1(\tau, y)$. Note that these estimations are of the form

$\left| \frac{\partial^i V_1(\tau, y)}{\partial \tau^i} \right| \leq (|q_1(y)| + |q_2(y)|\tau) \exp(-\tau); i = 1, 2, \dots$, where $q_1(y), q_2(y)$ are the known functions, moreover $q_1(y) \in S_y, q_2(y) \in S_y$.

Now we get estimations for the derivatives of $V_1(\tau, y)$ with respect to y and for mixed derivatives. We can define the function $\frac{\partial V_1}{\partial y}$ as the solution of a boundary value problem for the equation in variations that is obtained from (23) for $j = 1$ by differentiating with respect to y . We can note that the function $\frac{\partial V_1}{\partial y}$ is also determined by formula (42), only in this formula the function $\varphi_j(y)$ should be replaced by $\varphi'_j(y)$, and function $\int_z^{+\infty} Q_j(\xi, y) d\xi$ by the following function:

$$\int_z^{+\infty} Q'_{1y}(\xi, y) d\xi + B'_y(z, y) \frac{\partial V_1(z, y)}{\partial z}.$$

Consequently, this time by obtaining estimations, instead of (44) we use the estimation

$$\left| \int_z^{+\infty} Q'_{1y}(\xi, y) d\xi + B'_y(z, y) \frac{\partial V_1(z, y)}{\partial z} \right| \leq (|q_1(y)| + |q_2(y)|z) \exp(-z).$$

As a result, for $\frac{\partial V_1}{\partial y}$ we get the estimation:

$$\left| \frac{\partial V_1}{\partial y} \right| \leq (|q_1(y)| + |q_2(y)|\tau + |q_3(y)|\tau^2) \exp(-\tau), \text{ where } q_i(y) \in S_y; i = 1, 2, 3.$$

If we differentiate the both sides of the formula for $\frac{\partial V_1}{\partial y}$ with respect to τ , we can get the following estimation:

$$\left| \frac{\partial^2 V_1}{\partial y \partial \tau} \right| \leq (|q_1(y)| + |q_2(y)|\tau + |q_3(y)|\tau^2) \exp(-\tau).$$

It should be noted that at each differentiation of $V_1(\tau, y)$ with respect to y , the power of the polynomial with respect to τ , standing at the right side of the estimation increases by a unit. The estimation for $V_1(\tau, y)$ in the general case has the form

$$\left| \frac{\partial^i V_1(\tau, y)}{\partial \tau^i \partial y^{i_2}} \right| \leq (|q_{10}(y)| + |q_{11}(y)|\tau + \dots + |q_{1i_2+1}(y)|\tau^{i_2+1}) \exp(-\tau),$$

where $q_{1j}(y) \in S_y; j = 0, 1, \dots, i_2 + 1$ are the known functions.

Using the obvious form for $Q_2(\tau, y)$ and taking into account the known estimations for V_0, V_1 and their derivatives, we can show that

$$|Q_2(\tau, y)| \leq (|q_1(y)| + |q_2(y)|\tau + |q_3(y)|\tau^2) \exp(-\tau). \quad (47)$$

Having put in (42) $j = 2$, and taking into attention estimation (47), the validity of the following estimations is proved in the same way as above

$$\left| \frac{\partial^i V_2(\tau, y)}{\partial \tau^{i_1} \partial y^{i_2}} \right| \leq (|q_{20}(y)| + |q_{21}(y)|\tau + \dots + |q_{2i_2+3}(y)|\tau^{i_2+3}) \exp(-\tau).$$

Continuing this process, and each time taking into account the obvious form of the right side of the equation for V_j , we get the estimation

$$\left| \frac{\partial^i V_j(\tau, y)}{\partial \tau^{i_1} \partial y^{i_2}} \right| \leq \left(\sum_{s=0}^{i_2+j+1} |q_{js}(y)|\tau^s \right) \exp(-\tau); \quad j = 1, 2, \dots, n, \quad (48)$$

where $q_{js} \in S_y$ are the known functions.

Multiply all the functions V_j ; $j = 0, 1, \dots, n+1$ by a smoothing multiplier and leave previous denotation for the obtained new functions. At the expense of smoothing multipliers all the functions V_j ; $j = 0, 1, \dots, n+1$ vanish for $x = 0$.

Therefore, it follows from (16) that the constructed sum $\tilde{U} = W + V$ in addition to boundary condition (24) also satisfies the condition

$$(W + V)|_{x=0} = 0. \quad (49)$$

From (16) and (25) we get that this sum satisfies the following boundary condition as well

$$\lim_{|y| \rightarrow +\infty} (W + V) = 0. \quad (50)$$

Having denoted $U - \tilde{U} = z$, we get the following asymptotic expansion in small parameter of the solution of problem (1), (2):

$$U = \sum_{i=0}^n \varepsilon^i W_i + \sum_{j=0}^{n+1} \varepsilon^j V_j + z. \quad (51)$$

where z is a remainder term.

Now estimate the remainder term.

4. Estimation of Remainder Term

Putting together (4) and (19), we get that $\tilde{U}$ satisfies the equation

$$L_\varepsilon \tilde{U} = 0(\varepsilon^{n+1}). \quad (52)$$

Subtracting (52) from (1), we have

$$-\varepsilon^p \left[\left(\frac{\partial u}{\partial x} \right)^p - \left(\frac{\partial \tilde{u}}{\partial x} \right)^p \right] - \varepsilon^p \left[\left(\frac{\partial u}{\partial y} \right)^p - \left(\frac{\partial \tilde{u}}{\partial y} \right)^p \right] - \varepsilon \Delta z +$$

$$+\frac{\partial z}{\partial x}-\frac{\partial^2 z}{\partial y^2}+az=\varepsilon^{n+1}F(\varepsilon, x, y), \quad (53)$$

where $\|F(\varepsilon, x, y)\|_{L_2(\Pi)} \leq C$ for any $\varepsilon \in [0, \varepsilon_0)$, and $C > 0$ is independent of ε .

From (2), (24), (49), (50) and (51) it follows that z satisfies the boundary conditions:

$$z|_{x=0} = z|_{x=1} = 0, \quad \lim_{|y| \rightarrow +\infty} z = 0. \quad (54)$$

Multiplying the both sides of (53) by $z = U - \tilde{U}$ and integrating by parts allowing for boundary conditions (54), after some transformations we get the estimation

$$\begin{aligned} & \varepsilon^p \iint_{\Pi} \left[\left(\frac{\partial z}{\partial x} \right)^{p+1} + \left(\frac{\partial z}{\partial y} \right)^{p+1} \right] dx dy + \varepsilon \iint_{\Pi} \left[\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] dx dy + \iint_{\Pi} \left(\frac{\partial z}{\partial y} \right)^2 dx + \\ & + C_1 \iint_{\Pi} z^2 dx dy \leq C_2 \varepsilon^{2(n+1)}, \end{aligned} \quad (55)$$

where $C_1 > 0, C_2 > 0$ are the constants independent of ε .

5. Conclusion

Combining the obtained results, we arrive at the following statement.

Theorem. Let $f(x, y)$ be a function given in Π , have continuous derivatives with respect to x to the $(n+1)$ -th order inclusively, be infinitely differentiable with respect to y and satisfy equation (7). Then for the generalized solution of problem (1), (2) it holds asymptotic representation (51), where the functions W_i are determined by the first iterative process, V_j is a boundary layer type function near the boundary $x = 1$, z is a remainder term and estimation (55) is valid for it.

References

- [1] Byeon Jaeyoung, Singularly perturbed nonlinear Dirichlet problems with a general nonlinearity, Trans. Amer. Math. Soc., 362 (4) (2010), 1981-2001.
- [2] Cheng Yan, Asymptotic solution of a boundary problem for the semilinear singularly perturbed elliptical equation, Shuxue Zazhi = J. Math., 1 (2005), 25-29.

- [3] Del Pino Manuel, Felmer Patricio, Localizing spikelayer patterns singularly perturbed elliptic problems, *Tohoku Math. Publ.*, 8 (1998), 35-42.
- [4] M.M.Sabzaliev, The asymptotic form of the solution of boundary value problem for singular perturbed quasilinear parabolic differential equation, *Proceedings of Mathematics and Mechanics of NAS Azerbaijan*, 21 (2004), 169-176.
- [5] M.M.Sabzaliev, The asymptotic form of the solution of boundary value problem for singular perturbed quasilinear parabolic differential equation, *Proceedings of Mathematics and Mechanics of NAS Azerbaijan*, 21 (2004), 169-176.
- [6] ZHU Zhen-bo, NI Ming-kang, Dirichlet problem of semilinear singular perturbation elliptic equations, *Huadong shifan daxue xuebao. Ziran Kexue Ban = I. East China Norm. Univ. Sci.*, 5 (2008), 72-77.
- [7] ZHU Zhen-bo, NI Ming-kang, Dirichlet problem of semilinear singular perturbation elliptic equations, *Huadong shifan daxue xuebao. Ziran Kexue Ban = I. East China Norm. Univ. Sci.*, 5 (2008), 72-77.
- [8] V.Yu. Lunin, On asymptotics of solutions of the first boundary problem for quasilinear elliptic equations. *Vestnik Moskow. Univ.* 3 (1976), 43-51.

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