Periodic Orbits of Weakly Exact Magnetic Flows

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Abstract

For a weakly exact magnetic flows with a bounded primitive on a closed Riemannian manifold, we prove the existence of periodic orbits in almost all energy levels below of the Mañé’s critical value.

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1 Introduction and results

Let $(M,g)$ be a closed Riemannian manifold, denote by $\omega_0$ the symplectic form on $TM$, obtained by pulling back the canonical symplectic form of $T^*M$ via $g$. Let $\Omega$ be a closed 2-form on $M$ and $\omega := \omega_0 + \pi^*\Omega$, where $\pi : TM \to M$ is the canonical projection. Consider $F : TM \to \mathbb{R}$, $F(x, v) = \frac{1}{2}g_x(v, v)$. The magnetic flow of $(g, \Omega)$ is the Hamiltonian flow of $F$ with respect to $\omega$.

We say that a magnetic flow is weakly exact, if the pull-back $\widetilde{\Omega}$ of $\Omega$ to the universal cover $\widetilde{M}$ is exact. Suppose that $\widetilde{\Omega}$ is exact with a smooth 1-form $\theta$ such that $\widetilde{\Omega} = d\theta$. We consider the Lagrangian on $\widetilde{M}$ given by $L(x, v) = \frac{1}{2}\tilde{g}_x(v, v) + \theta_x(v)$, where $(\widetilde{M}, \tilde{g})$ is a Riemannian universal covering of $(M, g)$, thus the flow of $L$ coincide with the lift of the magnetic flow $(g, \Omega)$ therefore is complete.

Recall that on a boundaryless, complete Riemannian manifold $N$ an autonomous Lagrangian is a smooth function, $L : TN \to \mathbb{R}$ such that $L$ is convex and superlinear when restricted to any fiber (see [2]), i.e., $L$ is superlinear if for all $A \in \mathbb{R}$ there is $B \in \mathbb{R}$ such that $L(x, v) \geq A|v| - B$ for all $(x, v) \in TN$. Note if a weakly exact magnetic flow has a bounded primitive $\theta$, then our Lagrangian $L$ satisfies the above conditions (see [2] for details).
The energy $E : TN \to \mathbb{R}$ of $L$ is given by $E(x, v) := \frac{\partial L}{\partial v}(x, v) - L(x, v)$, $E$ is a first integral for the flow $\varphi_t$ of $L$. The $L$-action of an absolutely continuous curve $\gamma : [a, b] \to \tilde{M}$ is defined by $S_L(\gamma) := \int_a^b L(\gamma(t), \dot{\gamma}(t)) dt$. The Mañé’s critical value is given by $c(L) := \inf \{ k \in \mathbb{R} | S_{L+k}(\gamma) \geq 0 \}$, where $\gamma$ is any absolutely continuous closed curve defined on any closed interval $[a, b]$. It is well-known that $c(L) \geq e_0 := \min \{ k \in \mathbb{R} | \pi(E^{-1}(k)) = \tilde{M} \}$. For weakly exact magnetic flow $e_0 = 0$.

Our interest is to study the existence problem for periodic orbits of weakly exact magnetic flows, we study the case of lower energy levels where very little is known, recall that, the exact case, was treated by G. Contreras [1], for some related results in small energy levels see [4], [3]. The last results use symplectic techniques which do not give a precise estimates of the energy of the periodic orbits. Being able to express the weakly exact magnetic flow as a Lagrangian allows us to use all the variational techniques, thus we have

**Theorem 1.1** A magnetic flow given by a magnetic field whose pullback to the universal cover has bounded primitive admits a contractible periodic orbit with energy $k$ and positive $(L + k)$-action for almost $k \subset (e_0, c(L))$ with respect to the Lebesgue measure, where $L$ is the corresponding Lagrangian on the universal cover.

**Remarks:** The methods for to prove ours results are an adaptation from those used in [1]. I thank to G. Contreras for his suggestion to work this case.

### 2 Variational Setting

Let $\mathcal{H}^1(\tilde{M}) := \{ x : [0, 1] \to \tilde{M} \mid \text{a. c. curve with } \int_0^1 |\dot{x}(s)|^2 ds < \infty \}$, it is a Hilbert manifold and its tangent space at the curve $x(s)$ is given by weakly differentiable vector fields along $x(s)$ with covariant derivative in $L^2$. We consider in the Hilbert manifold $\mathcal{H}^1(\tilde{M}) \times \mathbb{R}^+$ the Riemannian metric

$$
\langle (\xi, \alpha), (\eta, \beta) \rangle_{(x, T)} := \alpha \beta + f^2(T) \langle \xi(0), \eta(0) \rangle + f(T) \int_0^1 \langle \frac{D}{ds} \xi(s), \frac{D}{ds} \eta(s) \rangle_{x(s)} ds,
$$

where $\frac{D}{ds}$ is the covariant derivative along $x(s)$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is smooth, positive function with $f(T) := T$ if $T \leq 1$ and equal to 1 if $T \geq 10$. Observe that this metric is locally equivalent to the metric obtained with $f = 1$. Let $\Lambda_{\tilde{M}}$ be the set of closed curves in $\mathcal{H}^1(\tilde{M}) \times \mathbb{R}^+$ it is a Hilbert submanifold of $\mathcal{H}^1(\tilde{M}) \times \mathbb{R}^+$. Given $k \in \mathbb{R}$, we define the action functional $\tilde{S}_k : \mathcal{H}^1(\tilde{M}) \times \mathbb{R}^+ \to \mathbb{R}$ as

$$
\tilde{S}_k(x, T) := \int_0^1 \left[ L(x(s), \frac{\dot{x}(s)}{T}) + k \right] T ds = \int_0^1 \left[ \frac{1}{2T} |\dot{x}(s)|^2 + \theta_x(x(s)) + kT \right] ds.
$$
where \((x, T) \in \mathcal{H}^l(\tilde{M}) \times \mathbb{R}^+\), if \(y(t) := x(\frac{t}{T}), t \in [0T]\) then \(\tilde{S}_k(x, T) = S_{L+k}(y)\). From a Smale theorem, we have that \(\tilde{S}_k\) is \(C^2\) on \(\mathcal{H}^l(\tilde{M}) \times \mathbb{R}^+\) (see [5]) and the derivative of \(\tilde{S}_k\) is given by

\[
d_{(x,T)}\tilde{S}_k(\xi, \alpha) = \xi \cdot L_v(y, \dot{y})\big|_0^T + \int_0^T \xi \cdot \left[\frac{d}{dt}L_v - L_x\right](y, \dot{y})ds + \frac{\alpha}{T} \int_0^T [k - E(y, \dot{y})]ds,
\]

where \(y(t) := x(\frac{t}{T}), \xi(t) := \xi(\frac{t}{T})\), for \(0 \leq t \leq T\) and \(E(x, v) = vL_v(x, v) - L(x, v) = \frac{1}{2}v^2\). Note in a critical point, the second term will be zero which implies that \(y(t)\) is a solution of the Euler-Lagrange equation, therefore \(y(t)\) has constant energy and the third term implies that this energy is equal to \(k\).

Now, taking \((x_1, T) \in \Lambda_{\tilde{M}} \times \mathbb{R}^+\) and \(h\) a deck transformation. We consider \(x_2 := h \circ x_1\) then \(\tilde{S}_k(x_1, T) = \tilde{S}_k(x_2, T)\). Indeed, let \(p : \tilde{M} \to M\) be the canonical projection then the conditions \(\tilde{\Omega} = p^*\Omega\) and \(p = p \circ h\) imply that \(\tilde{\Omega} = h^*\tilde{\Omega}\). Now, since \(\tilde{M}\) is simply connected the closed curves \(x_i\), can be extended to a smooth map \(\phi_i : \mathbb{D}^2 \to \tilde{M}\) such that \(\phi_i|_{\partial \mathbb{D}^2} = x_i\) respectively for \(i = 1, 2\). For any such extension \(\phi_i\) we have \(\int_{\mathbb{D}^2} \phi_i^*\tilde{\Omega} = \int_{\mathbb{S}^1} \theta\). In particular, if we take \(\phi_2 := h \circ \phi_1\) then \(\int_{x_2} \theta = \int_{x_1} \theta\). Finally, by taking a metric \(\tilde{g}\) on \(\tilde{M}\) such that \((\tilde{M}, \tilde{g})\) is a Riemannian universal covering of \((M, g)\), the deck transformations are isometries and we get \(\tilde{S}_k(x_1, T) = \tilde{S}_k(x_2, T)\).

Let \(\Lambda_0\) be the set of closed curves with trivial homotopy on \(M\). For \(k \in \mathbb{R}\), we define the action \(S_k : \Lambda_0 \times \mathbb{R}^+ \to \mathbb{R}\) as follow, given \((x, T) \in \Lambda_0 \times \mathbb{R}^+\) consider a lift \(\tilde{x}\) of \(x\) to \(\tilde{M}\) and take \(S_k(x, T) := \tilde{S}_k(\tilde{x}, T)\), from above remarks it is well defined.

## 3 Mountain pass and Palais-Smale condition

The following appears in [1], Lemma 5.1 the same argument still working

**Lemma 3.1** Let \(\theta\) be a bounded 1-form in \(\tilde{M}\) and \(W \subset \tilde{M}\) be a neighbourhood of the point \(x_0\). Then there exists an open ball \(U \subset W\) centred at \(x_0\) and \(c > 0\) such that, if \(\gamma\) is a closed curve in \(U\) then \(\left|\int_\gamma \theta_x\right| \leq c \cdot \text{length}(\gamma)^2\).

Now, we show that for energy levels \(0 = e_0 < k < c(L)\), the action functional \(S_k\) exhibits a mountain pass on \(\Lambda_0\).

**Theorem 3.2** Let \(k > e_0 = 0\). There exists \(c > 0\) such that if \(\Gamma : [0, 1] \to \Lambda_0\), is a path joining constant curve \(\Gamma(0) = x_0 : [0, T_0] \to x_0 \subset M\) to any closed curve \(\Gamma(1)\) with negative \((L+k)\)-action then \(\sup_{s \in [0, 1]} S_{L+k}(\Gamma(s)) > c > 0\).

**Proof:** Let \(\{V_i\}_{i=1}^N\) be a finite cover of \(M\) by open balls such that \(\bigcup U_{i_0} = p^{-1}V_i\) where \(U_{i_0}\) are open balls given by lemma 3.1 with the corresponding constant
$c_i$. Note that $c_i > 0$ only depends on $V_i$, we call this constant $c_i$. Let $r_0 > 0$ be such that any ball of radius $r_0$ in $M$ is contained in some $V_i$, write $b := \max_{1 \leq i \leq N} c_i$ and set $0 < l_0 < \min\{r_0, \sqrt{\frac{k}{2b^2}}\}$.

Claim: There exists $s_0 \in [0, 1]$ such that $\text{length}(\Gamma(s_0)) = l_0$.

Proof of Claim: If for some $s_1 \in [0, 1], \Gamma(s_1)$ is not contained in some $V_i$ the claim is clear, therefore suppose that $\Gamma(s)$ is contained in some $V_i$ for all $s \in [0, 1]$. Suppose that $\text{length}(\Gamma(s)) < l_0$ for all $s \in [0, 1]$ as $l_0 < r_0$ then for all $s \in [0, 1]$, $\Gamma(s)$ is contained in some $V_i$. Therefore the lifting $\tilde{\Gamma}(s)$ is contained in some $U_i$, for all $s \in [0, 1]$. Writing $\gamma_1 := \Gamma(1) : [0, T_1] \rightarrow M$ and $l_1 := \text{length}(\gamma_1)$ we have

$$0 > S_{L+k}(\gamma_1) \geq \frac{1}{2} \int_0^{T_1} |\dot{\gamma}_1|^2 - |\int \theta_x| + kT_1 \geq \frac{1}{2} \int_0^{T_1} |\dot{\gamma}_1|^2 - bl_1^2 + kT_1.$$ 

By Schwartz inequality $l_1^2 = (\int_0^{T_1} |\dot{\gamma}_1| dt)^2 \leq T_1 \cdot \int_0^{T_1} |\dot{\gamma}_1|^2 dt$, therefore $0 > S_{L+k}(\gamma_1) \geq (\frac{1}{2T_1} - b)l_1^2 + kT_1$, but $k > 0$ and $T_1 > 0$ then $\frac{1}{2T_1} - b < 0$, i.e., $T_1 > \frac{1}{2b}$ thus $l_1^2 > \frac{kT_1}{2b} > \frac{kT_1}{b} > \frac{k}{2b} > \frac{l_0^2}{b}$. The claim follows from continuity of $s \rightarrow \text{length}(\Gamma(s))$ and that $\text{length}(\Gamma(0)) = 0$.

We continue the proof of theorem 3.2. Take $g(t) := (\frac{1}{2} - b)l_0^2 + kt$ since $\text{length}(\Gamma(s_0)) = l_0 < r_0$ then $\Gamma(s_0)$ is contained in some $V_i$. Taking a lift $\tilde{\Gamma}(s_0)$ contained in $U_i$, we can apply Lemma 3.1, therefore if $\Gamma : [0, 1] \rightarrow \Lambda_0$ as $l_0 < \sqrt{\frac{k}{2b}} < \sqrt{\frac{2k}{b}}$, we have

$$S_{L+k}(\Gamma(s_0)) \geq g(T') \geq \min_{t \in \mathbb{R}^*} g(t) = l_0[\sqrt{2k} - bl_0] =: c > 0. \quad \Box$$

We will say that $S_k$ satisfies the Palais-Smale condition on $\Lambda_0$. If every sequence $(x_n, T_n)$ in the same connected component of $\Lambda_0$ with $|S_k(x_n, T_n)|$ bounded and $\lim_n \|d(x_n, T_n)S_k\|(x_n, T_n) = 0$ has a convergent subsequence. We say that $S_k$ satisfies the Palais-Smale condition at the level $a$ (resp. levels in $[a, b]$). If the above condition is valid restricted to $(x_n, T_n)$ with $\lim_n S_k(x_n, T_n) = a$ (resp. $\lim_n S_k(x_n, T_n) \in [a, b]$). In order to study the Palais-Smale condition of $S_k$ we need some preliminary results, from the convexity of $L$ result

**Lemma 3.3** Suppose that $L(x, v) = \frac{1}{2} |v|^2_x + \theta_x(v)$. Given any coordinate chart there are positive numbers $\delta, B, a$ and $D$ such that for $x, y$ with $d(x, y) < \delta$ we have

$$a|v - w|^2 \leq (L_v(x, v) - L_v(y, w)) \cdot (v - w) + [B(|v| + |w|) + D] |v - w| d(x, y).$$

**Lemma 3.4** If $x_n \in \Lambda_0, S_k(x_n, T_n) < A_1$ and $T_n \leq A_2$ there is $B > 0$ with $\frac{l_n^2}{T_n} \leq \int_0^{T_n} |\dot{y}_n|^2 dt = \frac{1}{T_n} \int_0^{1} |\dot{x}_n|^2 dt < B$, where $l_n := \text{length}(x_n)$ and $y_n(t) = x_n(\frac{t}{T_n})$. In particular if $T_n \rightarrow 0$ then $\lim_n l_n = 0$. 


Proof: Take $l_n := \text{length}(y_n)$, by Schwartz inequality $l_n^2 = (\int_0^{T_n} |\dot{y}_n|dt)^2 \leq T_n \cdot \int_0^{T_n} |\dot{y}_n|^2dt$, thus we obtain the first inequality. Now, since $L(x,v) = \frac{1}{2} |v|^2_x + \theta_x(v)$ is superlinear, i.e., for all $A \in \mathbb{R}$ there is $C \in \mathbb{R}$ such that $L(x,v) \geq A|v| - C$ for all $(x,v) \in T\bar{M}$, using that the actions $S_k(x_n,T_n)$ are bounded a calculation gives the second inequality. $\square$

For $p_1, p_2$ in a boundaryless, complete manifold $N$ and $T > 0$ take

$$C_T(p_1,p_2) := \{ \gamma \in C_{ac}([0,T], N) \mid \gamma(0) = p_1, \gamma(T) = p_2 \}.$$ 

**Theorem 3.5** ([2], th. 3-1.11.) Let $L$ be a convex, superlinear Lagrangian on $N$. Let $N$ be boundaryless, complete Riemannian manifold then for any $p_1, p_2 \in N$, $T > 0$, $b \in \mathbb{R}$ the set $A(b) := \{ \gamma \in C_T(p_1,p_2) \mid S_L(\gamma) \leq b \}$, is compact in the $C^0$-topology.

The next result gives some conditions which imply the Palais-Smale condition.

**Proposition 3.6** If a sequence $(x_n,T_n)_{n \in \mathbb{N}} \subset \Lambda_0$ satisfies

$$|S_k(x_n,T_n)| < A_1, \quad \|d(x_n,T_n)S_k\| < \frac{1}{n} \quad \text{and} \quad 0 < \liminf_n T_n < +\infty,$$

then there exists a convergent subsequence.

**Proof:** From compactness of $M$ given $(x_n,T_n) \in \Lambda_0$ taking a subsequence we can assume that $q_0 := \lim_n x_n(0) = \lim_n x_n(1)$, and that $T := \lim_n T_n \in \mathbb{R}^+$ exist, also suppose that $d(x(0),q_0) < \delta$ for $\delta > 0$ small enough. Let $y_n(t) := x_n \left( \frac{t}{T_n} \right)$ and $\alpha_n : [0,1] \to M$ be the geodesic joining $\alpha_n(0) := q_0$, $\alpha_n(1) = x_n$, and take $\beta_n : [T_n + 1, T + 3] \to M$ geodesics such that $\beta_n(T_n + 1) = y_n(T_n)$, $\beta_n(T + 3) = q_0$ then $|\dot{\alpha}_n| \leq 1$ and $|\dot{\beta}_n| \leq 1$. We define

$$w_n(t) := \begin{cases} \alpha_n(t) & \text{if } 0 \leq t \leq 1, \\ y_n(t-1) & \text{if } 1 \leq t \leq T_n + 1, \\ \beta_n(t) & \text{if } T_n + 1 \leq t \leq T + 3. \end{cases}$$

All the closed curves $w_n : [0,T + 3] \to M$ are based at $q_0$ and their actions are uniformly bounded. Now, applying theorem 3.5 with $N := \tilde{M}$, $p_1 = p_2 = \tilde{q}_0$ then the set $w_n$ is relatively compact in the $C^0$-topology. Thus there is a convergent subsequence of $w_n$ in the $C^0$ topology. From now, we will work with a convergent subsequence of $\{x_n\}$. We shall assume without loss of generality that the limit point of $\{x_n\}$ is contained in a coordinate chart and work in $\mathbb{R}^{2n}$. Taking $z_n := \frac{x_n}{T_n}$ since $0 < T < \infty$ by lemma 3.4 there is $K > 0$ with

$$\|x_n\|_{H^1} \leq K \quad \text{and} \quad \|z_n\|_{H^1} \leq K,$$

as $\lim_n \|d(x_n,T_n)S_k\|_{H^1 \times \mathbb{R}} = 0$ we get

$$\int_0^1 \left[ T_n L_x(x_n, \dot{z}_n) - T_m L_x(x_m, \dot{z}_m) \right]|(x_n-x_m) + |L_v(x_n, \dot{z}_n) - L_v(x_m, \dot{z}_m)\| |(\dot{x}_n - \dot{x}_m)| ds,$$

is less than $\frac{\epsilon}{4}$ for $n, m > N$. Since $L(x,v) = \frac{1}{2} |v|^2_x + \theta_x(v)$ then there exists a constant $c > 0$ such that $\|L_x\| < c|v|^2_x$. Therefore the first term is bounded by
\[ T_n + T_m \] \( (cK^2) \| x_n - x_m \|_\infty \) thus the second term is small. By lemma 3.3 and the above remark we have
\[
a \int_0^1 |\dot{z}_n - \dot{z}_m|^2 \, ds \leq \frac{\epsilon}{2} + B \int_0^1 (|\dot{z}_n| + |\dot{z}_m|) |\dot{z}_n - \dot{z}_m| |x_n - x_m| + D |\dot{z}_n - \dot{z}_m| |x_n - x_m| \, ds,
\]
thus
\[
a \| \dot{z}_n - \dot{z}_m \|_2^2 \leq \frac{\epsilon}{2} + (4BK^2 + 2KD) |x_n - x_m|_\infty,
\]
as \( x_n \) converges in the \( C^0 \)-topology we get that \( z_n \) converges in the \( \mathcal{H}^1 \)-norm. Therefore \( x_n \) converges in the \( \mathcal{H}^1 \times \mathbb{R}^\tau \)-norm because \( T \neq 0 \) as we wished to show. \( \Box \)

4 Minimax method and Proof of theorem 1.1

This section follows closely Contreras' ideas in [1]. Let \( G \) be an open set in a Riemannian manifold and \( f : G \to \mathbb{R} \) a \( C^2 \) function. The vector field \(- \nabla f\) is not necessarily Lipschitz then the gradient flow \( \psi_s \) of \(- f\) might be only a local flow. For \( p \in G \) define \( \alpha(p) := \sup \{ t > 0 \mid s \to \psi_s(p) \text{ is defined on } s \in [0, t] \} \).

We say that the flow \( \psi_s \) is relatively complete on \([a \leq f \leq b]\) if for \( a \leq f(p) \leq b \), either \( \alpha(p) = +\infty \) or \( f(\psi_s(p)) \leq a \) for some \( 0 \leq s < \alpha(p) \).

We shall use the following mountain pass theorem

**Proposition 4.1** ([1], cor. 6.5) Let \( G \) be a \( C^2 \) Riemannian manifold and \( f : G \to \mathbb{R} \) a \( C^2 \) function. Let \( p, q \in G \) and \( c := \inf_{\Gamma \in C(p,q)} \sup_{s \in [0,1]} \{ f(\Gamma(s)) \} \), where \( C(p,q) := \{ \Gamma : [0,1] \to G \mid \Gamma \in C^0, \Gamma(0) = p, \Gamma(1) = q \} \). Suppose that

i).- \( c \in \mathbb{R} \) and the gradient flow of \(- f\) is relatively complete on \( |f - c| \leq \epsilon \)
for some \( \epsilon > 0 \).

ii).- \( \max \{ f(p), f(q) \} < c \).

iii).- There are closed subsets \( B \subset A \subset G \) such that

a).- \( f \) satisfies the Palais-Smale condition restricted to \( A \), at level \( c \).

b).- For some \( \epsilon_1 > 0 \), \( A \) contains the \( \epsilon_1 \)-neighbourhood of \( B \).

c).- For all \( \epsilon > 0 \), there are \( \lambda \in (0, \epsilon) \) and \( \Gamma \in C(p,q) \) such that \( \Gamma([0,1]) \subset (B \cap [f \leq c + \epsilon]) \cup [f \leq c - \lambda] \).

Then \( c \) is a critical value of \( f \) moreover \( K_{c,A} := \{ x \in G \mid x \in A, df(x) = 0, f(x) = c \} \), contains a point which is not a strict local minimizer.

Now we analyse whether \( S_k \) satisfies the conditions of the previous proposition

**Lemma 4.2** For all \( k \in \mathbb{R} \), if \( 0 \notin [a, b] \subset \mathbb{R} \), then the gradient flow of \(- S_k\) on \( A_0 \) is relatively complete on \([a \leq S_k \leq b]\).
Proof: The proof is a straightforward application to $L(x, v) = \frac{1}{2}|v|^2_x + \theta_x(v)$ of proof of lemma 6.7 in [1].

The next follows parallel arguments in [1], but applied to our context

**Proposition 4.3** ([1], p. 71) Take $c(k) := \inf_{\Gamma \in C(\gamma_0, \gamma_1)} \sup_{s \in [0, 1]} \{S_k(\Gamma(s))\}$, where $\gamma_0, \gamma_1 \in \Lambda_0$, $C(\gamma_0, \gamma_1) := \{\Gamma : [0, 1] \to \Lambda_M \mid \Gamma \in C^0, \Gamma(0) = \gamma_0, \Gamma(1) = \gamma_1\}$ and $k \in \mathbb{R}$. Suppose that for some $k_0 \in \mathbb{R}$ we have $\lim_{k \to k_0^+} c(k) \neq 0$ and that $c(k_0) > \max\{S_{k_0}(\gamma_0), S_{k_0}(\gamma_1)\}$. Then, there exists $\epsilon > 0$ such that for Lebesgue almost all $k \in (k_0, k_0 + \epsilon), c(k)$ is a critical value for $S_k$ on $\Lambda_0$ with a critical point which is not a strict local minimizer.

**Proof:** Observe that function $k \to c(k)$ is non-decreasing. Since $S_k$ is continuous on $k$ there is $\epsilon > 0$ such that

$$\max\{S_k(\gamma_0), S_k(\gamma_1)\} < c(k_0) \leq c(k) \neq 0 \ \forall \ k_0 < k < k_0 + \epsilon.$$ (1)

By Lebesgue’s theorem there is a total measure subset of $k_0 < k < k_0 + \epsilon$, where $c(\cdot)$ is locally Lipschitz. Given $k$ in this set and $M_k$ its Lipschitz constant. Let $B \subset A \subset \Lambda_0$ be the closed subset defined by $B := \{(x, T) \in \Lambda_0 \mid T < M_k + 2\}$ and $A := \{(x, T) \in \Lambda_0 \mid T < M_k + 3\}$. From (1), $c(k) \neq 0$ by prop. 3.6 we get that $S_k$ satisfies the Palais-Smale condition restricted to $A$ at level $c(k)$.

**Claim:** There exists $\Gamma \in C(\gamma_0, \gamma_1), \epsilon > 0$ and $\lambda > 0$ such that

$$\Gamma([0, 1]) \subset (B \cup (S_k \leq c(k) - \lambda)) \cap (S_k \leq c(k) + \epsilon).$$

Assuming the Claim, from proposition 4.1 we have that $S_k$ has a critical point in $A$ which is not a strict local minimizer this prove the proposition 4.3.

**Proof of Claim:** Fix $k$ such that $c(\cdot)$ is locally Lipschitz in $k$ we take a sequence $k_n \geq k$ with $k_n \to k$, then $S_{k_n}, S_k$ have a mountain pass geometry with the same set of paths in $C(\gamma_0, \gamma_1)$. Let $\Gamma_n \in C(\gamma_0, \gamma_1)$ be a path such that

$$\max_{s \in [0, 1]} S_{k_n}(\Gamma_n(s)) \leq c(k_n) + (k_n - k).$$

Now, since $c(\cdot)$ is continuous in $k$ and $k \to S_k(\gamma)$ is increasing then

$$\max_{s \in [0, 1]} S_k(\Gamma_n(s)) \leq \max_{s \in [0, 1]} S_{k_n}(\Gamma_n(s)) \leq c(k_n) + (k_n - k) \to c(k).$$

Take $s \in [0, 1]$ with $S_k(\Gamma_n(s)) > c(k) + (k_n - k)$ then $\Gamma_n(s) = (x, T)$ where

$$T := \frac{S_{k_n}(\Gamma_n(s)) - S_k(\Gamma_n(s))}{k_n - k} \leq M_k + 2,$$

for $n$ enough large. Given $\epsilon > 0$ take $n$ with $c(k_n) - c(k) + (k_n - k) < \epsilon$ and $\lambda := k_n - k$ then $\Gamma([0, 1]) \subset (B \cup (S_k \leq c(k) - \lambda)) \cap (S_k \leq c(k) + \epsilon)$, which proves the Claim. \qed
Proof of theorem 1.1: If $0 = \epsilon_0 < k < c(L)$ by definition of $c(L)$ there is $(x_1, T_1) \in \Lambda_0$ such that $S_k(x_1, T_1) < 0$. Now taking a constant curve $(x_0, T_0)$, from theorem 3.2 there exists a mountain pass geometry when we consider families of curves going from a constant curve with arbitrarily small action $(x_0, T_0)$ to a curve with negative action $(x_1, T_1)$. From proposition 4.3 there is $\epsilon > 0$ such that for almost all $k \in (k, k + \epsilon)$ there is a critical point of $S_k$ in $\Lambda_0$, this is a periodic orbit with trivial homotopy and positive $(L + k)$-action.

\[\Box\]

References


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