Hyperbolic Functions and the Heat Balance Integral Method

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Abstract

Implementations of the heat balance integral method are discussed in which hyperbolic functions are used in place of the familiar polynomial approximants.

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1 Introduction

A large number of problems in various areas of applied science appear as moving boundary or phase change problems. These can arise in heat conduction in conjunction with a change of phase and initial and moving boundary conditions, and need to be solved in a time-dependent space domain with a moving boundary condition. Since the moving boundary is a function of time and its location has to be determined as a part of the solution, such problems are

inherently non-linear. In general, the non-linearity associated with the moving boundary significantly complicates the analysis of this class of problems [1]. In this study, a simple example of a moving boundary problem, the one-dimensional melting ice problem is taken into consideration with one phase. The motion and location of the interface are unknown a priori and thus must be determined as part of the solution. Since material properties change following phase transformation and since a discontinuity in temperature gradient exists at the interface, it follows that each phase must be assigned its own temperature function.

Semi-analytic technique for generating approximate functional solutions is Goodman's heat balance integral method [2]. The profile must satisfy spatial boundary conditions [3]. One of the drawbacks of the heat balance integral method is how to choose the approximating function. In this paper the use of hyperbolic functions is explored within the context of a phase-change problem.

2 A Model Problem

Here we are interested in the temperature distribution $u(x, t)$ in the liquid region,

$0 < x < s(t)$ and in the location of the liquid/solid (moving) interface. The temperature u is governed by the heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < s(t), \quad t > 0, \quad (2.1)$$

subject to the conditions

$$u(s, t) = 0, \quad t = 0, \quad (2.2)$$

$$u(0, t) = 1, \quad t > 0. \quad (2.3)$$

The location of the liquid/solid interface is given by the heat balance equation known as the Stefan condition

$$-\beta \frac{ds}{dt} = \frac{\partial u}{\partial x}, \quad u = 0, \quad x = s(t), \quad t > 0. \quad (2.4)$$

Initially (at $t = 0$) there is no liquid region which implies the condition

$$s(0) = 0, \quad t = 0. \quad (2.5)$$

Referring to (2.4) and (2.5), this problem has the following exact solution for the liquid temperature distribution $u(x, t)$ and the interface location $s(t)$, respectively

$$u(x, t) = 1 - \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\pi}}\right)}{\operatorname{erf}(\alpha)}, \quad 0 \leq x \leq s(t), \quad t \geq 0, \quad (2.6)$$

$$s(t) = 2\alpha\sqrt{t}, \quad t \geq 0, \quad (2.7)$$

and α is the solution of the transcendental equation

$$\sqrt{\pi}\beta\alpha\operatorname{erf}(\alpha)e^{\alpha^2} = 1.$$

3 HBI Solution

Assume

$$u(x, t) = a + b \cosh\left(\frac{\pi x}{2s}\right) + c \sinh\left(\frac{\pi x}{2s}\right), \quad (3.1)$$

is a solution where a, b and c are constants to be found. Equations (2.2) and (2.3) imply that

$$1 = a + b \quad \text{and} \quad 0 = 2.509b + 2.301c. \quad (3.2)$$

$$\frac{du}{dx} = \frac{\pi}{2s} \left[b \sinh\left(\frac{\pi x}{2s}\right) + c \cosh\left(\frac{\pi x}{2s}\right) \right]. \quad (3.3)$$

$$\frac{du}{dt} = -\frac{\pi x}{2s^2} \left[b \sinh\left(\frac{\pi x}{2s}\right) + c \cosh\left(\frac{\pi x}{2s}\right) \right] \frac{ds}{dt}. \quad (3.4)$$

$$\left(\frac{du}{dx}\right)_{x=s} = \frac{\pi}{2s} [2.301b + 2.509c]. \quad (3.5)$$

Equations (2.1) and (3.5) gives

$$\frac{\pi}{2s} [2.301b + 2.509c] = -\beta \frac{ds}{dt} \implies s \frac{ds}{dt} = -\frac{\pi}{2\beta} [2.301b + 2.509c]. \quad (3.6)$$

Integrating both sides of equations (2.1) and using (2.4) gives

$$\int_0^s \frac{\partial u}{\partial t} dx = -\beta \frac{ds}{dt} - \frac{\pi c}{2s}. \quad (3.7)$$

$$\begin{aligned}
\int_0^s \frac{\partial u}{\partial t} dx &= \int_0^s -\frac{\pi x}{2s^2} \left[b \sinh\left(\frac{\pi x}{2s}\right) + c \cosh\left(\frac{\pi x}{2s}\right) \right] \frac{ds}{dt} dx \\
&= \left[\left(-2.509 + \frac{4.602}{\pi} \right) b + \left(\frac{3.018}{\pi} - 2.301 \right) c \right] \frac{ds}{dt}. \quad (3.8)
\end{aligned}$$

It follows from equations (3.7) and (3.8) that

$$s \frac{ds}{dt} = \frac{-\pi c}{2 \left[\left(-2.509 + \frac{4.602}{\pi} \right) b + \left(\frac{3.018}{\pi} - 2.301 \right) c + \beta \right]}. \quad (3.9)$$

We know that $s = \sqrt{\frac{\pi b t}{\beta}} = 2\alpha\sqrt{t}$, therefore $\alpha = \frac{1}{2}\sqrt{\frac{\pi b}{\beta}}$. For $\beta = 1$, the parameter values are

$$a = 0.5260, \quad b = 0.4740, \quad c = -0.7455, \quad \alpha = 0.6101 \quad \text{and} \quad s = 1.2202.$$

Table 1 shows the solution for $t = 1$. The tabulated values suggest that a

x/s	Analytic	Trigonometric [4]	Hyperbolic (3.1)
0	1	1	1
0.2	0.7753	0.8204	0.7855
0.4	0.5573	0.6235	0.5968
0.8	0.1654	0.2386	0.2226
s	1.2402	1.1268	1.1884

Table 1: HBI Solutions

hyperbolic profile provides a better accuracy.

4 Conclusion

A hyperbolic profile does provide more accuracy than a trigonometric profile.

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