**On the Asymptotic and L^p Properties of
Non-Oscillatory Solutions to the Second Order
Differential Equation $x'' + a(t)f(x)x' + b(t)g(x) = 0$**

Allan Kroopnick

University of Maryland University College

The Graduate School

3501 University Boulevard East

Adelphi, Maryland 20783, USA

allan.kroopnick@faculty.umuc.edu

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Abstract

This note discusses the asymptotic properties of the nonlinear differential equation $x'' + a(t)f(x)x' + b(t)g(x) = 0$. The behavior of non-oscillatory solutions are shown to be bounded and in $L^p[0, \infty)$ under specified conditions. Furthermore, the derivative are shown to be in $L^2[0, \infty)$ and the solutions as well as their derivatives shown to approach 0 as $t \rightarrow \infty$ implying stability. Finally, several examples illustrating these results are also given.

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In this note, using some fundamental tools from analysis we will show that any non-oscillatory solutions to the second-order nonlinear differential equation,

$$(1) \quad x'' + a(t)f(x)x' + b(t)g(x) = 0,$$

approaches 0 as $t \rightarrow \infty$. This will be true of their derivatives, too. In other words, the solutions are asymptotic stable. We shall prove this without the use of Liapunov functions [1, pp. 151-152] which is the usual method. The conditions on $a(t)$ and $b(t)$ are that they belong to $C^1[0, \infty)$, are non-negative on $[0, \infty)$, and both possess non-positive derivatives. Furthermore, $f(x)$ must be positive and continuous on $C(-\infty, \infty)$ and $g(x)$, too, must be continuous on $C(-\infty, \infty)$ with the property that $xg(x) > 0$ for $x \neq 0$ and, finally, $G(x) = \int^x g(u)du \rightarrow \infty$ as $|x| \rightarrow \infty$. Moreover, we shall show that these solutions are L^p -solutions when $a(t) > a_0 > 0$, $b(t) > b_0 > 0$, $f(x) > f_0 > 0$ for positive constants a_0, b_0, f_0 , and $xg(x) \geq kx^p$ for some positive constant k . By an L^p -solution, we mean a solution to (1) such that $\int_0^\infty |x(t)|^p dt < \infty$ where $p \geq 1$. This result extends the work of Kroopnick [2] and Straus [3] which discussed the boundedness of solutions but did address the asymptotic behavior for non-oscillatory solutions. We now formally state and prove our theorem.

Theorem: Give the differential equation (1). Suppose, $a(t)$ and $b(t)$ are elements of $C^1[0, \infty)$ with non-positive derivatives and, furthermore, both $f(x)$ and $g(x)$ are in $C(-\infty, \infty)$, then any non-oscillatory solution is bounded as $t \rightarrow \infty$. Moreover, if $a(t), b(t), f(x)$ are greater than positive constants a_0, b_0, f_0 and $xg(x) > 0$ for $x \neq 0$ then any non-oscillatory solution approaches 0 as $t \rightarrow \infty$. Finally, if $xg(x) \geq k|x|^p$ where $p \geq 2$, then the solution is an element of $L^p[0, \infty)$.

Proof. We begin our discussion by multiplying equation (1) by $x'(t)$ and then integrating from 0 to t . Thus we obtain,

$$(2) \quad x'(t)^2/2 + \int_0^t a(s)f(x(s))x'(s)^2 ds + b(t)G(x(t)) - \int_0^t b'(s)G(x(s))ds = \\ x'(0)^2/2 + b(0)G(x(0))$$

where we have integrated the third term in equation (1) by parts.

Looking at equation (2), it follows that both $x(t)$ and $x'(t)$ are bounded as $t \rightarrow \infty$.

Otherwise, the LHS of (2) would become infinite as $t \rightarrow \infty$ which is impossible. Should $a(t)$ and $f(x)$ remain greater than some positive constants a_0 and f_0 respectively, then we may conclude that $x'(t)$ is square-integrable, and, therefore, in $L^2[0, \infty)$.

We next show that $x(t)$ is in $L^p[0, \infty)$ when $xg(x) \geq kx^p$. Multiply equation (1) by $x(t)$ and integrate from 0 to t thereby obtaining,

$$(3) \quad x(t)x'(t) - \int_0^t x'(s)^2 ds + a(t)F(x(t)) - \int_0^t a'(s)F(x(s))ds + \int_0^t b(s)g(x)x ds = \\ a(0)F(x(0)) + x(0)x'(0). \quad (F(x) = \int^x f(u)du).$$

Here we have integrated by parts the first two terms of equation (1). Examining equation (3) shows us that as long as $b(t) > b_0 > 0$, then $x(t)$ is an element in $L^p[0, \infty)$ under the hypotheses of the theorem.

We next show that both $|x(t)|$ and $|x'(t)|$ approach 0 as $t \rightarrow \infty$. Without loss of generality, assume $x(t) > 0$ for $t \geq 0$. A similar argument works for $x(t) < 0$. In order to see that both $x(t)$ and $x'(t)$ approach 0, we first observe that $x'(t)$ must be of fixed sign. Otherwise, whenever $x'(t) = 0$, we have from equation (1) $x''(t) = -b(t)g(x) < 0$, so we have an infinite number of consecutive relative maxima one after the other which is impossible because a negative second derivative means $x'(t)$ is always decreasing between consecutive zeroes of $x'(t)$. Moreover, if we have two consecutive critical points, then $x''(t) = 0$ between these two points which we have shown is not possible. Furthermore, $x'(t)$ must be negative or else $x(t)$ will increase without bound and therefore $x(t)$ could not be in $L^p[0, \infty)$ should $x'(t)$ become positive. Therefore, by the monotonicity of both $x(\cdot)$ and $x'(\cdot)$ and the L^p property of $x(\cdot)$, we must have that $x(\cdot)$ and $x'(\cdot)$ approach 0 as $t \rightarrow \infty$.

Example 1. Consider the nonlinear differential equation,

$$(4) \quad x'' + 3x'/(1+t) + x^3 = 0.$$

A non-oscillatory solution to (3) is $x(t) = (1+t)^{-1}$. This solution is non-oscillatory and an element of $L^4[0, \infty)$ since $xg(x) = x^4$. Furthermore, its derivative $x'(t) = -(1+t)^{-2}$ is easily seen to be in $L^2[0, \infty)$ even though $a(t) = 3/(1+t)$ is not bounded away from zero.

Example 2. Consider the nonlinear differential equation,

$$(5) \quad x'' + (1+x^2)x' + x^3 = 0.$$

A non-oscillatory solution to (5) is $x(t) = e^{-x}$. This solution is non-oscillatory and an element of $L^4[0, \infty)$ since again $xg(x) = x^4$. Furthermore, its derivative $x'(t) = -e^{-x}$ is as before easily seen to be in $L^2[0, \infty)$.

Remark. In the linear case, the existence of non-oscillatory solutions to (1) are guaranteed by the Sturm Comparison theorem [4, pp.408-410] under suitable conditions. Furthermore, in the case of constant coefficients, $x''+ax'+bx=0$, nonoscillatory solutions occur whenever $a^2 - 4b \geq 0$.

The Sturm-Picone Comparison Theorem [5] can then be invoked for the case where the coefficients are variable. See references [6] and [7] and their bibliographies for a further discussion of nonoscillatory solutions in the linear case.

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