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On Nonlinear Fractional Integrodifferential Equations with Non Local Condition in Banach Spaces

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Abstract

The aim of the present paper is to establish the existence, uniqueness and boundedness of solutions to the nonlocal fractional integrodifferential equations in Banach spaces.

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1 Introduction

Let X be a general Banach space with norm $\|.\|$ and $C(J,X)$ be the Banach space of all continuous functions from $J=[0,T]$ into X endowed with the norm $\|x\|_C=\sup\{\|x(t)\|: t\in J\}$. We consider the class of nonlinear fractional integrodifferential equations with nonlocal condition of the type

$$ {}^cD^q x(t) = f\left(t, x(t), \int_0^t k(t,s)x(s)\,ds\right), \quad t>0; \tag{1}$$

$$ x(0)+g(x) = x_0 \in X; \tag{2}$$

where ${}^cD^q$ is the Caputo fractional derivative of order q, the function $f: J\times X\times X \to X$ is strongly measurable with respect to t and is continuous with respect to x, $k: J\times J\to R$, $g: C(J,X)\to X$ and $x_0\in X$.

In [1] Byszewski initiated the nonlocal condition proving the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [2] and Deng [4], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena.

The study of fractional differential and integral equations is linked to the wide applications of fractional calculus in physics, mechanics, signal processing, electromagnetics, biology, economics and many more. The theory of fractional calculus has been available and applicable to various fields of study. The investigation of the theory of fractional differential and integral equations has only been started quite recently [5, 6, 7, 8, 11]. The problems of existence, uniqueness and other properties of solutions of these equations (1) - (2) or their special forms have been studied by many authors by using different techniques, see [3, 7] and some of the references cited therein. Recently, in an interesting paper [15] XiWang Dong, JinRong Wang and Yong Zhou have investigated the existence, uniqueness and boundedness of solutions of special form of (1) - (2). Also in [16] YanLong Yang and JinRong Wang have studied the existence, uniqueness and other properties of solutions of special form of (1) - (2) when $q=1$. Our ideas were motivated by results of V. Lakshmikantham and A. S. Vatsala [7].

The aim of the present paper is to prove the existence, uniqueness and boundedness of solution of nonlinear fractional integrodifferential equations (1) - (2). The main tool employed in our analysis is based on the theory of fractional calculus and fixed point theorems.

The paper is organized as follows: Section 2, presents the preliminaries. Section 3, deals with the main results. Finally, in section 4, we discuss an example to illustrate the theory.

2 Preliminaries

Before proceeding to the statement of our main results, we setforth definitions, preliminaries and hypotheses that will be used in our subsequent discussion. For more details see [5, 14].

Definition 1 *The fractional integral of order γ with the lower limit zero for a function f is defined as*

$$I^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}}\, ds, \qquad t > 0, \gamma > 0, \tag{3}$$

provided the right side is point-wise defined on [0,∞), where $\Gamma(\cdot)$ is the gamma function.

Definition 2 *The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f : [0, \infty) \to R$ can be written as*

$$^{L}D^{q} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}}\, ds, t > 0, n-1 < \gamma < n. \tag{4}$$

Definition 3 *The Caputo derivative of order γ for a function $f : [0, \infty) \to R$ can be written as*

$$^{c}D^{q} f(t) = {}^{L}D^{q}\left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k} f^{(k)}(0) \right]. \tag{5}$$

Remark 1 .

1. *If $f(t) \in C^n[0, \infty]$, then for $t > 0, n-1 < \gamma < n$,*

$$^{c}D^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^n(s)}{(t-s)^{(\gamma+1-n)}}\, ds = I^{(n-\gamma)} f^n(t). \tag{6}$$

2. *The Caputo derivative of a constant is equal to zero.*

3. *If f is an abstract function with values in X, then integrals which appear in Definitions (3) and (4) are taken in Bochner's sense.*

We need the following results in our subsequent discussion.

Lemma 2.1 *(Bochner theorem) A measurable function $f : J \to X$ is a Bochner integral if $\|f\|$ is lebegue integrable.*

Lemma 2.2 *(Mazur lemma) If K is a compact subset of X, then its convex closure $\overline{conv}K$ is compact.*

Lemma 2.3 *(Ascoli-Arzela theorem) Let $S = \{s(t)\}$ is a function family of continuous mappings $s : J \to X$. If S is uniformly bounded and equicontinuous, and for any $t^* \in J$, the set $\{s(t^*)\}$ is relatively compact, then there exists a uniformly convergent function sequence $\{s_n(t)\}$ $(n = 1, 2, ..., t \in J)$ in S.*

Theorem 2.4 *(Krasnoselskii) Let B be a closed convex and nonempty subsets of X. Suppose that L and N are in general nonlinear operators which maps B into X such that:*
(i) $Lu + Nv \in B$ whenever $u, v \in B$;
(ii) L is a contraction mapping ;
(iii) N is compact and continuous.
Then there exists $w \in B$ such that $w = Lw + Nw$.

Theorem 2.5 *[14] Suppose $\beta > 0$,$\tilde{a}(t)$ is a nonnegative function locally integrable on J and $\tilde{g}(t)$ is a nonnegative, nondecreasing continuous function defined on $\tilde{g}(t) \le M, t \in J$, and suppose $x(t)$ is nonnegative and locally integrable on J with*

$$x(t) \le \tilde{a}(t) + \tilde{g}(t) \int_0^t \frac{x(s)}{(t-s)^{1-\beta}}\, ds, \quad t \in J.$$

Then

$$x(t) \le \tilde{a}(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(\tilde{g}(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} \tilde{a}(s) \right] ds, \quad t \in J.$$

Remark 2 *Under the hypothesis of Theorem 2.5, let $\tilde{a}(t)$ be a nondecreasing function on J. Then we have*

$$x(t) \le \tilde{a}(t) E_\beta(\tilde{g}(t)\Gamma(\beta)t^\beta), \tag{7}$$

where E_β is the Mittag-Leffler function defined by

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta+1)}. \tag{8}$$

For convenience, we list the following hypotheses used in our further discussion.

(H_1) For each $x, y \in X, f(t, x, y)$ is strongly measurable w.r.t. t on J.

(H_2) For each $t \in J, f(t, x, y)$ is continuous w.r.t. x and y on X.

(H_3) For arbitrary $x, y \in X$ there exist a constant $a_f > 0$ such that

$$\|f(t, x, y)\| \le a_f(1 + \|x\| + \|y\|)$$

and also for arbitrary $x \in C(J, X)$, there exists a constant $a_g \in (0, 1)$ such that $\|g(x)\| \le a_g(1 + \|x\|_C)$.

(H_4) For arbitrary $x_1, x_2, y_1, y_2 \in X$ satisfying $\|x_1\|, \|x_2\|, \|y_1\|, \|y_2\| \leq \rho$ there exists a constant $L_f(\rho) > 0$, such that

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq L_f(\rho)\,[\|x_1 - x_2\| + \|y_1 - y_2\|]$$

and also for arbitrary $x, y \in C(J, X)$, there exist a constant $L_g \in (0, 1)$ such that $\|g(x) - g(y)\| \leq L_g(\|x - y\|_C)$.

(H_5) For any $t \in J$, the set

$$K = \left\{(t-s)^{q-1} f\left(s, x(s), \int_0^s k(s,\tau)x(\tau)\,d\tau\right) : x \in C(J, X), s \in [0, t]\right\}$$

is relatively compact.

3 Main Results

In this section we state and prove some new results related to existence and uniqueness of solutions to the nonlocal problems for the fractional integrodifferential equation in Banach spaces.

Theorem 3.1 *Suppose that the hypotheses* $(H_1) - (H_3)$ *are satisfied. If* $x \in C(J, X)$ *is a solution of the fractional integral equation*

$$x(t) = x_0 - g(x) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s), \int_0^s k(s,\tau)x(\tau)\,d\tau)\,ds \qquad (9)$$

if and only if x *is a solution of the system (1) - (2).*

Proof : Define $B_r = \{x \in C(J, X) : \|x\|_C \leq r\}$ for any $r > 0$. Making use of hypotheses $(H_1) - (H_2)$, we have $f(t, ., .)$ is measurable function on J. Now for $x \in B_r$ and $t \in J$, we obtain

$$\int_0^t (t-s)^{q-1} \left\| f\left(s, x(s), \int_0^s k(s,\tau)x(\tau)\,d\tau\right)\right\| ds \leq a_f(1 + r + \, r\,T\,k_T) \left[\frac{T^q}{q}\right],$$

where $k_T = \sup\{|k(s,\tau)| : 0 \leq \tau \leq s \leq T\}$. Thus

$$\left\|(t-s)^{q-1} f\left(s, x(s), \int_0^s k(s,\tau)x(\tau)\,d\tau\right)\right\|$$

is Lebegue integrable with respect to $s \in [0, t]$, for all $t \in J$ and $x \in B_r$. Then from Bochner's theorem (Lemma 2.1) it follows that

$$(t-s)^{q-1} f\left(s, x(s), \int_0^s k(s,\tau)x(\tau)\,d\tau\right)$$

is Bochner integrable with respect to $s \in [0,t]$ for all $t \in J$. Let $G(\tau,s) = (t-\tau)^{-q}|\tau - s|^{q-1}$. Since $G(\tau,s)$ is a nonnegative measurable function on $D = [0,t] \times [0,t]$ for $t \in J$, we have

$$\begin{aligned}\int_D G(\tau,s)\,ds\,d\tau &= \int_0^t (t-\tau)^{-q}\left(\int_0^\tau (\tau-s)^{q-1}\,ds\right)d\tau \\ &\quad + \int_0^t (t-\tau)^{-q}\left(\int_\tau^t (s-\tau)^{q-1}\,ds\right)d\tau \le \frac{2T}{q(1-q)}\end{aligned}$$

and using hypothesis (H_3), we obtain $(t-\tau)^{-q}(\tau-s)^{q-1} f\left(s, x(s), \int_0^s k(s,\tau)x(\tau)\,d\tau\right)$ is a Lebesgue integrable function and hence we get

$$^{L}D^q\left[I^q f\left(t, x(t), \int_0^t k(t,s)x(s)ds\right)\right] = f\left(t, x(t), \int_0^t k(t,s)x(s)\,ds\right).$$

We claim that $x(t)$ is absolutely continuous on J. For any disjoint family of open intervals $\{(a_i,b_i)\}_{i=1}^n$ on J with $\sum_{i=1}^n (b_i - a_i) \longrightarrow 0$, we have

$$\begin{aligned}\sum_{i=1}^n \|x(b_i) - x(a_i)\| &\le \frac{a_f(1+r+\,r\,T\,k_T)}{\Gamma(q)}\sum_{i=1}^n \int_{a_i}^{b_i}(b_i-s)^{q-1}\,ds \\ &\quad + \frac{a_f(1+r+\,r\,T\,k_T)}{\Gamma(q)}\sum_{i=1}^n \int_0^{a_i}\{(a_i-s)^{q-1} - (b_i-s)^{q-1}\}\,ds \\ &\le \frac{2a_f(1+r+\,r\,T\,k_T)}{\Gamma(q+1)}\sum_{i=1}^n \{(b_i-a_i)^q\} \longrightarrow 0.\end{aligned}$$

Thus $x(t)$ is differential for almost all $t \in J$. According to the Remark (1), we have

$$\begin{aligned}{}^{c}D^q x(t) &= {}^{c}D^q\left[I^q f\left(s, x(s), \int_0^s k(s,\tau)x(\tau)\,d\tau\right)\right] \\ &= {}^{L}D^q\left[I^q f\left(t, x(t), \int_0^t k(t,\tau)x(\tau)\,d\tau\right)\right] \\ &\quad - \left[I^q f\left(t, x(t), \int_0^t k(t,\tau)x(\tau)\,d\tau\right)\right]_{t=0}\frac{t^{-q}}{\Gamma(1-q)}.\end{aligned}$$

Since, $(t-s)^{q-1} f\left(s, x(s), \displaystyle\int_0^s k(s,\tau)x(\tau)\,d\tau\right)$ is Lebesgue integrable w.r.t. $s \in [0,t]$, for all $t \in J$, we know that $I^q f(t, x(t), \int_0^t k(t,\tau)x(\tau)\,d\tau)_{t=0} = 0$ which implies that ${}^{c}D^q x(t) = f(t, x(t), \int_0^t k(t,s)x(s)ds)$, a.e. for $t \in J$. Moreover, $x(0) + g(x) = x_0$. Thus, $x \in C(J,X)$ is a solution of system (1) - (2). On the other hand, if $x \in C(J,X)$ is a solution of system (1) - (2), then x satisfies the equation (9).

Theorem 3.2 *Suppose system (1) - (2) has a solution x on an interval J. If the hypothesis (H_3) be satisfied, then there exists a constant $\rho > 0$ such that $\|x(t)\| \leq \rho$ for all $t \in J$.*

Proof : By Theorem 3.1, the solution of integrodifferential equations (1) - (2) is equivalent to the solution of integral equation (9). Using hypothesis (H_3), we have

$$\begin{aligned}\|x(t)\| &\leq \|x_0\| + a_g + a_g \|x\|_C + \frac{a_f T^q}{\Gamma(q+1)} \\ &\quad + \frac{a_f}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|x(s)\| \, ds + \frac{a_f k_T}{\Gamma(q)} \int_0^t \frac{(t-\tau)^q}{q} \|x(\tau)\| d\tau \\ &\leq \|x_0\| + a_g + a_g \|x\|_C + \frac{a_f T^q}{\Gamma(q+1)} + \frac{a_f}{\Gamma(q)} \left[1 + \frac{k_T T}{q}\right] \int_0^t (t-\tau)^{q-1} \|x(\tau)\| d\tau.\end{aligned}$$

Hence

$$\|x\|_C \leq \frac{\Gamma(q+1)(\|x_0\| + a_g) + a_f T^q}{(1-a_g)\Gamma(q+1)} + \frac{a_f}{(1-a_g)\Gamma(q)} \left[1 + \frac{k_T T}{q}\right] \int_0^t (t-\tau)^{q-1} \|x\|_C d\tau.$$

Applying the singular Gronwall inequality stated in Theorem 2.5, we obtain

$$\|x\|_C \leq \frac{\Gamma(q+1)(\|x_0\| + a_g) + a_f T^q}{(1-a_g)\Gamma(q+1)} \left[\sum_{n=0}^{\infty} \frac{\left(a_f T^q \left[1 + \frac{k_T T}{q}\right]\right)^n}{(1-a_g)^n \Gamma(nq+1)}\right],$$

where, $\sum_{n=0}^{\infty} \frac{\left(a_f T^q \left[1 + \frac{k_T T}{q}\right]\right)^n}{(1-a_g)^n \Gamma(nq+1)}$ is the well known Mittag-Leffler function. Thus there exists a constant $\rho > 0$ such that $\|x(t)\| \leq \rho, \quad$ for $\quad t \in J$.

Theorem 3.3 *Let the hypothesis $(H_1) - (H_4)$ be satisfied and suppose that the conditions*

$$a_g + \frac{a_f T^q (1 + T k_T)}{\Gamma(q+1)} < 1 \tag{10}$$

and

$$\gamma_{T,q,\rho} = L_g + \frac{T^q L_f(\rho)[1 + T k_T]}{\Gamma(q+1)} < 1 \tag{11}$$

are satisfied, then there exist an unique solution for the system (1) - (2).

Proof : Let $C_\rho = \{x \in C(J, X) : \|x(t)\| \leq \rho, t \in J\}$, where

$$\left[\frac{\|x_0\| + a_g + \left(\frac{a_f T^q (1+T k_T)}{\Gamma(q+1)}\right)}{1 - \left(a_g + \frac{a_f T^q (1+T k_T)}{\Gamma(q+1)}\right)}\right] \leq \rho.$$

Define a operator $F : C_\rho \to C_\rho$ as follows

$$(Fx)(t) = x_0 - g(x) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, x(s), \int_0^s k(s,\tau)x(\tau)\, d\tau)\, ds. \quad (12)$$

By Theorem 3.1, it is obvious that F is well defined on C_ρ in the sense of Bochner integrable. First we prove that $Fx \in C_\rho$, for $x \in C_\rho$.
For every $x \in C_\rho$, we have

$$\begin{aligned}
\|(Fx)(t+\delta) &- (Fx)(t)\| \\
&\leq \frac{a_f(1+\|x(s)\| + \rho T k_T)}{\Gamma(q)} \int_0^t \left[(t-s)^{q-1} - (t+\delta-s)^{q-1}\right] ds \\
&\quad + \frac{a_f(1+\|x(s)\| + \rho T k_T)}{\Gamma(q)} \int_t^{t+\delta} (t+\delta-s)^{q-1}\, ds \\
&\leq \frac{a_f(1+\rho+\rho T k_T)}{\Gamma(q)} \left[\frac{t^q}{q} - \frac{(t+\delta)^q}{q} + \frac{\delta^q}{q}\right] + \frac{a_f(1+\rho+\rho T k_T)}{\Gamma(q)} \left[\frac{\delta^q}{q}\right] \\
&\leq \frac{2a_f(1+\rho+\rho T k_T)}{\Gamma(q+1)} \delta^q.
\end{aligned}$$

As $\delta \to 0$ we observe that the right-hand side of the above inequality tends to zero. This shows that $Fx \in C(J, X)$.
Now, for all $t \in J$ and $x \in C_\rho$, we have

$$\begin{aligned}
\|(Fx)(t)\| &\leq \|x_0\| + a_g(1+\rho) + \frac{a_f(1+\rho+\rho T k_T)}{\Gamma(q)} \int_0^t (t-s)^{q-1}\, ds \\
&\leq \|x_0\| + (1+\rho)\left[a_g + \frac{a_f T^q (1+T k_T)}{\Gamma(q+1)}\right]. \quad (13)
\end{aligned}$$

Making use of condition (10) in equation (13), we obtain $\|Fx(t)\| \leq \rho$, which implies that $Fx \in C_\rho$.
Making hypothesis (H_4) for any $x, y \in C_\rho$, we have

$$\begin{aligned}
\|(Fx)(t) - (Fy)(t)\| &\leq L_g \|x-y\|_C + \frac{L_f(\rho)}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|x(s) - y(s)\|\, ds \\
&\quad + \frac{L_f(\rho) T k_T}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|x(s) - y(s)\|\, ds \\
&\leq \left[L_g + \frac{T^q L_f(\rho)[1+T k_T]}{\Gamma(q+1)}\right] \|x-y\|_C \leq \gamma_{T,q,\rho} \|x-y\|_C .
\end{aligned}$$

Since $\gamma_{T,q,\rho} < 1$, F is a contraction map on C_ρ and by applying Banach's contraction mapping principle the operator F has a unique fixed point on C_ρ. Hence the system (1) - (2) has an unique solution.

Theorem 3.4 *Suppose that the hypothesis* $(H_1) - (H_3)$ *and* (H_5) *holds. If the condition (10) hold, then system (1) - (2) has at least one solution.*

Proof : We subdivide the operator F defined by (12) into two operators P and Q on C_ρ as follows

$$(Px)(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, x(s), \int_0^s k(s,\tau)x(\tau)\,d\tau)\,ds, t \in J;$$
$$(Qy)(t) = x_0 - g(y), t \in J,$$

where C_ρ is given in Theorem3.3. Therefore, to prove the existence of a solution of system (1) - (2) is equivalent to prove that the operator $P + Q$ has a fixed point on C_ρ. The proof is divided into several steps.

Step 1. $Px + Qy \in C_\rho$: For every pair $x, y \in C_\rho$, we have

$$\begin{aligned} \|(Px)(t) + (Qy)(t)\| &\leq \|x_0\| + a_g(1+\rho) + \frac{a_f(1+\rho)(1+Tk_T)T^q}{\Gamma(q+1)} \\ &\leq \|x_0\| + (1+\rho)\left[a_g + \frac{a_f T^q(1+Tk_T)}{\Gamma(q+1)}\right]. \end{aligned}$$

Making use of condition (10) in above equation, we obtain $\|(Px)(t)+(Qy)(t)\| \leq \rho$, which implies that $Px + Qy \in C_\rho$.

Step 2. Q is a contraction mapping on C_ρ : For every $y_1, y_2 \in C_\rho$,

$$\|Qy_1 - Qy_2\| = \|g(y_1) - g(y_2)\| \leq L_g \|y_1 - y_2\|_C .$$

From hypothesis (H_4), $L_g \in (0,1)$ and hence Q is a contraction mapping.

Sep 3. P is a continuous operator: Let $\{x_n\}$ be a sequence of C_ρ such that $x_n \to x$ in C_ρ. Then by hypotheses (H_2) and (H_3), for all $t \in J$, we have

$$\begin{aligned} \|(Px_n)(t) &- (Px)(t)\| \\ &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} L_f(\rho)\left[\|x_n(s) - x(s)\| + k_T \int_0^s \|x_n(\tau) - x(\tau)\|\,d\tau\right] \\ &\to 0, \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thus, $Px_n \to Px$ as $n \to \infty$ which implies that P is continuous.

Step 4. P is a compact operator: Let $\{x_n\}$ be a sequence of C_ρ.

$$\|(Px_n)(t)\| \leq \frac{a_f(1+\rho+\ \rho T\, k_T)}{\Gamma(q)}\int_0^t (t-s)^{q-1}\,ds \leq \frac{a_f(1+\rho+\ \rho T\, k_T)T^q}{\Gamma(q+1)}.$$

Thus $\{x_n\}$ is uniformly bounded. Now we prove that $\{Px_n\}$ is equicontinuous. For $0 \leq t_1 < t_2 \leq T$, we get

$$\|(Px_n)(t_1) \quad - \quad (Px_n)(t_2)\|$$

$$\begin{aligned} &\leq \frac{a_f}{\Gamma(q)}\int_0^{t_1}\left[(t_1-s)^{q-1}-(t_2-s)^{q-1}\right](1+\rho+\rho T\,k_T)\,ds \\ &\quad+\frac{a_f}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}(1+\rho+\rho T\,k_T)\,ds \\ &\leq \frac{2a_f(1+\rho+\rho T\,k_T)}{\Gamma(q+1)}(t_2-t_1)^q. \end{aligned}$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero. Therefore $\{Px_n\}$ is equicontinuous. In view of the condition (H_5) and the Lemma 2.2, we know that $\overline{conv}K$ is compact. For any $t^* \in J$, we have

$$\begin{aligned} (Px_n)(t^*) &= \frac{1}{\Gamma(q)}\int_0^{t^*}(t^*-s)^{q-1}f(s,x_n(s),\int_0^s k(s,\tau)x_n(\tau)\,d\tau)\,ds \\ &= \frac{1}{\Gamma(q)}\lim_{k\to\infty}\sum_{i=1}^{k}\frac{t^*}{k}\left(t^*-\frac{it^*}{k}\right)^{q-1} f\left(\frac{it^*}{k},x_n\left(\frac{it^*}{k}\right),\int_0^{\frac{it^*}{k}}k\left(\frac{it^*}{k},\tau\right)x_n(\tau)\,d\tau\right) \\ &= \frac{t^*}{\Gamma(q)}\xi_n, \end{aligned}$$

where $\xi_n = \lim_{k\to\infty}\sum_{i=1}^{k}\frac{1}{k}\left(t^*-\frac{it^*}{k}\right)^{q-1}f\left(\frac{it^*}{k},x_n\left(\frac{it^*}{k}\right),\int_0^{\frac{it^*}{k}}k\left(\frac{it^*}{k},\tau\right)x_n(\tau)\,d\tau\right)$. Since $\overline{conv}K$ is convex and compact, we know that $\xi_n \in \overline{conv}K$. Hence, for any $t^* \in J$, the set $\{Px_n\}(n=1,2,...)$ is relatively compact. From Ascoli-Arzela theorem every $\{Px_n(t)\}$ contains a uniformly convergent subsequence $\{Px_{n_k}(t)\}(k=1,2,3,...)$ on J. Thus, the set $\{Px : x \in C_\rho\}$ is relatively compact. Therefore, the continuity of P and relatively compactness of the set $\{Px : x \in C_\rho\}$ imply that P is a completely continuous operator. By Krasnoselskii's fixed point theorem, we get that $P+Q$ has a fixed point on C_ρ. Hence system (1) - (2) has at least one solution. This completes the proof.

4 Application

In this section we give the application of our main results established in previous section. We consider the following nonlinear fractional integrodifferential equations with non local condition

$$ {}^cD^{\frac{1}{2}}x(t) = \frac{e^{-t}|x(t)|}{(5+e^t)(1+|x(t)|)}+\frac{1}{9}\int_0^t\frac{1}{(2+t)^2}x(s)ds,\ t\in J=[0,1], \tag{14}$$

$$ x(0) + \sum_{j=1}^{m}\lambda_j x(t_j)=0,\quad \lambda_j>0,\quad 0<t_1<t_2<...<t_m<1. \tag{15}$$

Problem (14)-(15) is of the form (1) - (2) with $q = \frac{1}{2}$,

$$f(t, x(t), K_1x(t)) = \frac{e^{-t}|x(t)|}{(5+e^t)(1+|x(t)|)} + \frac{1}{9}K_1x(t), \quad \text{and} \quad g(x) = \sum_{j=1}^{m}\lambda_j x(t_j),$$

where $K_1x(t) = \int_0^t \frac{1}{(2+t)^2}x(s)ds$. For $x_1, x_2 \in X$ and $t \in J$, we have

$$\begin{aligned} \|f(t, x_1, K_1x_1) &- f(t, x_2, K_1x_2)\| \\ &\leq \frac{e^{-t}}{(5+e^t)}\|x_1(t) - x_2(t)\| + \frac{1}{9}\|K_1x_1(t) - K_1x_2(t)\| \\ &\leq \frac{1}{6}\left[\|x_1 - x_2\| + \|K_1x_1 - K_1x_2\|\right]. \end{aligned} \tag{16}$$

Also

$$\|g(x_1) - g(x_2)\| \leq \sum_{j=1}^{m}\lambda_j \|x_1(t_j) - x_2(t_j)\| \leq \sum_{j=1}^{m}\lambda_j \max_{t_j \in J}\|x_1(t_j) - x_2(t_j)\|. \tag{17}$$

Similarly, for all $x \in X$ and each $t \in J$,

$$\|f(t, x, K_1x)\| \leq \frac{1}{6}\left\|\frac{|x(t)|}{(1+|x(t)|)}\right\| + \frac{1}{9}\|K_1x(t)\| \leq \frac{1}{6}\left[\|x\| + \|K_1x\|\right] \tag{18}$$

and

$$\|g(x)\| = \|\sum_{j=1}^{m}\lambda_j x(t_j)\| \leq \sum_{j=1}^{m}\lambda_j\|x(t_j)\| \leq \sum_{j=1}^{m}\lambda_j \max_{t_j \in J}\|x(t_j)\|. \tag{19}$$

From equations (16)-(19), we observe that the assumptions of Theorem 3.3 can be satisfied by choosing a sufficiently small values of λ_j such that $\sum_{j=1}^{m}\lambda_j + \frac{10}{24\sqrt{\pi}} < 1$, and hence the problem (14)-(15) has an unique solution.

References

[1] L. Byszewski; Theorems about existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problems , *J. Math. Anal. Appl.*, **162**(1991), 494-505.

[2] L. Byszewski, V. Lakshmikantham; Theorems about existence and uniqueness of solutions of a nonlocal abstract Cauchy problems in a Banach spaces, *Anal. Appl.*, **40**(1991), 11-19.

[3] K. Balchandran, J.Y. Park; Nonlocal Cauchy problem for abstract fractional semilinear evolution equations, *Nonlinear. Anal.* 71(2009), 4471-4475.

[4] K. Deng; Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions, *J. Math. Anal. Appl.* 179(1993), 630-637.

[5] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; Theory and applications of fractional differential equations, *North-Holland Mathematics Studies, vol.204, Elsevier Science B. V., Amsterdam* , (2006).

[6] V. Lakshmikantham, S. Leela, J. V. Devi; Theory of fractional dynamic systems, *Cambridge Scientific Publishers,* (2009).

[7] V. Lakshmikantham, A. S. Vatsala; Basic theory of fractional differential equations, *Nonlinear Anal.* (2008), 2677-2682.

[8] K. S. Miller, B. Ross; An introduction to the fractional calculus and differential equations, *John Wiley, New York*, (1993).

[9] G. M. N'Guerekata; A Cauchy problem for some fractional differential abstract differential equation with nonlocal conditions, *Nonlinear Anal.* 70(2009), 1873-1876.

[10] G. M. N'Guerekata, Corrigendum; A Cauchy problem for some fractional differential equation, *Commun. Math. Anal.*, **7**(2009), 11-11.

[11] I. Podlubny; Fractional differential equations, *Academic Press, San Diego*, (1999).

[12] JinRong Wang,W.Wei, Y. Yang; On some impulsive fractional differential equations in Banach spaces, *Opuscula Math.,* 30 (2010) 507-525.

[13] JinRong Wang, X. Yan, X.-H. Zhang, T.-M. Wang, X.-Z. Li; A class of nonlocal integrodifferential equations via fractional derivative and its mild solutions, *Opuscula Math.,* 31 (2011), 119-135.

[14] H. Ye, J. Gao, Y. Ding; A generalized Gronwall inequality and its application to a fractional differential equation, *J. Math. Anal. Appl.*, 328(2007), 1075-1081.

[15] XiWang Dong, JinRong Wang, Yong Zhou; On Nonlocal Problems for fractional differential equations in Banach spaces, *Opuscula Math.,* 31(3), (2011), 341-347.

[16] YanLong Yang and JinRong Wang; On some existence results of mild solutions for nonlocal Integrodifferential Cauchy problems in Banach spaces,*Opuscula Math.*, 31(3), (2011), 443-455. .

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